Chronological orderings of interval digraphs

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Abstract

An *interval digraph* is the intersection digraph of a family of ordered pairs of intervals on the real line; the family is *an interval representation* of the digraph. It is well known that an interval digraph has many representations that differ in the order of the endpoints of the intervals on the line. This paper generalizes the corresponding results on interval graphs by Skrien [Chronological orderings of interval graphs, Discrete Appl. Math. 8 (1984) 69–83] and describes how, given an interval digraph, the order of intervals of one representation differs from another.

Keywords: Interval digraph/bigraph; Ferrers digraph/bigraph; Ferrers dimension; Chronological ordering; Partitionable zero matrix

1. Introduction

An undirected graph $G(V, E)$ is an *interval graph* if every vertex $v \in V$ is assigned an interval I_v on the real line such that $uv \in E$ iff $I_u \cap I_v \neq \phi$. A directed graph $D(V, E)$ is an interval digraph if there is a family $\mathscr{F} = \{(S_v, T_v) : v \in V\}$ of ordered pairs of intervals on the real line such that $uv \in E$ iff $S_u \cap T_v \neq \phi$. The family $\mathscr{F}$ is an *interval representation* of D. S_v and T_v are the *source interval* and the *sink* (*terminal*) *interval*, respectively corresponding to the vertex v.

An interval representation of an interval graph is not unique. An interval graph may have many interval representations depending on lengths of the intervals and on the relative positions of the intervals. To study the possible relative positions of the intervals in an interval representation of an interval graph is the problem of chronological ordering of an interval graph.

An interval bigraph is a bigraph $(U, V; E)$ (with bipartition U and V) for which there are two families $\{S_x | x \in U\}$ and $\{T_y | y \in V\}$ of intervals such that x and y are adjacent whenever $S_x \cap T_y \neq \phi$. A digraph D is an interval digraph iff the corresponding bipartite graph $B(D)$ is an interval bigraph [4]. In the present paper, we will use both the terms interval digraph and interval bigraph interchangeably to suit our perspective and convenience.

The problem of chronological orderings of an interval graph arise naturally in archaeological seriation problems [2,4] where the archaeologists wish to determine the interval of time in which the various styles of artifacts obtained in different places were in use. Assuming each style was in use in only one interval of time, the archaeologists could determine which intervals of use overlapped. But in attempting to construct an interval graph for the artifacts, the data suggesting the inclusion (or omission) of certain edge might be insufficiently compelling. (A similar problem was encountered by the geneticist Benzer [1] in testing intersections of regions of chromosome.) Problem of chronological orderings is an attempt to construct the possible interval graphs that agree with the practical information. It is known that an interval digraph is a generalization of an interval graph [7]. Motivated by the study of this problem on interval graphs, we study in this paper the corresponding problems on interval digraphs.

Here, we will study the problem of chronological orderings from two view points. We describe and formulate them below.

2. Preliminaries

Let $B(U, V; E)$ be an interval bigraph ($|U|=|V|=n$) and let $\{S_x|x \in U\}$ and $\{T_y|y \in V\}$ be an interval representation of the bigraph B. Let K denote the reference set of $4n$ end points of the intervals $S_x=[a_x, b_x]$ and $T_y=[c_y, d_y]$. Without loss of generality, we will suppose throughout the paper that all the $4n$ end points are all distinct. It is clear that the given interval representation of B induces a linear order on K.

Now consider K as a set of $4n$ vertices and a complete oriented graph $D(K, T)$ on the set K, where an edge pq $(p, q \in K)$ has an orientation from p to q if $p<q$. Since real numbers are transitive, every interval representation of B induces a transitive tournament on K. Again a transitive tournament on K determines a linear ordering on K uniquely; so our problem is: Given an interval bigraph B, to study those transitive tournaments on the vertex set K whose linear orderings will describe the interval bigraph B. These linear orderings will be called the *chronological orderings* of B.

Note that the representation of an interval bigraph is independent of the relation between two source intervals or between two sink intervals. So in our problem we draw our attention to the complete oriented bigraph (we will call bitournament) whose partite sets are $A \cup B$ and $C \cup D$, where A and B denote the left and the right ends of the source intervals, respectively, and C and D are those of the sink intervals, respectively. We now pose the problem as follows: Given an interval bigraph B and a subgraph S of the complete oriented bigraph on the point sets $A \cup B$ and $C \cup D$, can the partial information provided by S be extended to a transitive orientation of the complete graph on $K=A\cup B\cup C\cup D$ which is a chronological ordering of the bigraph B?

We next look at the problem of chronological orderings from another perspective. For an interval bigraph $B(U, V; E)$ with an interval representation by the family $\{S_u|u \in U\}$ and $\{T_v|v \in V\}$, it can be seen that a pair of intervals (S_u, T_v) is related in one of the following ways:

1. S_u properly contains T_v or S_u is contained properly in T_v.
2. S_u overlaps T_v on the left or overlaps T_v on the right.
3. S_u follows T_v or T_v follows S_u.

