A-optimal diallel crosses for test versus control comparisons

By ASHISH DAS
Indian Statistical Institute, New Delhi 110 016, India

SUDHIR GUPTA
Northern Illinois University, Dekalb, IL 60115, USA

AND SANPEI KAGEYAMA
Hiroshima University, Higashi-Hiroshima 739-8524, Japan

Summary

A-optimality of block designs for control versus test comparisons in diallel crosses is investigated. A sufficient condition for designs to be A-optimal is derived. Type S_0 designs are defined and A-optimal type S_0 designs are characterized. A lower bound to the A-efficiency of type S_0 designs is also given. Using the lower bound to A-efficiency, type S_0 designs are shown to yield efficient designs for test versus control comparisons.

Some key words: Type S design; Type S_0 design; A-optimality; Inbred lines; A-efficiency.

1. Introduction

Although designs for varietal trials and factorial experiments have been extensively investigated in literature over the past several decades, it was not until recently that some progress in the design of diallel cross experiments has been made, see e.g. Gupta and Kageyama (1994), Dey and Midha (1996), Mukerjee (1997), Das, Dey and Dean (1998). Designs for control versus test comparisons where the treatments form different levels of a factor have also been extensively investigated in the literature; see Majumdar (1996). The problem of deriving designs appropriate for diallel crosses is quite different from the set-up of designs for varietal trials and factorial experiments. Therefore, here we continue the work of Gupta and Kageyama (1994) for studying optimal block designs for control versus test comparisons among the lines with respect to their general combining ability effects. Recently Choi, Gupta and Kageyama (2002) introduced a class of designs, called type S block designs, for control-test comparisons in a diallel cross experiment. Let $p\ (> 2)$, b, and k denote the number of test lines, number of blocks and block size respectively. In Section 2 we define a sub-class of type S designs, called type S_0 designs, and derive a sufficient condition for designs to be A-optimal. Henceforth, by optimal we mean A-optimal, and by efficiency we mean A-efficiency. We also characterize optimal type S_0 designs in Section 2, and show the optimality of some of the designs of Choi, Gupta and Kageyama (2002). A new method of constructing type S_0 designs is also provided there. A type S_0 design satisfying the derived sufficient condition for optimality does not always exist. Therefore, a lower bound to the efficiency of a type S_0 design is defined in Section 3. For type S_0 designs in a practical range $p \leq 30$, $b \leq 50$, $k \leq p$, 70 designs out of a total of 247 possible types S_0 designs are optimal. Of the remaining 177 designs that do not satisfy the sufficient condition for optimality, 175 designs have lower bound to efficiency at least 0.80. Thus, type S_0 designs provide highly efficient designs for control-test comparisons. For the sake of brevity, table of efficient type S_0 designs is not presented in this paper, and it will be reported elsewhere.

2. Optimal designs

We consider diallel cross experiments involving $p+1$ inbred lines, giving rise to a total of $n_c = p(p+1)/2$ distinct crosses. Let a cross between lines i and j be denoted by $(i,j), i < j = 0,1,\ldots,p$. Suppose line 0 is a control or a standard line and lines $1,\ldots,p$ are test lines. Let s_{dj} denote the total number of times that the jth line occurs in the crosses in the design d, $j = 0,1,\ldots,p$. Further let $s_d = (s_{d0}, s_{d1}, \ldots, s_{dp})'$ and let n denote the total number of crosses in the design. Following e.g., Gupta and Kageyama (1994), the model under the block design set-up, for a design d involving $p+1$ inbred lines and b blocks each containing k crosses, is assumed to be

$$Y_d = \mu 1_n + \Delta_{1d}\tau + \Delta_{2d}\beta + \varepsilon,$$

where Y_d is the $n \times 1$ vector of responses, μ is the overall mean, 1_t is the $t \times 1$ vector of 1's, $\tau = (\tau_0, \tau_1, \ldots, \tau_p)'$ is the vector of $p+1$ general combining ability effects, $\beta = (\beta_1, \beta_2, \ldots, \beta_b)'$ is the vector of b block effects and Δ_{1d} (Δ_{2d}) is the corresponding observation versus line (block) design matrix, that is, the (h,l)th element of Δ_{1d} (Δ_{2d}), is 1 if the hth observation pertains to the lth line (block), and is zero otherwise, and ε is the $n \times 1$ vector of independent random errors with zero expectation and a constant variance σ^2. The coefficient matrix of the reduced normal equations for estimating the vector of general combining ability effects is then given by

$$C_d = G_d - \frac{1}{k} N_d N_d' \tag{2.1}$$

where $N_d = (n_{dij})$, $i = 0, 1, \ldots, p$; $j = 1, \ldots, b$, is the $(p+1) \times b$ line versus block incidence matrix, $G_d = (g_{dii'}), g_{dii} = s_{di}$, and for $i \neq i', g_{dii'}$ is the number of times the cross (i,i') appears in the design.

