LETTER TO THE EDITOR

Spontaneous $\mathcal{PT}$ symmetry breaking and pseudo-supersymmetry

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Abstract
The phenomena of spontaneous $\mathcal{PT}$ symmetry breaking, associated with non-Hermitian Hamiltonians, are investigated. It is shown that spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of *pseudo-supersymmetry*. We also discuss in detail the resulting structure.

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1. Introduction

Non-Hermitian quantum mechanics has drawn a lot of attention for almost a decade now, because of the intrinsic interest of such potentials [1] admitting real spectrum under certain conditions, as well as their possible applications [2–4]. Among the various non-Hermitian models, a particular class with $\mathcal{PT}$ symmetry is of special interest, since their energy spectrum exhibits a characteristic feature—the energies are real for unbroken $\mathcal{PT}$ symmetry (when the potential as well as the wavefunctions are invariant under the combined action of space inversion ($\mathcal{P}$) and time reversal ($\mathcal{T}$)) while they switch to complex conjugate pairs for spontaneously broken $\mathcal{PT}$ symmetry (i.e., the wavefunctions lose their $\mathcal{PT}$ symmetry, although the potential still retains it) [5–7]. At the same time, various studies have shown that $\mathcal{PT}$ symmetry is neither a necessary nor a sufficient condition for the existence of a real spectrum. The criteria for the energies to be real (or in complex conjugate pairs) are the η-pseudo-Hermiticity of these non-Hermitian Hamiltonians [8].

The phenomenon of spectral discontinuity has been the subject of study of a number of works, both for Hermitian models [9, 10] as well as non-Hermitian ones [6, 8, 11–13], employing a variety of techniques. In particular, it has been observed that it occurs when a set of parameters in the potential reaches certain critical values. While the nonanalytic behaviour of the energy spectrum was interpreted in terms of supersymmetry breaking in Hermitian systems [10], an interplay was established between $\mathcal{PT}$ symmetry and supersymmetry in a certain class of non-Hermitian models [12–14]. In the present letter, we shall show that the spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of *pseudo-supersymmetry*, and establish the significant role played by a set of parameters a (in the non-Hermitian potential) in this respect. We shall make a detailed study with the help of a couple of exactly solvable examples, and also study the nature of the wavefunctions.

2. Theory

To begin with let us briefly recall some bare facts about $\mathcal{PT}$ symmetry. A non-Hermitian Hamiltonian $H(x; a)$, given by (a denoting a set of parameters)

$$H(x; a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x; a) \tag{1}$$

is said to be $\mathcal{PT}$ symmetric if

$$(\mathcal{PT})H(x; a) = H(x; a)(\mathcal{PT}) \tag{2}$$

where the *space inversion* operator $\mathcal{P}$ and the *time reversal* operator $\mathcal{T}$ are defined by their action on the position, momentum and identity operators, respectively, as

$$\mathcal{P}x\mathcal{P} = -x, \qquad \mathcal{P}p\mathcal{P} = \mathcal{T}p\mathcal{T} = -p, \qquad \mathcal{T}(i.1)\mathcal{T} = -i.1 \tag{3}$$

We note that for unbroken $\mathcal{PT}$ symmetry, the Hamiltonian $H(x; a)$ and the wavefunctions $\psi(x; a)$ are both invariant under the $\mathcal{PT}$ transformations [6, 7]

$$H^*(-x; a) = H(x; a), \qquad \psi^*(-x; a) = \pm\psi(x; a). \tag{4}$$

On the other hand a non-Hermitian Hamiltonian H is said to be η-pseudo-Hermitian (thus possessing real or complex conjugate pairs of energies), if [8]

$$H = H^\sharp = \eta^{-1} H^\dagger \eta \tag{5}$$

where η is a linear, Hermitian, invertible operator.

Let a non-Hermitian Hamiltonian $H_1(x; a)$

$$H_1(x; a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x; a) \tag{6}$$

be defined in such a way that the potential $V_1(x; a)$ has an even real part $V_+(x; a)$ and an odd imaginary part $V_-(x; a)$:

$$V_1(x; a) = V_+(x; a) + \mathrm{i}V_-(x; a), \qquad V_\pm(\pm x) = \pm V_\pm(x). \tag{7}$$

Evidently, $H_1(x; a)$ is $\mathcal{PT}$ symmetric,

$$\mathcal{PT} H_1(x; a) = H_1(x; a)\mathcal{PT} \tag{8}$$

and for such a Hamiltonian, η may be represented by the parity operator $\mathcal{P}$, i.e., $H_1(x; a)$ is $\mathcal{P}$-pseudo-Hermitian.