Let $S_u=[a_u, b_u]$ and $T_v=[c_v, d_v]$. We use the following notations to classify these relations:

(1a) $uv \in C_1$ if $T_v \subset S_u$ that is, if $a_u<c_v<d_v<b_u$.
(1b) $uv \in C_2$ if $S_u \subset T_v$ that is, if $c_v<a_u<b_u<d_v$.
(2a) $uv \in O_1$ if $a_u<c_v<b_u<d_v$.
(2b) $uv \in O_2$ if $c_v<a_u<d_v<b_u$.
(3a) $uv \in F_1$ if $a_u<b_u<c_v<d_v$.
(3b) $uv \in F_2$ if $c_v<d_v<a_u<b_u$.

It is clear that in the bigraph $B(U, V; E)$, $uv \in E$ iff $uv \in C_i$ or O_i $(i=1, 2)$ and $uv \in \overline{E}$ iff $uv \in F_i$ $(i=1, 2)$.

Let now $T(U, V; A)$ denote a bitournament with partite sets U and V, the orientation being as follows: (i) $uv \in A$ if $uv \in C_1 \cup O_1 \cup F_1$ (ii) $vu \in A$ if $uv \in C_2 \cup O_2 \cup F_2$. Clearly an interval representation of a bigraph $B(U, V; E)$ determines a bitournament T on the partite sets U and V completely depending upon the relative positions of the end points of the intervals.

If $|U|=|V|=n$ and if K denotes the reference set of all $4n$ distinct end points then a linear order of K which induces the bitournament $T(U, V; A)$ and the interval bigraph $B(U, V; E)$ is called a chronological ordering of B.

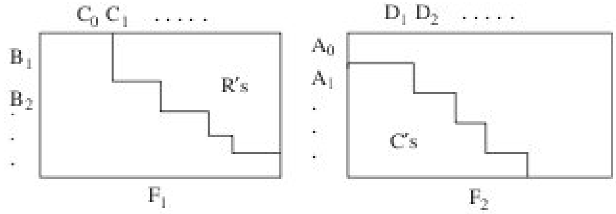

Fig. 1. Ferrers digraph decomposition of $\overline{D}$.

With these notations we pose the following: Given an interval bigraph B and a complete bitournament $T(U, V; A)$ with its classification of edges, is there any interval representation of B which has a chronological order corresponding to T?

In this paper, we will deal with the second problem in Theorem 2 while the earlier problem will be considered in Theorem 3.

The results obtained in this paper relies on the interval digraph characterization in terms of Ferrers digraph decomposition [5]. We recall the theorem below.

Theorem 1 (*Sen et al.* [5]). *The following conditions are equivalent*:

(A) *D is an interval digraph.*
(B) *The rows and columns of $A(D)$ can be (independently) permuted so that each 0 can be replaced by one of $\{R, C\}$ in such a way that every R has only R's to its right and every C has only C's below it.*
(C) *D is the intersection of two Ferrers digraphs whose union is complete.*

Let F_1 and F_2 be the Ferrers digraphs whose edges are R's and C's respectively in the complement of D and let their adjacency matrices be so arranged that R's are clustered in the upper right and C's in the lower left as in Fig. 1 below.

Let $A_0, A_1, \ldots, A_{p-1}$ be the source partitions and $D_1, D_2, \ldots, D_p$ the terminal partitions for F_2; also $B_1, B_2, \ldots, B_q$ and $C_0, C_1, \ldots, C_{q-1}$ for F_1. With these partitions the interval representation is constructed by assigning integer numbers a_i, b_i, c_i, d_i to the classes A_i, B_i, C_i, D_i, respectively. Then the required intervals are $S_v = [a_v, b_v]$, $T_v = [c_v, d_v]$, where $(a_v, b_v, c_v, d_v) = (a_i, b_j, c_k, d_l)$ if v belongs to A_i, B_j, C_k, D_l subject to the following requirements:

(i) $a_i \leqslant b_j$ if $A_i \cap B_j \neq \phi$ and $c_k \leqslant d_l$ if $C_k \cap D_l \neq \phi$
(ii) a, b, c, d are strictly increasing sequences.
(iii) $a_i = d_i + 1$ and $c_i = b_i + 1$ for $i \geqslant 1$.

To this end, an auxiliary directed graph M is constructed on the vertices A_i, B_j, C_k and D_l. Begin with directed paths $A_0, A_1, \ldots, A_{p-1}$; $B_1, B_2, \ldots, B_q$; $C_0, C_1, \ldots, C_{q-1}$ and $D_1, D_2, \ldots, D_p$. Add edges A_iB_j when $A_i \cap B_j \neq \phi$, C_kD_l when $C_k \cap D_l \neq \phi$, B_iC_i for $1 \leqslant i \leqslant q-1$ and D_iA_i for $1 \leqslant i \leqslant p-1$. In [5] it was proved that M is acyclic and as such determines a linear ordering on its vertices. An integer numbering on the vertices according to the linear order produces the required interval representation.