Let $\mathcal{D}(p+1,b,k)$ denote the set of all connected designs with p test lines, one control line and bk crosses arranged in b blocks each of size k. A design $d \in \mathcal{D}(p+1,b,k)$ is said to be optimal for control-test comparisons if it minimizes $\sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0})$, where $\hat{\tau}_{di} - \hat{\tau}_{d0}$ denotes the best linear unbiased estimator (BLUE) of $\tau_i - \tau_0$ using d. Let $P = (-1_p \quad I_p)$ where I_t denotes the identity matrix of order t. Then the covariance matrix for the BLUE's $(\hat{\tau}_{d1} - \hat{\tau}_{d0}, \hat{\tau}_{d2} - \hat{\tau}_{d0}, \ldots, \hat{\tau}_{dp} - \hat{\tau}_{d0})$ of the control-test contrast is $\sigma^2 P C_d^- P'$. If one partitions C_d as

$$C_d = \begin{pmatrix} c_{d00} & \gamma_d' \\ \gamma_d & M_d \end{pmatrix} \tag{2.2}$$

then it can be shown (see Gupta, 1989) that $(PC_d^- P')^{-1} = M_d$, i.e., M_d is the information matrix for the control-test contrasts. For a design d in $\mathcal{D}(p+1,b,k)$, using Kiefer's (1975) technique of averaging, we obtain

$$tr(PC_d^- P') \geq tr(P\bar{C}_d^- P'); \tag{2.3}$$

see also Majumdar and Notz (1983) and Jacroux and Majumdar (1989). Here $\bar{C}_d = \frac{1}{p!}\sum_{\pi}\pi C_d \pi'$, the summation being taken over all $(p+1)\times(p+1)$ permutation matrices π that correspond to permutations of the p test treatments only. Partitioning $\bar{C}_d$ as in (2.2), we see that $\bar{M}_d = (P\bar{C}_d^- P')^{-1}$ is a completely symmetric matrix. In general, there may be no design in $\mathcal{D}(p+1,b,k)$ for which $\bar{M}_d$ is the information matrix for the control-test contrasts. If there is such a design, then for this design, call it d^*, $M_{d^*} = \bar{M}_{d^*}$ is completely symmetric and γ_{d^*} of (2.2) is a vector with all entries equal. That is, d^* belongs to a class of designs, called type S block designs, introduced by Choi, Gupta and Kageyama (2002).

Definition 2.1. A design $d \in \mathcal{D}(p+1,b,k)$ is called a type S block design if there are positive integers g_0, g_1, λ_0 and λ_1, such that for $i \neq i' = 1, \ldots, p$,

$$g_{d0i} = g_0,\ g_{dii'} = g_1,\ \Sigma_{j=1}^{b} n_{d0j}n_{dij} = \lambda_0,\ \Sigma_{j=1}^{b} n_{dij}n_{di'j} = \lambda_1.$$

We denote a type S block design with parameters p, b, k, g_0, g_1, λ_0 and λ_1 by $S(p,b,k,g_0,g_1,\lambda_0,\lambda_1)$. For an $S(p,b,k,g_0,g_1,\lambda_0,\lambda_1)$ design d it holds that $s_{d0} = pg_0$, $s_{d1} = s_{d2} = \cdots = s_{dp} = g_0 + (p-1)g_1$, $bk = (s_{d0} + ps_{d1})/2$,

$$Var(\hat{\tau}_{di} - \hat{\tau}_{d0}) = \frac{k\{a_1 - (p-2)b_1\}\sigma^2}{(a_1+b_1)\{a_1-(p-1)b_1\}},\ \ i = 1,\ldots,p\ ,$$

$$Cov(\hat{\tau}_{di} - \hat{\tau}_{d0},\ \hat{\tau}_{di'} - \hat{\tau}_{d0}) = \frac{kb_1\sigma^2}{(a_1+b_1)\{a_1-(p-1)b_1\}},\ \ i \neq i' = 1,\ldots,p\ ,$$

where $a_1 = \lambda_0 - kg_0 + (p-1)b_1$ and $b_1 = \lambda_1 - kg_1$.

