On a problem of Simons and Johnson

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Abstract

This note solves a problem (related to proving a convergence which is stronger than convergence in total variation norm in the contexts like that of the celebrated Poisson convergence of binomial probabilities) stated towards the end of Simons and Johnson [1971. On the convergence of binomial to Poisson distribution. Ann. Math. Statist. 42, 1735–1736], using an approach different from theirs. The present method is simple, goes beyond the context of Poisson convergence and is expected to be more widely applicable.

1. Introduction and main result

Let

$$b_r(n;p) = \binom{n}{r} p^r (1-p)^{n-r}, \quad p_r(\lambda) = \exp(-\lambda)\lambda^r/r!.$$

Assuming $np_n = \lambda \in (0, \infty)$, Simons and Johnson (1971) showed that

$$\sum_{r=0}^{\infty} h(r)|b_r(n;p_n) - p_r(\lambda)| \to 0 \quad \text{as } n \to \infty, \tag{1}$$

whenever

$$\sum_{r=0}^{\infty} h(r)p_r(\lambda) < \infty \quad \text{and } h(r) \geqslant 0 \ \forall r. \tag{2}$$

Taking $h(r) = \exp(tr)$ for $t > 0$, they concluded that if $p_n = \lambda/n$,

$$\sum_{r=0}^{\infty} \exp(tr)|b_r(n;p_n) - p_r(\lambda)| \to 0 \quad \text{as } n \to \infty, \tag{3}$$

and posed the problem of showing analogs of (3) in cases other than the Poisson convergence of binomial probabilities. Chen (1974) generalized the result of Simons and Johnson (1971) to the case of Poisson binomial, and Wang (1991) generalized the same to the case of compound Poisson, both using essentially the approach of Simons and Johnson (1971); note that this result is stronger than the fact that convergence holds in the total variation norm. However, there seems to be no similar result in contexts other than the usual Poisson convergence. It is worth pointing out that analogs of (3) can be established in a large number of cases in a natural and straightforward way. The key idea is to use Lemma 2 which is a variation of a classical result of Scheffe (1947), and is of independent interest (we supply a proof since it is very short); for this purpose, we first state a useful consequence of Fatou's Lemma which will be used in Example 5 also.

Lemma 1. *For each* $n \geqslant 1$, *let* k_n *and* K_n *be* $\mathscr{A}$*-measurable real-valued functions on* $(\Omega, \mathscr{A}, \mu)$. *If* $k_n \leqslant K_n$, $K_n \to K$ *a.e.* $[\mu]$, K *is* μ*-integrable,* $\int K_n(\omega)\mu(\mathrm{d}\omega)$ *exists for all sufficiently large n and*

$$\int K_n(\omega)\mu(\mathrm{d}\omega) \to \int K(\omega)\mu(\mathrm{d}\omega),$$

then

$$\limsup \int k_n(\omega)\mu(\mathrm{d}\omega) \leqslant \int (\limsup k_n)(\omega)\mu(\mathrm{d}\omega).$$

If, moreover, $0 \leqslant k_n$ *and* $k_n \to 0$ *a.e.* $[\mu]$, *then*

$$\int k_n(\omega)\mu(\mathrm{d}\omega) \to 0.$$

Proof. Apply Fatou's lemma to $(K_n - k_n)$. $\square$

Lemma 2. *For each* $n \geqslant 1$, *let* f_n, f *and* h *be* $\mathscr{A}$*-measurable real-valued functions defined on* $(\Omega, \mathscr{A}, \mu)$; *let* $0 < \alpha < \infty$. *If*

$$f_n \to f \ \text{a.e.} \ [\mu] \tag{4}$$

and

$$\int |h(x)|^\alpha |f_n(x)|^\alpha \mu(\mathrm{d}x) \longrightarrow \int |h(x)|^\alpha |f(x)|^\alpha \mu(\mathrm{d}x) < \infty, \tag{5}$$

then

$$\int |h(x)|^\alpha |f_n(x) - f(x)|^\alpha \mu(\mathrm{d}x) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{6}$$

Proof. It suffices to prove the result when $h \equiv 1$; the general case then follows by considering hf_n and hf in place of f_n and f, respectively. But then the result follows by applying Lemma 1 with $k_n = |f_n - f|^\alpha$ and $K_n = 2^\alpha[|f|^\alpha + |f_n|^\alpha]$. $\square$