Note that in the auxiliary digraph M there are no edges connecting A_i's with C_j's and B_i's with D_j's. This is so, because in an interval digraph $D(V, E)$, $uv \in E$ iff $a_u < d_v$ and $c_v < b_u$ and $uv \in \overline{E}$ iff either $b_u < c_v$ or $d_v < a_u$.

Observe that for an interval representation of an interval digraph, the orders between b_u and c_v and those between a_u and d_v are fixed and we do not have any choice in these respects. But for the same, we are at liberty when we are to choose those between a_u and c_v and between b_u and d_v. In our first problem we are initially given a subdigraph S which may contain edges connecting these vertices. So the question here reduces to finding if there is a chronological ordering of D which agrees with these partial information, hitherto not taken care of.

For the second problem we start with a bitournament $T(U, V, A)$, classify the edges into six classes C_i, O_i, F_i $(i=1, 2)$ and the question is to find out if this complete set of information actually agrees with an interval representation of the interval bigraph B.

3. Chronological orderings

We still need another characterization of an interval digraph which involves the concept of generalized linear one's property [6]. A (0, 1) matrix A has the *generalized linear one's property* (*glop*) if it has a stair partition (L, U) such that 1's in U are consecutive and appear left most in each row and 1's in each column in L are consecutive and appear top most in each column. It is proved in [6] that a digraph is an interval digraph iff its adjacency matrix has an independent row and column permutations such that the resulting matrix has the generalized linear one's property.

A close scrutiny into the theorem will reveal that the rows and columns of the rearranged matrix that exhibit a *glop* are ordered in increasing order of the left ends of the source intervals $\{a_i\}$ and of the sink intervals $\{c_i\}$, respectively. Assuming that all the end points of the intervals are distinct, a position uv in the matrix belongs to the upper sector U or the lower sector L according as $a_u < c_v$ or $c_v < a_u$, respectively.

If we rearrange the rows and columns of the matrix in the increasing order of the right ends $\{b_i\}$ and $\{d_i\}$ of the intervals instead of the left ends, then the resulting matrix will again have a stair partition (L, U), but this time 1's in U are consecutive in each column appearing down most, while 1's in L are consecutive in each row appearing rightmost. A stair partition with the above arrangement of 1's and 0's can be seen again to characterize an interval digraph and will also be referred to as a generalized linear one's property; to distinguish between them, we will refer to the former as *first glo property* (*glop I*) while the latter the *second glo property* (*glop II*). Note that these two properties are equivalent as one can be obtained from the other by reversing the orders of rows and columns (Fig. 2).

Since $a_i < c_j$ means uv belongs to C_1, O_1, or F_1 and $c_j < a_i$ means uv belongs to C_2, O_2, or F_2, it is clear that in the matrix exhibiting glop I if P and P' are the Ferrers digraph corresponding to the top right and bottom left corner of the matrix then $P = F_1 \cup O_1 \cup C_1$ and $P' = F_2 \cup O_2 \cup C_2$.

Similarly, if Q and Q' are the Ferrers digraph corresponding to the top right and bottom left corner of the matrix exhibiting glop II then $Q = F_1 \cup O_1 \cup C_2$, $Q' = F_2 \cup O_2 \cup C_1$.

Next consider the product relations $QF_2^{-1}P$ and $Q'^{-1}F_1P'^{-1}$. It can be shown that $QF_2^{-1}P \subset F_1$ and $Q'^{-1}F_1P'^{-1} \subset F_2$.

We now sum up the above properties below. Let $B(D)=(U, V; E)$ be an interval bigraph and let $T=(U, V; C \cup O \cup F)$ be the corresponding bitournament, where its edge set is classified into six subsets C_i, O_i and F_i $(i = 1, 2)$; then

(1) $C \cup O = E$ and $F = \overline{E}$,
(2) a rearranged matrix of $B(D)$ has a glop I with the sectors U and L consisting exclusively of members of $P = F_1 \cup O_1 \cup C_1$ and $P' = F_2 \cup O_2 \cup C_2$, respectively,
(3) a rearranged matrix of $B(D)$ has a glop II with the sectors U and L consisting exclusively of members of $Q = F_1 \cup O_1 \cup C_2$ and $Q' = F_2 \cup O_2 \cup C_1$, respectively,
(4) $QF_2^{-1}P \subset F_1$ and $Q'^{-1}F_1P'^{-1} \subset F_2$.

We say in such a case that the bitournament T satisfies *a natural chronological ordering of B*.

