Intertwining operator in nonlinear pseudo-supersymmetry

R. Roychoudhury, B. Roy *

Physics & Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India

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Abstract

An intertwining operator linking a non-Hermitian Hamiltonian to the adjoint of its nonlinear pseudo-supersymmetric partner Hamiltonian has been found. Explicit realization of this intertwining operator, which gives rise to a new pair of isospectral Hamiltonians, is given for complex Scarf and Morse potential.

Recently, there has been growing interest in the study of non-Hermitian Hamiltonians which appear in different branches of physics [1]. An important subclass of non-Hermitian operators is the pseudo-Hermitian operators [2], i.e. those operators A, which satisfy

$$\eta A\eta^{-1} = A^\dagger, \tag{1}$$

where η is a linear invertible operator satisfying

$$\eta = \eta^\dagger. \tag{2}$$

Whenever (1) holds without the constraint (2), A is called weakly pseudo-Hermitian [3]. Many interesting properties of both pseudo-Hermitian and weakly pseudo-Hermitian operators have been examined by several authors [4,5]. Particularly, a generalization of supersymmetry [6], namely, pseudo-supersymmetry was developed in Ref. [7] that would apply for general pseudo-Hermitian Hamiltonians. In this Letter we attempt to find an intertwining operator connecting one complex Hamiltonian with the adjoint of its nonlinear pseudo-supersymmetric [12] partner Hamiltonian through a proposition stated and proved below. The motivation is to generate exactly solvable non-Hermitian potentials.

Theorem. *Let $H = AB$, A and B being first order differential operators, be a non-Hermitian diagonalizable Hamiltonian having discrete spectrum consisting of real or complex conjugate pairs of eigenvalues and the multiplicity of complex conjugate eigenvalues are the same. Then there exists an intertwining operator η such that*

$$\eta H = H_s^\dagger \eta, \tag{3}$$

where $H_s = BA$ is the partner Hamiltonian of H and $H_s^\dagger$ is the adjoint of H_s.

Proof. Since H and H_s are of the form $H = AB$ and $H_s = BA$, where the operators A and B are of the form $A = \frac{d}{dx} + W(x)$ and $B = -\frac{d}{dx} + W(x)$, $W(x)$ being any function of x, there exists an intertwining operator η_1 such that

$$\eta_1 H = H_s \eta_1. \tag{4}$$

An almost trivial first order solution for η_1 is B or A^{-1}, if the latter exists.

H is diagonalizable with a discrete spectrum. By this it is meant that H admits a complete biorthonormal system of eigenvectors $\{|\psi_n, a\rangle, |\phi_n, a\rangle\}$. The latter satisfy the following defining properties [8]:

$$H|\psi_n, a\rangle = E_n|\psi_n, a\rangle, \qquad H^\dagger|\phi_n, a\rangle = E_n^*|\phi_n, a\rangle, \tag{5}$$

$$\langle\phi_m, b|\psi_n, a\rangle = \delta_{mn}\delta_{ab}, \tag{6}$$

$$\sum_n \sum_{a=1}^{d_n} |\phi_n, a\rangle\langle\psi_n, a| = \sum_n \sum_{a=1}^{d_n} |\psi_n, a\rangle\langle\phi_n, a| = 1, \tag{7}$$

* Corresponding author.

E-mail addresses: raj@isical.ac.in (R. Roychoudhury), barnana@isical.ac.in (B. Roy).

where † stands for the adjoint of the corresponding operator, d_n is the multiplicity (degree of degeneracy) of the eigenvalue E_n, n is the spectral label and a and b are degeneracy label.

Since H_s is the partner of H in the sense described above, H_s also admits a complete biorthonormal system of eigenvectors and has either a real spectrum or complex conjugate pairs of eigenvalues and the multiplicity of complex conjugate eigenvalues are the same. Then, as shown by Mostafazadeh, H_s is η_2-pseudo-Hermitian [4], i.e.

$$\eta_2 H_s = H_s^\dagger \eta_2. \tag{8}$$

Using (8) we obtain from (4),

$$\eta_2\eta_1 H = \eta_2 H_s \eta_1 = H_s^\dagger \eta_2\eta_1. \tag{9}$$

Taking $\eta = \eta_2\eta_1$, we get $\eta H = H_s^\dagger \eta$. □

Now using the adjoint of (3) in (4) we get

$$\eta^\dagger \eta_1 H = \eta^\dagger H_s \eta_1 = H^\dagger \eta^\dagger \eta_1. \tag{10}$$

Therefore H is $\tilde{\eta} = \eta^\dagger\eta_1$ pseudo-Hermitian.

Taking adjoint of (8), we get

$$H_s^\dagger \eta_2^\dagger = \eta_2^\dagger H_s \quad \text{or} \quad (\eta_2^\dagger)^{-1} H_s^\dagger \eta_2^\dagger = H_s, \tag{11}$$

which shows that $H_s^\dagger$ is $(\eta_2^\dagger)^{-1}$ pseudo-Hermitian.