Definition 2.2. A type S_0 block design denoted by $S_0(p,b,k,g_0,g_1,\lambda_0,\lambda_1)$ is a type S block design with the property that $|n_{d0j} - n_{d0j'}| \leq 1, |n_{dij} - n_{di'j'}| \leq 1$ for $i,i' = 1,\ldots,p$; $j,j' = 1,\ldots,b$.

Using $[z]$ to denote the largest integer not exceeding z, we now introduce some notations that are used in the sequel.

$a(s) = (2bk-s)(2x+1) - pbx(x+1),\ x = [\frac{2bk-s}{pb}]$,

$h(s) = s(2y+1) - by(y+1),\ y = [\frac{s}{b}]$,

$g(s;p,b,k) = \frac{p}{s-h(s)/k} + \frac{(p-1)^2}{2bk-s-a(s)/k-(s-h(s)/k)/p}$.

For an $S_0(p,\ b,\ k,\ g_0,\ g_1,\ \lambda_0,\ \lambda_1)$ design it can be shown that $n_{d0j} = [\frac{pg_0}{b}]$ or $[\frac{pg_0}{b}]+1$, $2ks_0 = h(s_0) + p\lambda_0$, $2ks_1 = h(s_1) + (p-1)\lambda_1 + \lambda_0$, and $n_{dij} = [\frac{2bk-pg_0}{pb}]$ or $[\frac{2bk-pg_0}{pb}]+1$, $i = 1,\ldots,p$; $j = 1,\ldots,b$.

We require the following lemmas for characterizing optimal type S_0 designs.

Lemma 2.1. *If $d \in \mathcal{D}(p+1,b,k)$ then $\bar{M}_d$ has eigenvalues μ_{d1}, $\mu_{d2} = \cdots = \mu_{dp}$ with*

$$\mu_{d1} = \frac{ks_{d0} - \Sigma_{j=1}^{b} n_{d0j}^2}{pk},\quad \mu_{d2} = \frac{2bk - s_{d0} - \frac{1}{k}\Sigma_{i=1}^{p}\Sigma_{j=1}^{b} n_{dij}^2 - \mu_{d1}}{p-1}.$$

PROOF. From (2.1) and (2.2), the entries of M_d are

$$m_{dii'} = \begin{cases} s_{di} - \frac{1}{k}\Sigma_{j=1}^{b} n_{dij}^2 & (i = i') \\ g_{dii'} - \frac{1}{k}\Sigma_{j=1}^{b} n_{dij} n_{di'j} & (i \neq i') \end{cases}$$

and the sum of the entries in the ith row (or i'th column) is $\sum_{i'=1}^{p} m_{dii'} = -g_{d0i} + \frac{1}{k}\Sigma_{j=1}^{b} n_{d0j} n_{dij}$. The lemma then follows by noting that that $\bar{M}_d = \xi_1 I_p + \xi_2 1_p 1_p'$ with $\xi_2 = \frac{1}{p(p-1)} \sum_{1 \leq i \neq i' \leq p} m_{dii'}$ and $\xi_1 = \frac{1}{p}\sum_{i=1}^{p} m_{dii} - \xi_2$.

LEMMA 2.2 *[Cheng, 1978]. For given positive integers v and t, the minimum of $\sum_{i=1}^{v} n_i^2$ subject to $\sum_{i=1}^{v} n_i = t$, where n_i's are non-negative integers, is obtained when $t - v[t/v]$ of the n_i's are equal to $[t/v] + 1$ and $v - t + v[t/v]$ are equal to $[t/v]$. The corresponding minimum of $\sum_{i=1}^{v} n_i^2$ is $t(2[t/v] + 1) - v[t/v]([t/v] + 1)$.*

LEMMA 2.3. *Let $d \in \mathcal{D}(p + 1, b, k)$ and $n_{d01}, \ldots, n_{d0b}$ be fixed quantities. Then*

$$tr(M_d^{-1}) \geq \mu_{d1}^{-1} + (p-1)^2\{2bk - s_{d0} - a(s_{d0})/k - \mu_{d1}\}^{-1} (= \theta_d,\ say). \tag{2.4}$$

PROOF. Using Lemma 2.2 we have $\Sigma_{i=1}^{p}\Sigma_{j=1}^{b} n_{dij}^2 \geq a(s_{d0})$. Thus from (2.3) we get $tr(M_d^{-1}) \geq \mu_{d1}^{-1} + (p-1)\mu_{d2}^{-1} \geq \mu_{d1}^{-1} + (p-1)^2\{2bk - s_{d0} - a(s_{d0})/k - \mu_{d1}\}^{-1}$.