Now the Hamiltonian in (1) can always be factorized using the following ansatz [15]:

$$H_1 = BA + E_0^{(1)} \tag{9}$$

where A and B are defined by

$$\left.\begin{aligned} A &= \frac{\mathrm{d}}{\mathrm{d}x} + W(x; a) \\ B &= -\frac{\mathrm{d}}{\mathrm{d}x} + W(x; a) \end{aligned}\right\} \tag{10}$$

$W(x;a)$ being given in terms of the ground state eigenfunction $\psi_0^{(1)}(x;a)$ of H_1:

$$W(x;a) = -\frac{\psi_0^{(1)\prime}(x;a)}{\psi_0^{(1)}(x;a)}. \tag{11}$$

This allows H_1 to be identified with the well-known form

$$H_1 = -\frac{d^2}{dx^2} + W^2 - W' + E_0^{(1)} \tag{12}$$

where $E_0^{(1)}$ is the ground state energy of H_1.

One can then construct another Hamiltonian H_2, isospectral to H_1, by

$$H_2 = AB + E_0^{(1)} \tag{13}$$

which, in terms of $W(x;a)$, reduces to

$$H_2 = -\frac{d^2}{dx^2} + W^2 + W' + E_0^{(1)}. \tag{14}$$

Evidently, if $\psi_n^{(1)}$ is an eigenfunction of H_1 with energy eigenvalue $E_n^{(1)}$, then $\psi_n^{(2)} = A\psi_n^{(1)}$ is an eigenfunction of H_2 with the same eigenvalue $E_n^{(1)}$, except for the ground state, which is annihilated by A.

$$H_2 A\psi_n^{(1)} = (AB)A\psi_n^{(1)} = A(BA)\psi_n^{(1)} = A\big(H_1\psi_n^{(1)}\big) = E_n^{(1)}\big(A\psi_n^{(1)}\big). \tag{15}$$

Thus,

$$E_{n+1}^{(1)} = E_n^{(2)}, \qquad \psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)} - E_0^{(1)}}} A\psi_{n+1}^{(1)}. \tag{16}$$

Thus A and B play the role of intertwining operators for the partner Hamiltonians H_1 and H_2:

$$AH_1 = H_2 A, \qquad H_1 B = BH_2 \tag{17}$$

$A(B)$ converts an eigenfunction of H_1 (H_2) into an eigenfunction of H_2 (H_1), with the same energy. Additionally, $A(B)$ destroys (creates) an extra node in the eigenfunction.

For conventional Hermitian quantum systems, $W(x;a)$ is the superpotential and $B = A^\dagger$. However, for non-Hermitian systems in general, $B \neq A^\dagger$, as $W(x;a)$ is a complex function. In analogy with conventional quantum mechanics, and considering the η-pseudo-Hermiticity of the Hamiltonian, $W(x;a)$ may be termed as the *pseudo-superpotential*.

Let us now construct a matrix Hamiltonian $\mathcal{H}$, of the form

$$\mathcal{H} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}. \tag{18}$$

If we consider the following matrix representation for η [8]

$$\eta = \begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{pmatrix} \tag{19}$$

where $\eta_+(\eta_-)$ is a Hermitian linear automorphism of $H_2(H_1)$, it follows from (5), that the intertwining operators A and B must be related through

$$B = A^\sharp = \eta_+^{-1} A^\dagger \eta_- \tag{20}$$

Hence, the pseudo-superpotential $W(x;a)$ must obey the relationship

$$W(x;a) = \eta_+^{-1} W^*(x;a)\eta_- \tag{21}$$

which, for the $\mathcal{PT}$ symmetric Hamiltonian $H_1(x;a)$ considered here (with $\eta_\pm = \pm\mathcal{P}$), reduces to

$$(\mathcal{PT})W(x;a)(\mathcal{PT})^{-1} = -W(x;a). \tag{22}$$

Writing $W(x; a)$ in the form

$$W(x; a) = W_R(x; a) + \mathrm{i}W_I(x; a) \tag{23}$$

the condition (22) implies

$$\mathcal{P}W_R(x; a)\mathcal{P}^{-1} = -W_R(x; a), \qquad \mathcal{P}W_I(x; a)\mathcal{P}^{-1} = W_I(x; a). \tag{24}$$

Thus the matrix Hamiltonian $\mathcal{H}$ constructed above represents the pseudo-supersymmetric Hamiltonian, formed by the pseudo-supersymmetric partners H_1 and H_2,

$$\mathcal{H} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix} = \begin{pmatrix} AA^\sharp & 0 \\ 0 & A^\sharp A \end{pmatrix}. \tag{25}$$

