ON THE DISTRIBUTION OF THE MEANS OF SAMPLES DRAWN FROM A BESSEL FUNCTION POPULATION

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1. S. Bose[1] has made a critical study of the following Bessel function distribution : –

$$f(x)dx = C\, e^{-\alpha x} \,.\, x^{m/2} \,.\, I_m(q\sqrt{x})\, dx \qquad \dots \quad (1)$$

where
$$C = (2/q)^m \,.\, \alpha^{m+1} \,.\, e^{-q^2/4\alpha}, \quad \text{and} \quad q \geq 0,\ \alpha \geq 0,\ m > -1 \qquad \dots \quad (2)$$

This distribution first arose in a specialised form, in connection with the researches of the present author on the exact distribution of the D^2-statistics.[2] The object of the present paper is to find the distribution of the mean of a random sample of n, from this population. It appears that the distribution of means is of the same type as the mother population. Since the type III distribution is a special case of the distribution investigated here, Irwin's[3] distribution of the mean of a random sample of n from a type III population follows as a corollary.

2. The joint distribution of a sample

$$x_1,\ x_2,\ \dots\dots\dots\ x_n$$

from the population (1) can be written as

$$C^n \,.\, e^{-\alpha(x_1 + x_2 + \dots\dots\dots x_n)} \,.\, (x_1\ x_2\ \dots\dots\ x_n)^{m/2}$$

$$\times\ I_m(q\sqrt{x_1}) \,.\, I_m(q\sqrt{x_2}) \dots\dots I_m(q\sqrt{x_n}) \times dx_1\, dx_2 \dots\dots dx_n \qquad \dots \quad (3)$$

Now make the transformation

$$\begin{aligned} u_1 &= x_1 \\ u_2 &= x_1 + x_2 \\ u_3 &= x_1 + x_2 + x_3 \\ &\dots\dots\dots\dots\dots\dots \\ u_n &= x_1 + x_2 + x_3 + \dots\dots\dots x_n \end{aligned}$$

It is readily seen that

$$\frac{\partial(u_1, u_2, \dots\dots u_n)}{\partial(x_1, x_2, \dots\dots x_n)} = 1 \qquad \dots \quad (4)$$

so that (3) takes the form

$$C^n . e^{-au_n} . \{u_1(u_2-u_1)(u_3-u_2)\ldots\ldots(u_n-u_{n-1})\}^{m/2}$$
$$\times\ I_m(q\sqrt{u_1})\ I_m\{q\sqrt{(u_2-u_1)}\}\ I_m\{q\sqrt{(u_3-u_2)}\}\ \ldots\ldots\ldots$$
$$\ldots\ldots\ I_m\{q\sqrt{(u_n-u_{n-1})}\} . du_1\, du_2 \ldots\ldots\ldots du_n \qquad \ldots\ (5)$$

Now let, $$u_1 = u_2 \sin^2\theta \qquad \ldots\ (6)$$

Hence (5) can be written as

$$C^n . e^{-au_n} . u_2^m . \sin^m\theta . \cos^m\theta . \{(u_3-u_2)(u_4-u_3)\ldots\ (u_n-u_{n-1})\}^{m/2}$$
$$\times\ I_m(q\sin\theta_1\sqrt{u_2}) . I_m(q\cos\theta_1\sqrt{u_2}) . I_m\{q\sqrt{(u_3-u_2)}\}\ldots\ldots\ldots$$
$$\ldots\ldots\ I_m\{q\sqrt{(u_n-u_{n-1})}\} . 2u_2\sin\theta_1\cos\theta_1\, d\theta_1\, du_3\ldots\ldots\ du_n \qquad \ldots\ (7)$$

As $x_1, x_2, \ldots\ldots\ldots x_n$ vary from 0 to ∞, it is readily seen that θ_1 varies from 0 to $\pi/2$

Now we have to use Sonine's second finite integral [4]

$$\int_0^{\pi/2} J_\mu(z\sin\theta) . J_\nu(Z\cos\theta) . \sin^{\mu+1}\theta . \cos^{\nu+1}\theta\, d\theta$$
$$= \frac{z^\mu . Z^\nu . J_{\mu+\nu+1}\{\sqrt{(Z^2+z^2)}\}}{(Z^2+z^2)^{\frac{1}{2}(\mu+\nu+1)}} \qquad \ldots\ (8)$$

which is valid when both $R(\mu)$ and $R(\nu)$ exceed -1.