To get the analog of (1), one simply takes $\alpha = 1$ and f_n, f probability density functions (w.r.t. μ) and $h \geqslant 0$. In our approach, one has to show, in addition to an analog of (2), that

$$\int h(x) f_n(x)\mu(\mathrm{d}x) \longrightarrow \int h(x) f(x)\mu(\mathrm{d}x); \tag{7}$$

in this case, it may be noted that (6) implies (7). In the set-up of (3), assuming that $np_n \longrightarrow \lambda$ (*in place of* $p_n = \lambda/n$) and taking $h(x) = \exp(tx)$ where $t > 0$, we have

$$\text{LHS of (7)} = (1 - p_n + p_n \exp(t))^n \longrightarrow \exp[\lambda(\exp(t) - 1)] = \text{RHS of (7)},$$

and so (3) *holds whenever* $np_n \to \lambda$ which is a *useful extension* of the last result of Simons and Johnson (1971).

Example 1. Let $X_n \sim \chi_n^2$ and f_n be the Lebesgue-density function of $(X_n - n)/\sqrt{2n}$. Then it is well-known that $f_n(x) \to \exp(-x^2/2)/\sqrt{2\pi} = f(x)$, say; moreover,

$$\begin{aligned} \int \exp(tx) f_n(x)\,\mathrm{d}x &= \exp\left(-n^{1/2}t/\sqrt{2}\right)(1 - 2^{1/2}t/n^{1/2})^{-n/2}, \quad t < (n/2)^{1/2} \\ &\to \exp(t^2/2) = \int \exp(tx) f(x)\,\mathrm{d}x < \infty, \end{aligned}$$

and so for any $t>0$,

$$\int \exp(tx)|f_n(x)-f(x)|\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n\to\infty. \tag{8}$$

Example 2. Let X_n follow Student's t-distribution with n degrees of freedom; let f_n be the Lebesgue-density function of X_n. Then $f_n \longrightarrow f$ where $f(x)$ is as in Example 1. Take $h(x)=|x|^r$ for some $r>0$; then one can verify that

$$\int h(x)f_n(x)\,\mathrm{d}x \longrightarrow \int h(x)f(x)\,\mathrm{d}x<\infty$$

so that for any $r>0$

$$\int |x|^r|f_n(x)-f(x)|\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n\to\infty.$$

Clearly, in this Example, (8) does not hold for any $t>0$.

Example 3. Let

$$f_n(x)=\binom{n+x-1}{x}(1-q_n)^n q_n^x$$

for $x=0,1,2,\ldots,0<q_n<1$ and $nq_n\to\lambda\in(0,\infty)$. Then it is well-known that $f_n(x)\to p_x(\lambda)$ as $n\to\infty$; moreover,

$$\sum_{x=0}^{\infty}\exp(tx)f_n(x)=(1-q_n)^n(1-q_n\exp(t))^{-n}$$

$$\longrightarrow \exp[\lambda(\exp(t)-1)]=\sum_{x=0}^{\infty}\exp(tx)p_x(\lambda)<\infty,$$

so that $\sum_{x=0}^{\infty}\exp(tx)|f_n(x)-p_x(\lambda)|\to 0$ as $n\to\infty$.

Example 4. Let

$$f_n(r_1,\ldots,r_k)=\frac{n!}{r_1!\ldots r_k!}p_{1,n}^{r_1}\cdots p_{k,n}^{r_k}p_{k+1,n}^{r_{k+1}},$$

where $r_{k+1}=n-\sum_{i=1}^{k}r_i$ and $p_{k+1,n}=1-\sum_{i=1}^{k}p_{i,n}$ be the multinomial probabilities; let

$$f(r_1,\ldots,r_k)=\prod_{i=1}^{k}p_{r_i}(\lambda_i).$$

Let $np_{i,n}\to\lambda_i\in(0,\infty)$ for each $i=1,\ldots,k$. Then it is well-known, and easy to verify using induction on k, that for each $r_1,\ldots,r_k$

$$f_n(r_1,\ldots,r_k)\to f(r_1,\ldots,r_k) \quad \text{as } n\to\infty.$$

Also, one has

$$\begin{aligned}
&\sum_{r_1,\ldots,r_k=0}^{\infty}\exp(t_1r_1+\cdots+t_kr_k)f_n(r_1,\ldots,r_k)\\
&=(p_{1,n}\exp(t_1)+\cdots+p_{k,n}\exp(t_k)+p_{k+1,n})^n\\
&\to \prod_{i=1}^{k}\exp[\lambda_i(\exp(t_i)-1)]\\
&=\sum_{r_1,\ldots,r_k=0}^{\infty}\exp(t_1r_1+\cdots+t_kr_k)\prod_{i=1}^{k}p_{r_i}(\lambda_i)<\infty.
\end{aligned}$$