The pseudo-super-Hamiltonian $\mathcal{H}$ is part of a closed algebra containing both bosonic and fermionic operators, with commutation and anticommutation relations. Such a quantum system is generated by pseudo-supercharges Q and $Q^\sharp$, which change bosonic degrees of freedom into fermionic ones and vice versa:

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\sharp = \begin{pmatrix} 0 & 0 \\ A^\sharp & 0 \end{pmatrix} = \eta^{-1} Q^\dagger \eta. \tag{26}$$

The following commutation and anticommutation relations then describe the closed pseudo-superalgebra

$$\mathcal{H} = \{Q, Q^\sharp\}, \qquad Q^2 = Q^{\sharp 2} = 0, \qquad [Q, \mathcal{H}] = [Q^\sharp, \mathcal{H}] = 0. \tag{27}$$

Let the dependence of the potential $V_1(x; a)$ on the set of parameters a be such that spontaneous breakdown of $\mathcal{PT}$ symmetry occurs at some critical value of a, say a_c, and real energies change to complex conjugate pairs. In terms of the pseudo-superpotential, the condition (22) or (24) holds only for unbroken $\mathcal{PT}$ symmetry. In such a situation, the relationship (20) breaks down: $B \neq A^\sharp$. Consequently, the isospectrality of the partners is lost as A and $A^\sharp$ fail to intertwine the non-Hermitian Hamiltonians denoted by H_2 and H_1. Though one can still write $H_2 = AB$ formally, the anticommutator of the pseudo-supercharges fails to give the pseudo-super-Hamiltonian $\mathcal{H}$

$$\{Q, Q^\sharp\} \neq \mathcal{H}. \tag{28}$$

In analogy with the spontaneous breakdown of supersymmetry in conventional quantum mechanics (with vanishing zero energy ground state), this may be viewed as the explicit breakdown of pseudo-supersymmetry in non-Hermitian $\mathcal{PT}$ symmetric quantum systems. Thus the pseudo-supersymmetric algebra defined in (27) holds only for unbroken $\mathcal{PT}$ symmetry, when the pseudo-superpotential defined in (11) above obeys (24), and the energies are real. However, at the point of spontaneous breakdown of $\mathcal{PT}$ symmetry ($a = a_c$), when the energies of the system switch from real to complex conjugate pairs, both conditions (20) and (24) are violated, and the pseudo-supersymmetry of the system is explicitly broken.

We consolidate our observations with a couple of exactly solvable examples.

3. Explicit examples

3.1. $\mathcal{PT}$ symmetric Scarf II potential

The non-Hermitian $\mathcal{PT}$ symmetric Scarf II model may be described by the Hamiltonian

$$H_1(x; v_1, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - v_1 \operatorname{sech}^2 x - \mathrm{i}\left(v_1 + a + \frac{1}{4}\right) \operatorname{sech} x \tanh x, \qquad v_1 > 0 \tag{29}$$

where v_1 and a are real. The energy levels and the corresponding eigenfunctions are given by [6]

$$E_{nq}^{(1)}(v_1; a) = -\left\{n + \frac{1}{2} - \frac{1}{2}(s + qt)\right\}^2, \qquad n = 0, 1, 2, \ldots < \frac{1}{2}(|s + qt| - 1) \tag{30}$$

$$\psi_{nq}^{(1)}(x; v_1, a) = N_{nq}\left(\frac{1 - \mathrm{i}\sinh x}{2}\right)^{-\lambda_q}\left(\frac{1 + \mathrm{i}\sinh x}{2}\right)^{-\mu_q} P_n^{-2\lambda_q - \frac{1}{2}, -2\mu_q - \frac{1}{2}}(\mathrm{i}\,\sinh x) \tag{31}$$

where $s = \sqrt{2v_1 + a + \frac{1}{2}}$, $t = \sqrt{-a}$, $\lambda_q = -\frac{1}{4} + q\frac{s}{2}$, $\mu_q = -\frac{1}{4} + q\frac{t}{2}$ and q $(= \pm 1)$ is the quasiparity, giving rise to doublet solutions, which is a characteristic feature of this class of $\mathcal{PT}$ symmetric models. Normalization requirement restricts the signs allowed in λ_q and μ_q.