LEMMA 2.4. *Suppose $d \in \mathcal{D}(p + 1, b, k)$ satisfies $\Sigma_{i=1}^{p}\Sigma_{j=1}^{b} n_{dij}^2 = a(s_{d0})$ and has $s_{d0} > b[\frac{k}{2}]$. Then there exists a design d^* satisfying* (i) *$\Sigma_{i=1}^{p}\Sigma_{j=1}^{b} n_{d^*ij}^2 = a(s_{d^*0})$ and* (ii) *$s_{d^*0} \leq b[\frac{k}{2}]$ such that $\theta_{d^*} \leq \theta_d$ unless* (i) *$p = 5, k = 3$,* (ii) *$p = 4, k$ odd,* (iii) *$p = 3$.*

PROOF. We replace d by a d^* which is such that

$$n_{d^*0j} = n_{d0j} \text{ if } n_{d0j} \leq [\frac{k}{2}] \text{ and } n_{d^*0j} = k - n_{d0j} \text{ if } n_{d0j} > [\frac{k}{2}].$$

Clearly, $s_{d^*0} < s_{d0}$, and s_{d^*0} satisfies $s_{d^*0} \leq b[\frac{k}{2}]$. Also, $\mu_{d^*1} = \mu_{d1}$. The result then follows by noting that the function $\psi(s_{d0}) = 2bk - s_{d0} - a(s_{d0})/k$ decreases as s_{d0} increases except when (i) $p = 5, k = 3$, (ii) $p = 4, k$ odd and (iii) $p = 3$.

LEMMA 2.5. *Let $d \in \mathcal{D}(p + 1, b, k)$. Then*

$$\theta_d \geq pk\{(ks_0 - h(s_{d0}))^{-1} + (p-1)^2(pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0} + h(s_{d0}))^{-1}\} \tag{2.5}$$

where θ_d is the same as in (2.4).

PROOF. From Lemma 2.1 and equation (2.4) we have

$$\begin{aligned} \theta_d &= pk\{(ks_0 - \Sigma_{j=1}^{b} n_{d0j}^2)^{-1} + (p-1)^2(pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0} + \Sigma_{j=1}^{b} n_{d0j}^2)^{-1}\} \\ &= pk\{(ks_0 - q)^{-1} + (p-1)^2(w + q)^{-1}\}, \end{aligned}$$

where $q = \Sigma_{j=1}^{b} n_{d0j}^2$ and $w = pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0}$. For fixed s_{d0}, $\frac{s_{d0}^2}{b} \leq q \leq ks_{d0}$. We shall prove that

$$\delta\theta_d/\delta q \geq 0 \text{ for all } q \in [\frac{s_{d0}^2}{b}, ks_{d0}]. \tag{2.6}$$

Inequality (2.5) will then follow from (2.6) since a sharp lower bound for q is $h(s_{d0})$. To prove (2.6), it is enough to show that $(w+q)^2 \geq (p-1)^2(ks_{d0}-q)^2$. Equivalently, since both $ks_{d0}-q$ and $w+q$ are non-negative, we need only show that $w+q \geq (p-1)(ks_{d0}-q)$, i.e., $q \geq 2ks_{d0}+a(s_{d0})-2bk^2$. Thus, since $q \geq \frac{s_{d0}^2}{b}$, it is sufficient to show that $\frac{s_{d0}^2}{b} \geq 2ks_{d0}+a(s_{d0})-2bk^2$. Now we consider separately the two cases (i) $2k \leq p$ and (ii) $2k > p$.

Case (i): $2k \leq p$. Here $a(s_{d0}) = 2bk - s_{d0}$ since $x = [\frac{2bk-s_{d0}}{pb}] = 0$. Therefore, we need to show $\frac{s_{d0}^2}{b} \geq 2ks_{d0}+2bk-s_{d0}-2bk^2$, or $s_{d0}\{s_{d0}-b(2k-1)\}+2b^2k(k-1) \geq 0$. This inequality holds because $s_{d0} \leq bk/2$, a consequence of Lemma 2.4.

