DSR Relativistic Particle in a Lagrangian formulation and Non-Commutative Spacetime: A Gauge Independent Analysis

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Abstract:

In this paper we have constructed a coordinate space (or geometric) Lagrangian for a point particle that satisfies the Doubly Special Relativity (DSR) dispersion relation in the Magueijo-Smolin framework. At the same time, the symplectic structure induces a Non-Commutative phase space, which interpolates between κ-Minkowski and Snyder phase space. Hence this model bridges an existing gap between two conceptually distinct ideas in a natural way.

We thoroughly discuss how this type of construction can be carried out from a phase space (or first order) Lagrangian approach. The inclusion of external physical interactions are also briefly outlined.

The work serves as a demonstration of how Hamiltonian (and Lagrangian) dynamics can be built around a given non-trivial symplectic structure.

Quantum gravity motivates [1] us in accepting two observer-independent scales: a length scale (∼ Planck length?) and of course, the velocity of light. On the one hand, this gave rise to the novel idea of Doubly Special Relativity (DSR) [2]. [1] On the other hand, existence of a length scale is directly linked with the breakdown of spacetime continuum and the emergence of a Non-Commutative (NC) spacetime (below *e.g.* Planck length) [4]. The specific form of NC spacetime that one supposedly encounters in the context of DSR is known as κ-Minkowski spacetime [5]. There are manifestly distinct constructions of DSR dispersion relation [6, 5, 7, 8, 9], which appear as Casimir operators of distinct κ-Poincare algebra. From Quantum Group theoretic point of view, it has been argued [9, 10] that these κ-Poincare algebra are *dual* to different forms of κ-Minkowski NC phase space, all of which, indeed, have the *same* κ-Minkowski spacetime structure. Indeed, this state of affairs makes the connection between DSR and NC spacetime somewhat indirect.

In the present state of affairs, as described above, there exist two important gaps that make the physical picture of DSR particle incomplete. These are the following:
(i) A *minimal* $3+1$-dimensional *dynamical* model that induces the Magueijo-Smolin (MS) DSR dispersion relation is absent.
(ii) A *direct connection* in a dynamical framework between the DSR particle and its underlying NC spacetime is lacking.

The present work sheds light on both these areas simultaneously. As for (i), we provide an explicit geometric Lagrangian that obeys the exact DSR dispersion law in Magueijo-Smolin base. The problem (ii) is also taken care of since the particle phase space discussed here has an inherent NC structure, which turns out to be a novel one. It is a combination between κ-Minkowski (in MS base) and Snyder [4] NC phase space algebra. This algebra emerges as Dirac Brackets after an analysis of the constraints are performed in Hamiltonian framework [11]. Note that our conclusion deviates somewhat from the previously obtained Quantum Group motivated result [5, 6, 7, 9] of an association between the MS particle and κ-Minkowski phase space (in MS base). The latter derivation does not involve particle dynamics and is more mathematically inclined.

The present work is also an improvement from our previous attempt [12] where the NC phase space was recovered in a particular gauge.

The plan of the paper is the following: In Section I we will present the geometric Lagrangian of the DSR particle in Magueijo-Smolin base and demonstrate that it correctly reproduces the MS DSR relation. We also perform the constraint analysis to reproduce the interpolating NC phase space algebra. This is the main result of our work. In Section II we study the symmetry properties of this NC phase space and show that DSR mass-shell condition, reproduced here, is Lorentz invariant. Section III is devoted to a systematic analysis of the derivation of this particular geometric Lagrangian from a first order phase space Lagrangian [13]. In Section IV we summarize our results and point out some future directions of study.

Section I: The DSR Geometric Lagrangian

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[1] One of the motivations of studying DSR is that it induces modified dispersion relations which might be useful in explaining observations of ultra-high energy cosmic ray particles, that violate the Greisen-Zatsepin-Kuzmin bound [3].