At this point let us remark that since H is non-Hermitian, H_s and $H_s^\dagger$ are non-Hermitian in general, but in some particular cases they may be Hermitian [9]. The intertwining operator η_1 and η_2 may be Hermitian or non-Hermitian. In the former case we have pseudo-hermiticity [2] while in the latter we have weak pseudo-hermiticity [3].

Now following Mostafazadeh [7] it is possible to obtain a two-component realization of nonlinear pseudo-supersymmetry [12] in which the state vector $|\tilde{\psi}\rangle$, the nonlinear pseudo-supersymmetry generator $\mathcal{Q}$, and its pseudo-adjoint $\mathcal{Q}^\#$, the Hamiltonian $\tilde{H}$ and the operator η_M are respectively represented as

$$|\tilde{\psi}\rangle = \begin{pmatrix} |\tilde{\psi}_+\rangle \\ |\tilde{\psi}_-\rangle \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix}, \tag{12}$$

$$\mathcal{Q}^\# = \begin{pmatrix} 0 & \eta^\# \\ 0 & 0 \end{pmatrix}, \qquad \tilde{H} = \begin{pmatrix} H & 0 \\ 0 & H_s^\dagger \end{pmatrix},$$

$$\eta_M = \begin{pmatrix} \tilde{\eta} & 0 \\ 0 & (\eta_2^\dagger)^{-1} \end{pmatrix}, \tag{13}$$

where $\tilde{\eta} = \eta^\dagger\eta_1$, $\mathcal{Q}^\# = (\eta_M)^{-1}\mathcal{Q}^\dagger\eta_M$, and $\eta^\# = (\tilde{\eta})^{-1}\eta^\dagger \times (\eta_2^\dagger)^{-1}$.

The nonlinear pseudo-superalgebra is given by

$$\mathcal{Q}^2 = \mathcal{Q}^{\#2} = 0, \qquad \{\mathcal{Q}, \mathcal{Q}^\#\} = f(\tilde{H}), \tag{14}$$

where $\mathcal{Q}^\# = (\eta_M)^{-1}\mathcal{Q}^\dagger\eta_M$ and $f(\tilde{H})$ denotes any function of $\tilde{H}$. It is not difficult to see that $H, H_s^\dagger$ satisfy the intertwining relations

$$\eta H = H_s^\dagger \eta, \qquad \eta^\# H_s^\dagger = H\eta^\#. \tag{15}$$

As a consequence, H and $H_s^\dagger$ are isospectral, η maps the eigenvectors of H to $H_s^\dagger$ and $\eta^\#$ does the converse except for those eigenvectors that are eliminated by these operators. They also have identical degeneracy structure except possibly for the zero eigenvalue. In analogy with ordinary supersymmetric quantum mechanics they are called nonlinear pseudo-superpartner Hamiltonians. At this point it is perhaps pertaining to remark that in the above mathematical steps we have assumed that inverse of the operators like $\eta_2^\dagger$, η_M, etc. exist though to find the analytical expression for the inverse of the operators $\eta_2^\dagger$, η will be a nontrivial mathematical task itself and hence we have assumed the existence of the operators like $\eta^\#$. Our objective here is to obtain new classes of non-Hermitian Hamiltonians that are isospectral to the known ones. Below we give the realization for η in the case of complex Scarf and Morse potential to demonstrate the practical aspects of our finding. In both the cases we shall consider discrete spectrum only.

Example 1 *(Scarf potential [10]).* For Scarf potential

$$W(x) = -\tanh x + iV_2 \operatorname{sech} x, \tag{16}$$

where V_2 is an arbitrary parameter.

Therefore

$$\begin{aligned} H = AB &= -\frac{d^2}{dx^2} + V(x) = -\frac{d^2}{dx^2} + W^2 + W' \\ &= -\frac{d^2}{dx^2} - (V_2^2 + 2)\operatorname{sech}^2(x) - 3iV_2 \operatorname{sech} x \tanh x + 1, \end{aligned} \tag{17}$$

$$\begin{aligned} H_s = BA &= -\frac{d^2}{dx^2} + \tilde{V}(x) = -\frac{d^2}{dx^2} + W^2 - W' \\ &= -\frac{d^2}{dx^2} - V_2^2 \operatorname{sech}^2(x) - iV_2 \operatorname{sech} x \tanh x + 1 \end{aligned} \tag{18}$$

and

$$\begin{aligned} H_s^\dagger &= -\frac{d^2}{dx^2} + \tilde{V}(x)^\dagger = -\frac{d^2}{dx^2} + (W^2 - W')^\dagger \\ &= -\frac{d^2}{dx^2} - V_2^2 \operatorname{sech}^2(x) + iV_2 \operatorname{sech} x \tanh x + 1, \end{aligned} \tag{19}$$

where † denotes adjoint.