Case (ii): $2k > p$. Here $a(s_{d0}) \leq \frac{(2bk-s_{d0})^2}{pb} + \frac{pb}{4}$. Therefore, we need to show $\frac{s_{d0}^2}{b} \geq 2ks_{d0} + \frac{(2bk-s_{d0})^2}{pb} + \frac{pb}{4} - 2bk^2$, or $4(p-1)s_{d0}^2 - 8bk(p-2)s_{d0} + 8b^2k^2(p-2) - p^2b^2 \geq 0$. This inequality holds whenever $s_{d0} \leq bk/2$. It is easy to see that the inequality also holds for the particular cases (i) $p=5, k=3$, (ii) $p=4, k$ odd and (iii) $p=3$. Using Lemma 2.4, the inequality holds for the other cases as well.

Finally, the following theorem can be established using Lemmas 2.1 - 2.5.

THEOREM 2.1. *Suppose s_0 is an integer defined by*

$$g(s_0;p,b,k) = \min_{1\leq s\leq c} g(s;p,b,k), \tag{2.7}$$

where $c = bk$ if (i) $p=5, k=3$, (ii) $p=4$, k *odd or* (iii) $p=3$, *else* $c = b[\frac{k}{2}]$. *Then a type S_0 block design $S_0(p,b,k,g_0,g_1,\lambda_0,\lambda_1)$ with $g_0 = \frac{s_0}{p}$, $g_1 = \frac{s_1-g_0}{p-1}$, $\lambda_0 = \frac{2ks_0-h(s_0)}{p}$, $\lambda_1 = \frac{2ks_1-h(s_1)-\lambda_0}{p-1}$ and $s_1 = \frac{2bk-s_0}{p}$ is optimal in $\mathcal{D}(p+1,b,k)$.*

The integer s which minimizes $g(s;p,b,k)$ can easily be found using a computer.

Example 2.1. For $p=5$, $b=10$, $k=2$, the $g(s;5,10,2)$ is minimized for $s=10$. Thus the following $S_0(5,10,2,2,1,6,3)$ design d^* with $s_{d^*0} = s_0 = 10$ is optimal over $\mathcal{D}(6,10,2)$:

{(3,5) (0,1)}, {(1,4) (0,2)}, {(2,5) (0,3)}, {(1,3) (0,4)}, {(2,4) (0,5)},
{(4,5) (0,1)}, {(1,5) (0,2)}, {(1,2) (0,3)}, {(2,3) (0,4)}, {(3,4) (0,5)}.

Theorem 2.1 is useful in checking the optimality of type S_0 designs. Choi, Gupta and Kageyama (2002) gave some series of type S designs. Designs of their Series 1, 3 and 4 are type S_0 designs as they also satisfy the requirements of Definition 2.2. Among designs for $p \leq 30$, Series 1 designs for $p = 3,5,7,9$ are optimal. Clearly, not all type S_0 designs are optimal. In the next section we give a lower bound e_{Ad} to efficiency of a type S_0 design d, and show that type S_0 designs are highly efficient for control-test comparisons.

We now present a new method of constructing type S_0 designs. Following Gupta and Kageyama (1994), let d_n be a universally optimal block design for diallel crosses obtained using a nested balanced incomplete block design with parameters $v=p$, b_n, r_n, $k_n(<p)$, λ_n, the nest having block size two. Let B_i denote the ith block of the nested balanced incomplete block design, and let $\bar{B}_i$ denote the corresponding complementary block such that

the contents of B_i and $\bar{B}_i$ taken together form one replication of the lines $i = 1, \ldots, p$. Let $i_1, i_2, \ldots, i_{p-k_n}$ denote the contents of $\bar{B}_i$. Then, appending to the ith block of d_n the crosses $(0, i_1), (0, i_2), \ldots, (0, i_{p-k_n})$ yields a type S_0 design of the following theorem.

THEOREM 2.2. *The existence of a nested balanced incomplete block design with parameters* $v = p, b_n, r_n, k_n, \lambda_n$, *the nest having block size two, implies the existence of an* $S_0(p, b = b_n, k = p - k_n/2, g_0 = b_n(1 - k_n/p), g_1 = \frac{b_n k_n}{p(p-1)}, \lambda_0 = b_n(p - k_n), \lambda_1 = b_n)$.