We note that there is an overlapping area between condition (4) and earlier condition (2) and (3); for example, the fact that F_1 is a Ferrers digraph is a consequence of $F_1F_2^{-1}F_1 \subset F_1$, which follows from (4). Also note that the symbol R and C of Theorem 1 have been replaced here by F_1 and F_2 respectively for our convenience. Now we have the following theorem.

Theorem 2. *Let $B(D) = (U, V; E)$ be a bigraph and let $T = (U, V; C \cup O \cup F)$ be a bitournament, while the edge set of T is classified into three sets C, O and F. Let further $C = C_1 \cup C_2$, $O = O_1 \cup O_2$, $F = F_1 \cup F_2$, $C_1 \cap C_2 = \phi$,*

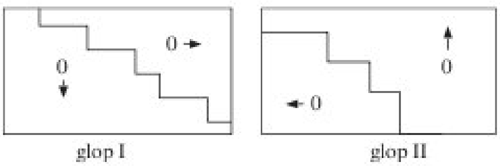

Fig. 2. Generalized linear one's properties I and II.

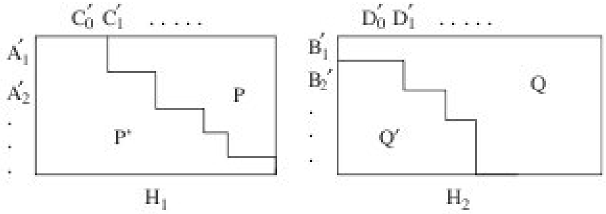

Fig. 3. Matrices H_1 and H_2.

$O_1 \cap O_2 = \phi$, $F_1 \cap F_2 = \phi$. Then B is an interval bigraph iff the bitournament T describes a natural chronological ordering of B.

Proof. *Necessary*. This part has been discussed earlier.

Sufficient. Let H_1 and H_2 be the rearranged adjacency matrices of the bigraph $B(D)$ exhibiting glop I and glop II, respectively (Fig. 3).

Let $A'_1, A'_2, \ldots, A'_s$ be the source and $C'_0, C'_1, \ldots, C'_{s-1}$ be the terminal partitions for the matrix H_1, where C'_0 and/or A'_s may be empty. Let again $B'_1, B'_2, \ldots, B'_r$ and $D'_0, D'_1, \ldots, D'_{r-1}$ denote the source and the terminal partitions for the matrix H_2, where D'_0 and/or B'_r may be empty.

Next let H'_1 and H'_2 denote the adjacency matrices of the Ferrers digraphs F_1 and F_2. Their rows and columns are so arranged that the 1's in H'_1 are all placed in upper right corner and those in H'_2 are all placed in lower left corner in the shape of the Ferrers diagrams as in Fig. 4.

The matrices H_1 and H_2 exhibiting glop I and glop II induce two interval representations of the bigraph $B(D)$. Also the matrices H'_1 and H'_2 together determine still another representation of $B(D)$. Note that the diagrams in Figs. 1 and 4 are same except for notations only. Our purpose here is to show that under the additional condition (4), there is an interval representation agreeing with all the three representations which has the chronological ordering corresponding to T.

Let us have a close look at the matrices H_1 and H'_2. In both the matrices F_2's are such that any position below an F_2 is an F_2. It follows that the rows within the same partition of the two matrices can be suitably rearranged so that they will have the same order without any loss of their intrinsic characteristics. By similar logic the columns of H_1 and H'_1 are of the same order and so are the rows of H_2 and H'_1 and columns of H_2 and H'_2.

Let the source and the terminal partitions of H'_1 be $B''_1, B''_2, \ldots, B''_q$ and $C''_0, C''_1, \ldots, C''_{q-1}$, respectively, where C''_0 and / or B''_q may be empty. Also let $A''_0, A''_1, \ldots, A''_{p-1}$ and $D''_1, D''_2, \ldots, D''_p$ be the source and the sink partitions respectively of H'_2. (These are the same partitions as in Theorem 1 with change in notations only.)

Now define additional partitions $A = \{A_0, A_1, \ldots, A_m\}$ which is the least common refinement (l.c.r) of the partitions $\{A'_i\}$ and $\{A''_j\}$, that is, each A_i is the intersection of some A'_k and some A''_l, indexing being carried out by the shared order of the row. Similarly for the other refinements $B = \{B_i\}$, $C = \{C_i\}$ and $D = \{D_i\}$. We call $\Omega = A \cup B \cup C \cup D$ nodes and construct an auxiliary digraph Z on the nodes of Ω. As in [5], we will suitably assign integer values $f : \Omega \to \mathcal{N}$ to the nodes, a_i, b_i, c_i, d_i to A_i, B_i, C_i, D_i, respectively. By assigning values $a(u) = a_i$, $b(u) = b_j$, $c(v) = c_k$, $d(v) = d_l$, where $u \in A_i \cap B_j$ and $v \in C_k \cap D_l$, we will see that the intervals $S_u = [a(u), b(u)]$ and $T_v = [c(v), d(v)]$ actually represents the bigraph $B(D)$ and also conforms to the chronological ordering of the given bipartite tournament B. We will construct the auxiliary digraph Z in such a way that we will put an edge $X \to Y$ in Z where we want the value of $f(X)$ to be less than $f(Y)$.