It follows from (29) and (31) that the Hamiltonian $H_1(x; v_1, a)$ is always invariant under the $\mathcal{PT}$ transformation irrespective of the value of a, while the wavefunctions $\psi_{nq}^{(1)}(x; v_1, a)$ are $\mathcal{PT}$ invariant only when

$$-\left(2v_1 + \tfrac{1}{2}\right) \leqslant a \leqslant 0. \tag{32}$$

The pseudo-superpotential corresponding to the Hamiltonian in (29) above, may be given by

$$\begin{aligned} W(x; a) &= (\lambda_q + \mu_q)\tanh x - \mathrm{i}(\lambda_q - \mu_q)\,\mathrm{sech}\,x \\ &= \frac{1}{2}\left(-1 + \sqrt{2v_1 + a + \frac{1}{2}} + q\sqrt{-a}\right)\tanh x - \frac{\mathrm{i}}{2}\left(\sqrt{2v_1 + a + \frac{1}{2}} - q\sqrt{-a}\right)\mathrm{sech}\,x. \end{aligned} \tag{33}$$

Obviously, (24) is satisfied for real λ_q and μ_q, which, in turn, is related to (32), and hence to unbroken $\mathcal{PT}$ symmetry, i.e. real energies. At the same time whenever a crosses a critical value a_c, i.e., a lies beyond the region specified in (32), and energies switch to complex conjugate pairs, two simultaneous phenomena are observed:

(i) the condition (24) is violated, thus inducing spontaneous breakdown of $\mathcal{PT}$ symmetry in $H_1(x; v_1, a)$;

(ii) the violation of (20) leading to the explicit breakdown of pseudo-supersymmetry.

If one keeps v_1 fixed, then from (30) one can show that though

$$\lim_{a \to 0^-} E_{nq}^{(1)}(a) = E_{nq}^{(1)}(a = 0) \tag{34}$$

the right-hand limit, viz., $\lim_{a\to 0^+} E_{nq}^{(1)}(a)$, does not exist. A similar situation occurs at $a = -(2v_1 + 1/2)$.

It would be interesting to study the nature and behaviour of the partner Hamiltonian $H_2(x; v_1, a)$, from (14).

(i) For a lying in the range as given in (32),

$$\begin{aligned} H_2(x; v_1, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left\{-\frac{3}{4} + \frac{s^2 + t^2}{2} - (s + qt)\right\}\mathrm{sech}^2\,x - \mathrm{i} \\ \times\left\{\frac{1}{2}(s^2 - t^2) - (s - qt)\right\}\mathrm{sech}\,x\tanh x. \end{aligned} \tag{35}$$

Evidently, as the condition (24) is obeyed in this case, the partner Hamiltonian $H_2(x; v_1, a)$ is also $\mathcal{PT}$ symmetric. It has real energies, isospectral to $H_1(x; v_1, a)$, with the possible exception of the ground state. Thus $H_1(x; v_1, a)$ and $H_2(x; v_1, a)$ form the pseudo-supersymmetric partners of the super-Hamiltonian $\mathcal{H}$, obeying the pseudo-supersymmetric algebra given in (27).

(ii) For values of a outside the range given in (32), $\mathcal{PT}$ symmetry is spontaneously broken in the Scarf II Hamiltonian $H_1(x; v_1, a)$. Let $a > 0$, so that $t = \mathrm{i}\alpha$. It can be seen that the partner Hamiltonian $H_2(x; v_1, a)$ is no longer $\mathcal{PT}$ symmetric:

$$H_2(x; v_1, a) = -\frac{d^2}{dx^2} - \left\{-\frac{3}{4} + \frac{s^2 - \alpha^2 - 2s}{2} - iq\alpha\right\} \operatorname{sech}^2 x - i \times \left\{\frac{1}{2}(s^2 + \alpha^2 - 2s) + iq\alpha\right\} \operatorname{sech} x \tanh x. \quad (36)$$

Thus the spontaneous breakdown of $\mathcal{PT}$ symmetry in the Scarf II Hamiltonian $H_1(x; v_1, a)$ is manifested as explicit $\mathcal{PT}$ symmetry breaking in the partner Hamiltonian $H_2(x; v_1, a)$, the two no longer being isospectral to each other. Though one can still write $H_2 = AB$ formally, the pseudo-supersymmetry is explicitly broken. Thus the spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of pseudo-supersymmetry.

The wavefunctions, too, behave quite strangely at these points of spectral discontinuities. So long as $\mathcal{PT}$ symmetry is unbroken, the wavefunctions are normalizable in the sense of $\mathcal{CPT}$ norm [11, 16]:

$$\langle\psi_m|\psi_n\rangle^{\mathcal{CPT}} = \int dx\, \psi_m^{\mathcal{CPT}}(x)\psi_n(x) = \delta_{m,n}, \qquad \psi_m^{\mathcal{CPT}}(x) = \int dy\, \mathcal{C}(x, y)\psi_m^*(y) \quad (37)$$

where $\mathcal{C}$ is the charge operator. The interesting point to be observed here is that, at the point of spontaneous breakdown of $\mathcal{PT}$ symmetry, though the wavefunctions remain well behaved, their $\mathcal{CPT}$ norm vanishes:

$$\int (\mathcal{CPT}\psi_n(x))\psi_n(x)\, dx \to 0. \quad (38)$$

This can be shown by straightforward calculations [17]. Thus, unlike the Hermitian models [9] where the effect of spectral discontinuities forces the eigenfunction to be non-square integrable, in the present case the eigenfunctions, though exhibiting proper behaviour at $\pm\infty$, become *self-orthogonal* [3].