Putting $z = ia$, $Z = ib$, where a and b are real, we have in particular

$$\int_0^{\pi/2} I_\mu(a\sin\theta) . I_\nu(b\cos\theta) . \sin^{\mu+1}\theta . \cos^{\nu+1}\theta\, d\theta$$
$$= \frac{a^\mu\, b^\nu . I_{\mu+\nu+1}\{\sqrt{(a^2+b^2)}\}}{(a^2+b^2)^{\frac{1}{2}(\mu+\nu+1)}} \qquad \ldots\ (9)$$

Now integrating (7) for θ_1 from 0 to $\pi/2$ by the help of (9), we have

$$\text{Const} \times e^{-au_n} . u_2^{m+1}\{(u_3-u_2)(u_4-u_3)\ldots\ldots\ (u_n-u_{n-1})\}^{m/2}$$
$$I_{2m+1}\{q(\sqrt{2u_2})\} . I_m\{q\sqrt{(u_3-u_2)}\} . I_m\{q\sqrt{(u_4-u_3)}\}\ldots\ldots\ldots$$
$$\ldots\ .\ I_m\{q\sqrt{(u_n-u_{n-1})}\} . du_2\, du_3 \ldots\ldots\ldots du_n \qquad \ldots\ (10)$$

Now let us set $u_2 = u_3 \sin^2 \theta_2$ and again integrate out for θ_2 from 0 to $\pi/2$ by the help of (9). We get

$$\text{Const} \times e^{-\alpha u_n} \cdot u_3^{(3m+3)/2} \{(u_4-u_3)\ldots\ldots\ldots(u_n-u_{n-1})\}^{m/2}$$

$$I_{3m+2}\{q\sqrt{(3u_3)}\} \cdot I_m\{q\sqrt{(u_4-u_3)}\}\ldots\ldots\ldots\ldots$$

$$\ldots\ldots I_m\{q\sqrt{(u_n-u_{n-1})} \cdot du_3\, du_4\ldots\ldots\ldots du_n \quad \ldots \quad (11)$$

Proceeding on in this manner we finally obtain for the distribution of u_n,

$$\text{Const} \times e^{-\alpha u_n} \cdot u_n^{(mn+n-1)/2} \cdot I_{mn+n-1}\{q\sqrt{(nu_n)}\}\, du_n \quad \ldots \quad (12)$$

If $\bar{x}$ is the sample mean clearly

$$u_n = n\bar{x} \quad \ldots \quad (13)$$

Hence the distribution of $\bar{x}$ is

$$C' \cdot e^{-n\alpha\bar{x}} \cdot \bar{x}^{(mn+n-1)/2} \cdot I_{mn+n-1}(nq\sqrt{\bar{x}})\, d\bar{x} \quad \ldots \quad (14)$$

Comparing with (1), and remembering that both in (1) and (14) the definite integral taken over the values 0 to ∞ of the variable must be unity, we have

$$C' = (2/nq)^{mn+n-1} \cdot (n\alpha)^{n(m+1)} \cdot e^{-nq^2/4\alpha} \quad \ldots \quad (15)$$

3. If now in (1) and (15) we make $q \to 0$ and remember that

$$\underset{z\to 0}{\text{Lt}} \frac{I_k(z)}{z^k} = \frac{1}{2^k\,\Gamma(k+1)} \quad \ldots \quad (16)$$

then (1) reduces to the type III form

$$\frac{1}{\Gamma(m+1)} \alpha^{m+1} \cdot e^{-\alpha x} \cdot x^m\, dx \quad \ldots \quad (17)$$

and (15) reduces to the following form

$$\frac{1}{\Gamma\{n(m+1)\}} \alpha^{n(m+1)} \cdot e^{-n\alpha\bar{x}} \cdot \bar{x}^{mn+n-1}\, d\bar{x} \quad \ldots \quad (18)$$

which therefore is the distribution of the mean of a random sample of n, from the type III population represented by (17), which is the distribution given by Irwin.[3]

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4. WATSON, G. N.: *Theory of Bessel Functions*, p. 376.