Our construction of the digraph Z relies heavily on the acyclic digraph M constructed in [5] to obtain an interval representation of a digraph D under the conditions that $\overline{D} = F_1 \cup F_2$, F_1 and F_2 being two disjoint Ferrers digraphs. We therefore focus our attention below on the construction of the acyclic digraph M (as in Theorem 1).

Begin with directed paths $A''_0, A''_1, \ldots, A''_{p-1}$; $B''_1, B''_2, \ldots, B''_q$; $C''_1, C''_2, \ldots, C''_q$; $D''_1, D''_2, \ldots, D''_p$. Add edges $A''_i B''_j$ when $A''_i \cap B''_j \neq \phi$, $C''_k D''_l$ when $C''_k \cap D''_l \neq \phi$, $B''_i C''_i$ for $1 < i \leqslant q-1$ and $D''_i A''_i$ for $1 < i \leqslant p-1$. Based on this digraph M, we now construct an associated digraph N on the node set Ω from the digraph M in the following manner: the nodes of Ω are all subsets of the sets of vertices corresponding to the nodes of M. Now begin with the directed

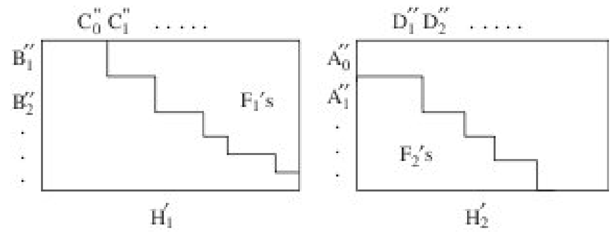

Fig. 4. Matrices H_1' and H_2'.

paths $A_1, A_2, \ldots, A_p$; $B_1, B_2, \ldots, B_q$; $C_1, C_2, \ldots, C_r$ and $D_1, D_2, \ldots, D_s$. Then join two nodes in Ω by an edge in N iff the corresponding sets to which they belong have an edge in M.

Next we add to N two other directed graphs N_1 and N_2 on the node sets $A \cup C$ and $B \cup D$ constructed as follows from the matrices H_1 and H_2, respectively.

For the digraph N_1 we put the edges $A_i C_j$ if the corresponding position in H_1 belongs to the Ferrers digraph $P = F_1 \cup O_1 \cup C_1$ that is, if $A_i \times C_j \subset P$ and otherwise (that is if $A_i \times C_j \subset P'$) then put the edges $C_j A_i$ in N_1.

Lastly for the digraph on the node set $B \cup D$, put the edges $B_i D_j$ or $D_j B_i$ in N_2 according as $B_i \times D_j \subset Q = F_1 \cup O_1 \cup C_2$ or $B_i \times D_j \subset Q' = F_2 \cup O_2 \cup C_1$, respectively.

We have now the desired auxiliary digraph $Z = N \cup N_1 \cup N_2$. Note that the initial digraph M has no edges between A_i and C_j and between B_i and D_j. All the digraphs N, N_1 and N_2 are mutually disjoint. Our aim is now to prove that the digraph $Z = N \cup N_1 \cup N_2$ is acyclic.

Once we can prove it, our problem is solved; because the acyclic digraph Z has been so constructed that it contains all the edges of the bitournament T, also the digraph Z agrees with all the three modes of representation of the interval bigraph $B(D)$ described a little earlier; so a linear order on the vertices of Z (obtained from the acyclicity of Z) can be suitably and consistently extended to determine an interval representation of $B(D)$, the chronological ordering of which corresponds to the bitournament T.

We will first prove that each of the digraphs N, N_1, N_2 are separately acyclic. We observed in Theorem 1 that M is acyclic and since the digraph N is obtained by replacing the edges of M by transitive tournament, N must be acyclic. Let if possible, the digraph N_1 has a cycle of the form $A_i C_j A_k C_l$.

From the construction of N_1, it follows that $A_i \times C_j \in P$, $A_k \times C_j \in P'$, $A_k \times C_l \in P$ and since P is a Ferrers digraph, it follows that $A_i \times C_l$ must belong to P. Hence $C_l A_i$ cannot be an edge in N_1. Using the same argument that P is a Ferrers digraph, it can be shown that N_1 cannot have any cycle of any form. Similarly we can show that N_2 is also acyclic.