3.2. $\mathcal{PT}$ symmetric oscillator

We next consider another non-Hermitian model, $\mathcal{PT}$ symmetrized in a different way; viz., the well-known $\mathcal{PT}$ symmetric oscillator, given by the Hamiltonian

$$H_1(x; a) = -\frac{d^2}{dx^2} + (x - i\epsilon)^2 + \frac{a - \frac{1}{4}}{(x - i\epsilon)^2} \quad (39)$$

where ϵ is a real number. The energy eigenvalues and the corresponding eigenfunctions are given by [18]

$$E_{nq}^{(1)}(a) = 4n + 2 - 2q\sqrt{a} \qquad n = 0, 1, 2, \ldots \quad (40)$$

$$\psi_{nq}(x; a) = N_{nq}\, e^{-\frac{(x-i\epsilon)^2}{2}} (x - i\epsilon)^{-q\sqrt{a}+\frac{1}{2}} L_n^{(-q\sqrt{a})}((x - i\epsilon)^2) \quad (41)$$

where the quasiparity $q(= \pm 1)$ again gives doublet solutions. Proceeding in a similar manner, the pseudo-superpotential, $W(x; a)$, and the partner, $H_2(x; a)$, turn out to be

$$W(x; a) = (x - i\epsilon) - \frac{-q\sqrt{a} + \frac{1}{2}}{(x - i\epsilon)} \quad (42)$$

$$H_2(x; a) = -\frac{d^2}{dx^2} + (x - i\epsilon)^2 + \frac{a - 2q\sqrt{a} + \frac{3}{4}}{(x - i\epsilon)^2} + 2. \quad (43)$$

Thus, it is easy to observe that the critical value of a here is $a_c = 0$. So long as

$$a \geqslant 0 \tag{44}$$

the condition (24) is satisfied, $\mathcal{PT}$ symmetry is unbroken in the $\mathcal{PT}$ oscillator, and the partner $H_2(x; a)$ Hamiltonian (in (43)) is also $\mathcal{PT}$ symmetric, both sharing same real energies, without possibly the ground state. Consequently, pseudo-supersymmetry is unbroken. On the other hand, for $a < 0$, $\mathcal{PT}$ symmetry is spontaneously broken in the original Hamiltonian, giving complex conjugate energies. The conditions (20) and (24) are violated, leading to the explicit breakdown of pseudo-supersymmetry. Furthermore, though

$$\lim_{a\to 0^+} E_{nq}^{(1)}(a) = E_{nq}^{(1)}(0) \tag{45}$$

the left-hand limit, viz., $\lim_{a\to 0^-} E_{nq}^{(1)}(a)$, does not exist. Additionally, though the wavefunctions remain well behaved at $\pm\infty$, their $\mathcal{CPT}$ norm goes to zero. Thus in this model too, the point of discontinuity of the spectrum is associated with the simultaneous breakdown of $\mathcal{PT}$ symmetry and pseudo-supersymmetry.

4. Conclusions

In the present letter we have established the relation between the spontaneous breakdown of $\mathcal{PT}$ symmetry and the explicit breakdown of pseudo-supersymmetry, at some critical value a_c of a set of parameters a in the Hamiltonian $H(x; a)$. In particular, we have shown that in a class of non-Hermitian, but $\mathcal{PT}$ symmetric Hamiltonians $H_1(x; a)$, the changing of energies from real to complex conjugate values is a direct consequence of the simultaneous breakdown of these two symmetries. The anticommutator of the pseudo-supercharges Q and $Q^\sharp$ fails to give the pseudo-super-Hamiltonian $\mathcal{H}$, as the Hamiltonian $H_2 = AA^\sharp$ is no longer isospectral to its partner $H_1 = A^\sharp A$. In fact, $\mathcal{PT}$ symmetry is explicitly broken in the partner $H_2(x; a)$. Furthermore, though the wavefunctions remain well behaved, they become self-orthogonal beyond a_c, as their $\mathcal{CPT}$ norm goes to zero. All the above observations hold in both the explicit examples considered here.

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