The point particle dispersion relation in Magueijo-Smolin base [8] is,

$$p^2 = m^2(1 - \frac{E}{\kappa})^2, \tag{1}$$

where E denotes the particle energy $E = p^0$. We posit the following geometric Lagrangian in order to describe the point particle obeying (1),

$$L = \frac{\kappa}{D} \frac{x^2}{(\eta x)^2}[\sqrt{B} + \frac{(\eta x)(x\dot{x})}{x^2} - (\eta \dot{x})]. \tag{2}$$

In the above the functions B and D stand for,

$$B = \{((\eta \dot{x}) - \frac{(x\dot{x})(\eta x)}{x^2})^2 + (\frac{(\eta x)^2}{x^2} + \frac{\kappa^2 - m^2}{m^2})(\dot{x}^2 - \frac{(x\dot{x})^2}{x^2})\},$$

$$D = 1 + \frac{(\kappa^2 - m^2)}{m^2} \frac{x^2}{(\eta x)^2}. \tag{3}$$

Our metric is $diag\ g^{00} = -g^{ii} = 1$. The constant vector $\eta_\mu \equiv \eta_0 = 1, \eta_i = 0$ is introduced to maintain relativistic notation. Clearly the presence of $E = \eta^\mu p_\mu$ in (1) makes the relation non-covariant from Special Theory of Relativity point of view. η_μ will also be useful later when we discuss the NC phase space algebra. The notation $a^\mu b_\mu = (ab)$ is followed throughout. The Lagrangian in (2) constitutes our principal result.

First we reproduce the DSR dispersion relation. The conjugate momentum

$$p^\mu = \partial L/\partial \dot{x}_\mu$$

is computed as,

$$p_\mu = \frac{\kappa x^2}{D(\eta x)^2}[\{\eta_\mu - \frac{(\eta x)}{x^2}x_\mu\} + \frac{1}{\sqrt{B}}\{(\eta \dot{x}) - \frac{(x\dot{x})(\eta x)}{x^2}\}\{\eta_\mu - \frac{(\eta x)}{x^2}x_\mu\}$$

$$+ D\frac{(\eta x)^2}{x^2}\{\dot{x}_\mu - \frac{(x\dot{x})}{x^2}x_\mu\}] \tag{4}$$

It is straightforward but tedious to check that (4) satisfies the relation,

$$p^2 = m^2(1 - \frac{(\eta p)}{\kappa})^2 = m^2(1 - \frac{E}{\kappa})^2. \tag{5}$$

Next we recover the NC phase space structure. This is achieved through a Hamiltonian analysis of constraints, as formulated by Dirac [11]. Notice that apart from the MS mass-shell condition (5), there is another primary constraint, that follows from (4):

$$\psi_2 \equiv (xp) \approx 0. \tag{6}$$

For this set of constraints

$$\psi_1 \equiv p^2 - m^2(1 - \frac{(\eta p)}{\kappa})^2 \approx 0 \ ; \quad \psi_2 \equiv (xp) \approx 0, \tag{7}$$

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we find that their Poisson Bracket is non-vanishing,

$$\{\psi_1, \psi_2\} = 2m^2(1 - (\eta p)/\kappa). \tag{8}$$

In the terminology of Dirac, non-commuting and commuting (in the sense of Poisson Brackets) constraints are termed as Second Class Constraints (SCC) and First Class Constraints (FCC) respectively. Presence of SCCs require a modification in the symplectic structure by way of replacing Poisson Brackets by Dirac Brackets. On the other hand, FCCs induce local gauge invariance.

In the presence of SCCs ψ_i with $\{\psi_1, \psi_2\} \neq 0$, the Dirac Brackets are defined in the following way,

$$\{A, B\}^* = \{A, B\} - \{A, \psi_i\}\{\psi_i, \psi_j\}^{-1}\{\psi_j, B\}, \tag{9}$$

where $\{\psi_i, \psi_j\}$ refers to the invertible constraint matrix. From now on we will always use Dirac brackets and refer them simply as $\{A, B\}$.