In our next step, we show that $N \cup N_1$ and $N \cup N_2$ are acyclic. Let if possible, $\{A_i, B_j, C_k\}$ be a cycle in $N \cup N_1$. Then $A_i \to B_j$ means that there is a vertex $u \in A_i \cap B_j$ and $B_j \to C_k$ means that there is a vertex v such that $uv \in F_1$. But again $C_k \to A_i$ implies that $uv \in P' = F_2 \cup O_2 \cup C_2$, which is not possible since $F_1 \cap F_2 = \phi$. So $N \cup N_1$ is acyclic. Similarly $N \cup N_2$ is acyclic.

Lastly we show that $Z = N \cup N_1 \cup N_2$ is acyclic. The possible cycle in Z which contains edges from N_1 to N_2 are $\{B_j D_l A_i C_k\}$ and $\{D_l B_j C_k A_i\}$. For a cycle $\{A_i C_k B_j D_l\}$ in Z, there must be vertices $u \in B_j$, $v \in D_l$, $w \in A_i$, $x \in C_k$ such that $uv \in Q$, $wv \in F_2$, $wx \in P$ which imply $ux \in F_1^c$ which is not possible because by condition (4) we have $ux \in F_1$. □

In interpreting Theorem 2 for an interval graph, we first observe that here the source intervals and sink intervals coincide so that $a_i = c_i$ and $b_i = d_i$. The bitournament $B(U, V; C, O, F)$ of Theorem 2 becomes a tournament $D(V; C, O, F)$ for an interval graph where $C = \{(u_p, u_j) : a_i < a_j < b_j < b_i\}$, $O = \{(u_p, u_j) : a_i < a_j < b_i < b_j\}$ and $F = \{(u_p, u_j) : a_i < b_i < a_j < b_j\}$.

For a chronological ordering of a bigraph $B(D) = (U, V; E)$ corresponding to the bitournament $B(U, V; C, O, F)$ the following results are matters of easy observations:

Corollary 1. *For* $(i, j) = (1, 2)$ *or* $(2, 1)$

(i) $C_i C_j C_i \subset C_i$,

(ii) $O_i O_j O_i \subset O_i + F_i$,
(iii) $(O_i + F_i)(O_j + F_j)(O_i + F_i) - O_i O_j O_i \subset F_i$.

The generalized one's property in a matrix corresponds to the quasidiagonal property [3] which is explained as follows: A symmetric (0, 1) matrix has a quasidiagonal property if the rows and the columns of the matrix have a (simultaneous) permutation such that the one's in each row appear consecutively rightward starting from the main diagonal. The corresponding characterization of an interval graph is that an undirected graph is an interval graph iff its adjacency matrix has quasidiagonal property. If we say this property to the *first quasidiagonal property*, then the *second quasi-diagonal property* can be similarly defined (when the one's in a column appear consecutively upward starting from the main diagonal).

Lastly we see that condition (4) of Theorem 2 becomes redundant for an interval graph as it is a follow up of quasidiagonal property of its adjacency matrix. The result for an interval graph corresponding to the last theorem is the following.

Corollary 2. *Let $G(V, E)$ be an undirected graph and let $D(V; C, O, F)$ be a tournament. Then G is an interval graph iff*

(1) $C \cup O = E$.
(2) *the rows and columns of $A(G)$, the adjacency matrix can be (simultaneously) rearranged that it has the first quasidiagonal property with $P = C \cup O \cup F$ in the upper triangle.*
(3) *the rows and columns of $A(G)$ can be (simultaneously) rearranged that it has the second quasidiagonal property with $Q = C^{-1} \cup O \cup F$ in the upper triangle.*

In another version of the chronological ordering of an interval graph, Skrien started from the linear orders of the left end points and right end points of the intervals separately and obtained conditions under which these linear orderings give a chronological ordering of G [8]. In terms of quasidiagonal property the following corollary is a consequence of Corollary 2.

Corollary 3. *Let $G(V, E)$ be an undirected simple graph and T_a and T_b be two linear orderings of the two sets $A_L = \{a_1, a_2, \ldots, a_n\}$ and $B_R = \{b_1, b_2, \ldots, b_n\}$. Then G is an interval graph which has a chronological ordering corresponding to T_a and T_b if and only if the following conditions hold*:

(1) *if the vertices of adjacency matrix of G are rearranged according to the order T_a of A_L, then it exhibits the first quasidiagonal property.*
(2) *if the vertices of adjacency matrix of G are rearranged according to the order T_b of B_R, then it exhibits the second quasidiagonal property.*
(3) 0*'s in the upper triangle (lower triangle) in the two quasidiagonal matrices have the same orientation.*

Let $B = (U, V; E)$ be an interval bigraph and let a rearranged biadjacency matrix exhibited an R, C-partition of 0's. let S be a partial acyclic orientation on the complete bigraph whose partite sets are $A \cup B$ and $C \cup D$. Let S have an extension to a transitive orientation of $D = (K, T)$ which is a chronological ordering of B then following properties can be easily verified:

(i) $\begin{array}{c} \\ u_i \\ u_j \end{array}\begin{array}{c} \begin{array}{cc} v_p & v_q \end{array} \\ \begin{pmatrix} 1 & R \\ 1 & C \end{pmatrix} \end{array} \Rightarrow a_j c_p \notin S,\ d_p b_i \notin S;$

(ii) $\begin{array}{c} \\ u_i \\ u_j \end{array}\begin{array}{c} \begin{array}{cc} v_p & v_q \end{array} \\ \begin{pmatrix} 1 & 1 \\ C & R \end{pmatrix} \end{array} \Rightarrow c_q a_i \notin S,\ b_i d_p \notin S;$

(iii) for the submatrix $\begin{array}{c} \\ u_i \\ u_j \end{array}\begin{array}{c} \begin{array}{cc} v_p & v_q \end{array} \\ \begin{pmatrix} 1 & R \\ 1 & 1 \end{pmatrix} \end{array} \quad \begin{cases} d_p b_i \in S \Rightarrow b_j d_p \notin S,\ c_q a_j \notin S, \\ c_q a_j \in S \Rightarrow a_j c_p \notin S,\ d_p b_i \notin S, \end{cases}$

(iv) for the submatrix $\begin{matrix} & v_p & v_q \\ u_i & 1 & 1 \\ u_j & C & 1 \end{matrix}$ $\begin{cases} a_j c_q \in S \Rightarrow c_q a_i \notin S, \; b_i d_p \notin S, \\ b_i d_p \in S \Rightarrow a_j c_q \notin S, \; d_q b_i \notin S. \end{cases}$

These properties will be called *natural chronological properties of* Adj(B) corresponding to S.

Theorem 3. *Let $B = (U, V; E)$ be an interval bigraph and let S be a partial acyclic orientation on the complete bigraph whose partite sets are $A \cup B$ and $C \cup D$. If $|U| = |V| = n$ and $K = A \cup B \cup C \cup D$ is the reference set of $4n$ vertices, then S can be extended to a transitive orientation of $D = (K, T)$ which is a chronological ordering of B if and only if the following condition holds*:

(a) *If $uv \in E$ and $S_u = [a, b]$, $T_v = [c, d]$ then $bc, da \notin S$.*
(b) *There exist two Ferrers relations R and C such that $\overline{E} = R \cup C$, $R \cap C = \phi$ with following properties*:
 (i) *If $u_i v_j \notin E$ and one of $a_i c_j, a_i d_j, b_i c_j, b_i d_j \in S$ then $u_i v_j \in R$.*
 (ii) *If $u_i v_j \notin E$ and one of $c_j a_i, c_j b_i, d_j a_i, d_j b_i \in S$ then $u_i v_j \in C$.*
(c) Adj(B) *satisfies natural chronological properties corresponding to S.*

We observe that in the above conditions there is a clear duality between the relations R and C (that is, corresponding to a property for R there is one for C and vice-versa).

Proof. *Necessary.* This part has been discussed earlier.

Sufficient. From condition (b) it follows that B is an interval bigraph. To obtain an interval representation of B that is an extension of S, we start from the auxiliary digraph M as described in Section 2. Now extend the digraph M to a digraph M' by adding edges to M in the following manner:

Add $B_i C_j$ for $i < j$, $C_j B_i$ for $j < i$, $D_i A_j$ for $i < j$ and $A_j D_i$ for $j < i$. If $A_i B_j$ is an edge of M, then add $A_i B_k$ for $k > j$. If $C_k D_l$ is an edge of M, then add $C_k D_m$ for $m > l$.

The digraph M' is acyclic because M is so. From M' generate another digraph W whose vertex set is the reference set K on $4n$ vertices $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$ by joining two vertices by an edge if and only if the corresponding sets to which they belong form an edge in M'.

For example, if $B_i C_j$ is an edge of M' then the complete bipartite digraph whose partite sets are B_i and C_j and the orientations are from a vertex of B_i to one of C_j, is a subdigraph of W. Since M' is acyclic, it is clear that W is also acyclic.

Observe that among the vertices $\{A_i\}, \{B_i\}, \{C_i\}, \{D_i\}$ of the graph M all the possible edges are there except the edges which are of the form $A_i C_j$, $C_j A_i$, $B_i D_j$ or $D_j B_i$ for which $u_i v_j \in E$. The present theorem in fact addresses the question regarding these particular edges. We partition the edges of S into two subset S_1 and S_2, where S_2 consists of those edges of the form ac or ca or bd or db for which $u_i v_j \in E$ and S_1 is the complement of S_2.

It is clear from the conditions on S given in the theorem that the edges of S_1 are all members of the edge set of W. In fact the conditions on S_1 are such that if, for instance, $x_i y_j \notin S_1$ then $y_j x_i$ is an edge of W.