So

$$\sum_{r_1,\ldots,r_k=0}^{\infty} \exp(t_1 r_1 + \cdots + t_k r_k)|f_n(r_1, \ldots, r_k) - f(r_1 \ldots, r_k)| \to 0 \quad \text{as } n \to \infty.$$

Example 5 (*Fisher's Transformation*). We continue with the notation of Example 1, and put

$$U_n = \sqrt{2X_n} - \sqrt{2n}$$

whose Lebesgue-density g_n is given by

$$g_n(u) = \frac{1}{2^{n-1}\Gamma(n/2)} \exp\left[-\frac{1}{4}\left(u + \sqrt{2n}\right)^2\right]\left(u + \sqrt{2n}\right)^{n-1} I_{(-\sqrt{2n},\infty)}(u),$$

I_A being the indicator function of the set A.

Then $(X_n - n)/\sqrt{2n} = U_n + U_n^2/(2\sqrt{2n})$. Also, for any real t,

$$\begin{aligned} I_n &:= \int_{-\sqrt{2n}}^{\infty} \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right]\left|g_n(u) - f\left(u + \frac{u^2}{2\sqrt{2n}}\right)\left(1 + \frac{u}{\sqrt{2n}}\right)\right| \mathrm{d}u \\ &= \int_{-\sqrt{n/2}}^{\infty} \exp(tv)|f_n(v) - f(v)|\,\mathrm{d}v \\ &\to 0 \quad \text{as } n \to \infty \text{ (by Example 1).} \end{aligned}$$

We shall show below that

$$J_n := \int_{-\infty}^{\infty} \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right]|g_n(u) - f(u)|\,\mathrm{d}u \to 0 \quad \text{as } n \to \infty, \tag{9}$$

which will imply that for any $t > 0$

$$\int_{-\infty}^{\infty} \exp(tu)|g_n(u) - f(u)|\,\mathrm{d}u \longrightarrow 0 \quad \text{as } n \to \infty. \tag{10}$$

But

$$\begin{aligned} J_n \leqslant I_n &+ \int_{-\sqrt{2n}}^{\infty} \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right]\left|f\left(u + \frac{u^2}{2\sqrt{2n}}\right)\left(1 + \frac{u}{\sqrt{2n}}\right) - f(u)\right| \mathrm{d}u \\ &+ \int_{-\infty}^{-\sqrt{2n}} \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right] f(u)\,\mathrm{d}u + \int_{-\infty}^{-\sqrt{n/2}} \exp(tu) f(u)\,\mathrm{d}u. \end{aligned} \tag{11}$$

To show the second term of the right side of (11) goes to zero, we apply Lemma 1 to

$$\begin{aligned} k_n(u) &= \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right]\left|f\left(u + \frac{u^2}{2\sqrt{2n}}\right)\left(1 + \frac{u}{\sqrt{2n}}\right) - f(u)\right| I_{(-\sqrt{2n},\infty)}(u), \\ K_n(u) &= \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right]\left[f\left(u + \frac{u^2}{2\sqrt{2n}}\right)\left(1 + \frac{u}{\sqrt{2n}}\right) I_{(-\sqrt{2n},\infty)}(u) + f(u)\right], \\ K(u) &= 2\exp(tu) f(u); \end{aligned}$$

then $0 \leqslant k_n \leqslant K_n$, $k_n \to 0$, $K_n \to K$ and K is integrable; moreover, $\int K_n(u)\,\mathrm{d}u \to \int K(u)\,\mathrm{d}u = 2\exp(t^2/2)$, since

$$\begin{aligned} &\int_{-\sqrt{2n}}^{\infty} \exp\left[t\left(u + \frac{u^2}{2\sqrt{2n}}\right)\right] f\left(u + \frac{u^2}{2\sqrt{2n}}\right)\left(1 + \frac{u}{\sqrt{2n}}\right) \mathrm{d}u \\ &\quad = \int_{-\sqrt{n/2}}^{\infty} \exp(tv) f(v) f(v)\,\mathrm{d}v \to \int_{-\infty}^{\infty} \exp(tv) f(v)\,\mathrm{d}v = \exp(t^2/2), \end{aligned}$$

and

$$\int_{-\infty}^{\infty} \exp\left[t\left(u+\frac{u^2}{2\sqrt{2n}}\right)\right] f(u)\,\mathrm{d}u = \sigma_n \exp(t^2\sigma_n^2/2) \to \exp(t^2/2),$$

where $\sigma_n^{-2} = 1-(2n)^{-1/2}$. Thus (9) holds.

We finally remark that we are not able to establish (10) for $t<0$.

References

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