Following the standard procedure [13] for finding β and utilizing the factorization property of η as illustrated in [14] we find

$$\begin{aligned} \eta &= -\left(\frac{d}{dx} - iV_2 \operatorname{sech} x\right)\left(-\frac{d}{dx} - \tanh x + iV_2 \operatorname{sech} x\right) \\ &= \eta_2\eta_1, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \eta_1 &= -\frac{d}{dx} - \tanh x + iV_2 \operatorname{sech} x \quad (\text{given in Eq. (4)}), \\ \eta_2 &= \frac{d}{dx} - iV_2 \operatorname{sech} x \quad (\text{given in Eq. (8)}), \end{aligned} \tag{21}$$

H and $H_s^\dagger$ of Eqs. (17) and (19) respectively are nonlinear pseudo-superpartner Hamiltonians and following the results obtained earlier in Ref. [10] it can be shown that they are isospectral, both having the real energy values (when $V_2 \in (\frac{3}{2}, \infty)$)

$$E_n = -\left(n + \frac{1}{2} - V_2\right)^2, \quad n = 0, 1, 2, \ldots < \left(V_2 - \frac{1}{2}\right). \tag{22}$$

The eigenfunctions corresponding to real eigenvalues are [10]

$$\psi_n(x) = N_n i^n \frac{\Gamma(n - V_2 + \frac{3}{2})}{n!\Gamma(\frac{3}{2} - V_2)} (\operatorname{sech} x)^{(V_2 - \frac{1}{2})} \times \exp\left[\frac{i}{2}\tan^{-1}(\sinh x)\right] P_n^{-V_2 - \frac{1}{2}, -V_2 + \frac{1}{2}}(i \sinh x), \tag{23}$$

where N_n is the normalization constant and $P_n^{a,b}(z)$ are the Jacobi polynomials [15].

Example 2 *(Complex Morse potential [3,11]).* In this case the function $W(x)$ is of the form

$$W = (A + iB)e^{-x} - C, \tag{24}$$

A, B and C being parameters. Therefore

$$\begin{aligned} V(x) &= W^2 + W' \\ &= (A + iB)^2 e^{-2x} - (2C + 1)(A + iB)e^{-x} + C^2, \\ \tilde{V}(x) &= W^2 - W' \\ &= (A + iB)^2 e^{-2x} - (2C - 1)(A + iB)e^{-x} + C^2, \\ \tilde{V}(x)^\dagger &= \left(W^2 - W'\right)^\dagger \\ &= (A - iB)^2 e^{-2x} - (2C - 1)(A - iB)e^{-x} + C^2. \end{aligned} \tag{25}$$

Here the intertwining operator η defined in Eq. (3) is

$$\eta = \left(-\frac{d}{dx} + (A + iB)e^{-x} - C\right)\left(e^{-\theta p}\right) = \eta_1\eta_2, \tag{26}$$

where

$$\begin{aligned} &\theta = \tan^{-1}(2B/A), \qquad p = -i\frac{d}{dx}, \\ &\eta_1 = -\frac{d}{dx} + (A + iB)e^{-x} - C \quad \left(\text{given in Eq. (4)}\right), \\ &\eta_2 = e^{-\theta p} \quad \left(\text{given in Eq. (8)}\right). \end{aligned} \tag{27}$$

$V(x)$ and $\tilde{V}(x)^\dagger$ given in Eq. (23) when substituted in H and $H_s^\dagger$ respectively give the nonlinear pseudo-superpartner Hamiltonians and they are isospectral, both having the energy eigenvalues [11]

$$E_n = -(n - C)^2, \quad n = 0, 1, 2, \ldots < C = \frac{V_2}{2\sqrt{V_1}} - 1, \tag{28}$$

where $V_1 = (A + iB)^2$ and $V_2 = (2C + 1)(A + iB)$. The eigenfunctions corresponding to $\tilde{V}(x)^\dagger$ can be written in terms of associated Laguerre polynomials as [11]

$$\psi_n(x) = z^{C-n} e^{-\frac{z}{2}} L_n^{2C-2n}(z), \quad z = 2\sqrt{V_1^*} e^{-x}, \tag{29}$$

'$*$' denoting complex conjugation.

A word of caution: as far as the application of our result in generating new exactly solvable complex potentials is concerned, the requirement that η_1 and η_2 appearing in Eqs. (21) and (27), must be invertible, may be relaxed.

To summarize, we have found an intertwining operator linking a non-Hermitian Hamiltonian to the adjoint of its nonlinear pseudo-supersymmetric partner Hamiltonian. Application of this intertwining operator to any non-Hermitian Hamiltonian will give rise to a new pair of isospectral Hamiltonians not reported before. Also in some cases a third or even higher order intertwining operator can be obtained for a given non-Hermitian Hamiltonian on using the result given in Eq. (10).

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