Now we come to the crucial stage where we prove that $W \cup S$ is acyclic. Once we can prove it, then acyclic digraph can be easily extended (arbitrarily but consistently with the interval representation of the interval bigraph) to a chronological ordering of B.

Since W is acyclic, any possible cycle in $W \cup S$ must contain at least one edge of $S_2 = S - W$ (that is an edge of the form $a_i c_j$ or $c_j a_i$ or $b_i d_j$ or $d_j b_i$ for which $u_i v_j \in E$).

The possible cycles in $W \cup S$ are (i) $(a_i c_p a_j)$, (ii) $(c_p a_i c_q)$, (iii) $(b_i d_p b_j)$, (iv) $(d_p b_i d_q)$, (v) $(a_i c_p b_j d_q)$, (vi) $(c_p a_i d_q b_i)$. It can be seen that any possible cycle in $W \cup S$ must be one of the above six forms.

Consider case (i). Let if possible, $a_i c_p \in S$, $c_p a_j \in S$ and $a_j a_i \in W$, so that there is a cycle $(a_i c_p a_j)$ in $W \cup S$. Now $c_p a_j \in S \Rightarrow u_j v_p \in E$ or $u_j v_p \in C$ (by condition b(ii)) and $a_i c_p \in S \Rightarrow u_i v_p \in E$ or $u_i v_p \in R$. So the submatrix $\begin{matrix} & v_p \\ u_i & \cdot \\ u_j & \cdot \end{matrix}$ has one of the forms $\binom{1}{C}$, $\binom{R}{C}$, $\binom{R}{1}$ or $\binom{1}{1}$.

For the first two alternatives, it is clear from R, C partition of the adjacency matrix and the corresponding construction of its interval representation in [6] that (the row u_i is above the row u_j in the matrix and accordingly) $a_i a_j \in W$ and so $a_j a_i \notin W$. We consider the possibility when we have $\begin{matrix} & v_p \\ u_i & \\ u_j & \end{matrix}\begin{pmatrix} R \\ 1 \end{pmatrix}$ and $a_j a_i \in W$. Since $a_j a_i \in W$, we must have a submatrix of the form $\begin{matrix} & v_q \\ u_i & \\ u_j & \end{matrix}\begin{pmatrix} R \\ 1 \end{pmatrix}$.

Then we have the submatrix $\begin{matrix} u_i \\ u_j \end{matrix}\overset{\begin{matrix} v_p & v_q \end{matrix}}{\begin{pmatrix} R & C \\ 1 & 1 \end{pmatrix}}$. By condition c(ii), $c_p a_j \notin S$. So this is not possible.

Lastly we consider the fourth alternative. Since $a_j a_i \in W$, we have the submatrix $\begin{matrix} u_i \\ u_j \end{matrix}\overset{\begin{matrix} v_p & v_q \end{matrix}}{\begin{pmatrix} 1 & C \\ 1 & 1 \end{pmatrix}}$.

By condition c(iv), $a_i c_p \in S \Rightarrow c_p a_j \notin S$. Thus we see that there cannot be any cycle of the form $(a_i c_p a_j)$ in $W \cup S$. Similarly we can show that there cannot be any cycle of the other five forms in $W \cup S$.

In the following lines, we actually construct the extension of S to a chronological ordering of B by constructing stair partitions of the adjacency matrix of B which will in fact exhibit the generalized linear one's property.

For this we start with the matrices having (R, C) partition. To construct the stair, place an $u_i v_j$ in the U-sector if $a_i c_j \in S$ and in the L-sector if $c_j a_i \in S$ and in conformity with this, draw a stair arbitrarily (For this one may have to rearrange the vertices belonging to a class in a partition). Since $\{a_i c_p a_j\}$ do not form a cycle in $W \cup S$, it is clear that if for a vertex v_p both the edges $a_i c_p$ and $c_p a_j$ belong to S, then $a_i a_j \in W$ and so the row u_i occurs above the row u_j in the matrix and similarly if $c_p a_i$ and $a_i c_q$ both belong to S then the column v_p occurs to the left of the column v_p in the matrix. Hence the stair is well-constructed.

Similarly start from the rearranged matrix with B_i's and D_j's as the row and column in their order and to obtain a stair partition (L, U) in the matrix, rearrange the vertices in the partitioned classes suitably to place $u_i v_j$ in U when $b_i d_j \in S$ and $v_j u_i$ in L when $d_j b_i \in S$. Now the stair partition satisfies the conditions (1), (2) and (3) of Theorem 2. Lastly because of the condition (c) in the theorem we can verify that $QF_2^{-1}P \subset F_1$ and $Q'^{-1}F_1P'^{-1} \subset F_2$.

Thus all the conditions of the Theorem 2 are satisfied. An interval representation thus constructed has a chronological ordering which is an extension of S. □

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