Several series of universally optimal block designs d_n are available in literature, see e.g. Das, Dey and Dean (1998). Using each of these series, a corresponding series of type S_0 designs can be derived using Theorem 2.2. For instance, Das, Dey and Dean (1998) gave Family 1 designs with parameters $p = 4t + 1,\ b_n = t(4t + 1),\ k_n = 4,\ \lambda_n = 3$, where t is a positive integer, and p is a prime or prime power. Using this family of designs, Theorem 2.2 yields a series of type S_0 designs with parameters $p = 4t+1,\ b = t(4t+1),\ k = 4t-1,\ g_0 = t(4t-3),\ g_1 = 1$, where $t \geq 1$, and p is a prime or prime power.

Morgan, Preece and Rees (2001) tabulated all possible nested balanced incomplete block designs for $p \leq 16$, and $r_n \leq 30$. Using designs in their table with nested design having block size 2, Theorem 2.2 yields a total of 24 type S_0 designs. Type S_0 designs obtained from designs in their table at serial numbers 6, 13.c, 40, 50.c and 62 are optimal as they satisfy the condition of Theorem 2.1. Of the remaining 19 designs, 7 designs have $e_{Ad} \geq 0.95$, 6 designs have $0.80 \leq e_{Ad} < 0.95$, 5 designs have $0.70 \leq e_{Ad} < 0.80$, and 1 design has $e_{Ad} = 0.66$. The e_{Ad} is the lower bound to efficiency as defined in the next section.

3. EFFICIENCY

The efficiency of a design $d \in \mathcal{D}(p+1, b, k)$ for control-test comparisons compared to an optimal design $d_A \in \mathcal{D}(p+1, b, k)$ is defined as

$$E_{Ad} = \frac{\sum_{i=1}^{p} Var(\hat{\tau}_{d_A i} - \hat{\tau}_{d_A 0})}{\sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0})}.$$

Clearly, $\sum_{i=1}^{p} Var(\hat{\tau}_{d_A i} - \hat{\tau}_{d_A 0}) \geq g(s_0, p, b, k)$, where s_0 is as in Theorem 2.1. Therefore, based on Theorem 2.1, a *lower bound* to the efficiency of a type S_0 design d with parameters $p, b, k, g_0, g_1, \lambda_0, \lambda_1$ is given by

$$e_{Ad} = g(s_0; p, b, k)/B_{0d}, \tag{3.1}$$

where $B_{0d} = g(s_{d0};, p, b, k)$. Since the efficiency of a design d is greater than or equal to e_{Ad}, high values of e_{Ad} indicate that the design d is highly efficient, and hence approximately optimal for control-test comparisons. The design d is optimal if $e_{Ad} = 1.0$.

Example 3.1. The following type S_0 design d for $p = 8,\ b = 10,\ k = 6$ with $e_{Ad} = 0.983$ is approximately optimal:

{(1,2) (3,5) (4,7) (0,6) (0,8) (0,1)}, {(2,3) (4,6) (5,8) (0,7) (0,1) (0,2)},
{(3,4) (5,7) (1,6) (0,8) (0,2) (0,3)}, {(4,5) (6,8) (2,7) (0,1) (0,3) (0,4)},
{(5,6) (1,7) (3,8) (0,2) (0,4) (0,5)}, {(6,7) (2,8) (1,4) (0,3) (0,5) (0,6)},
{(7,8) (1,3) (2,5) (0,4) (0,6) (0,7)}, {(1,8) (2,4) (3,6) (0,5) (0,7) (0,8)},
{(1,5) (2,6) (0,3) (0,4) (0,7) (0,8)}, {(3,7) (4,8) (0,1) (0,2) (0,5) (0,6)}.

The lower bound e_{Ad} was computed using (3.1) for all type S_0 designs in the practical range $p \leq 30,\ b \leq 50,\ k \leq p,\ g_0 \leq 10,\ 1 < g_0/g_1 \leq 5$. Of the 247 possible type S_0 designs in this range, 70 designs are optimal, i.e. have $e_{Ad} = 1.0$. Of the remaining 177 designs that do not satisfy condition (2.7) for optimality, 96 designs have $e_{Ad} \geq 0.95$, 52 designs have $0.90 \leq e_{Ad} < 0.95$, 27 designs have $0.81 \leq e_{Ad} < 0.90$, and 2 designs have $e_{Ad} = 0.77$, and 0.70 respectively. For the sake of brevity, these highly efficient type S_0 designs are not tabulated here, and they will be reported elsewhere.

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