For the model at hand, we find the only non-vanishing constraint matrix component to be

$$\{\psi_1, \psi_2\} = 2m^2(1 - (\eta p)/\kappa).$$

From (9) the Dirac brackets follow:

$$\{x_\mu, x_\nu\} = \frac{1}{\kappa}(x_\mu \eta_\nu - x_\nu \eta_\mu) + \frac{1}{m^2(1 - (\eta p)/\kappa)}(x_\mu p_\nu - x_\nu p_\mu),$$

$$\{x_\mu, p_\nu\} = -g_{\mu\nu} + \frac{1}{\kappa}\eta_\mu p_\nu + \frac{p_\mu p_\nu}{m^2(1 - (\eta p)/\kappa)}, \quad \{p_\mu, p_\nu\} = 0. \tag{10}$$

Performing an (invertible) transformation on the variables,

$$\tilde{x}_\mu = x_\mu - \frac{1}{\kappa}(x\eta)p_\mu, \quad x_\mu = \tilde{x}_\mu + \frac{(\eta\tilde{x})}{\kappa(1 - (\eta p)/\kappa)}p_\mu, \tag{11}$$

we find a novel and interesting form of algebra that interpolates between the Snyder [4] and κ-Minkowski form [2, 5, 9]:

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = \frac{1}{\kappa}(\tilde{x}_\mu \eta_\nu - \tilde{x}_\nu \eta_\mu) + \frac{\kappa^2 - m^2}{\kappa^2 m^2}(\tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu),$$

$$\{\tilde{x}_\mu, p_\nu\} = -g_{\mu\nu} + \frac{1}{\kappa}(p_\mu \eta_\nu + p_\nu \eta_\mu) + \frac{\kappa^2 - m^2}{\kappa^2 m^2}p_\mu p_\nu, \quad \{p_\mu, p_\nu\} = 0. \tag{12}$$

In absence of the $1/\kappa$-term one obtains the Snyder [4] algebra,

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = \frac{\kappa^2 - m^2}{\kappa^2 m^2}(\tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu),$$

$$\{\tilde{x}_\mu, p_\nu\} = -g_{\mu\nu} + \frac{\kappa^2 - m^2}{\kappa^2 m^2}p_\mu p_\nu, \quad \{p_\mu, p_\nu\} = 0. \tag{13}$$

On the other hand, without the $(\kappa^2 - m^2)/(\kappa^2 m^2)$ term, one finds the κ-Minkowski phase space in MS base [9, 14],

$$\{x^\mu, x^\nu\} = \frac{1}{\kappa}(x^\mu \eta^\nu - x^\nu \eta^\mu),$$

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$$\{x^\mu, p^\nu\} = -g^{\mu\nu} + \frac{1}{\kappa}(p^\mu\eta^\nu + p^\nu\eta^\mu) \; ; \{p^\mu, p^\nu\} = 0. \tag{14}$$

It is important to mention the base in which the κ-Minkowski algebra is expressed (see for example [9, 14]). The algebra (14) in detailed form is,

$$\{x^i, x^j\} = 0 \; ; \;\; \{x^0, x^i\} = -\frac{1}{\kappa}x^i,$$

$$\{x^0, p^i\} = \frac{1}{\kappa}p^i \; ; \; \{x^i, p^0\} = \frac{1}{\kappa}p^i \; ; \; \{x^0, p^0\} = -1 + \frac{2}{\kappa}p^0,$$

$$\{x^i, p^j\} = -g^{ij} \; ; \; \{p^\mu, p^\nu\} = 0. \tag{15}$$

For $\kappa = \infty$ one recovers the normal canonical phase space. Note that choice of other forms of bases, (such as bi-crossproduct or standard base), leads to forms of $\{x_\mu, p_\nu\}$ algebra that are distinct from (15) but in *all* the bases, the spacetime algebra is identical to that of (15) [15].

As we have mentioned in the Introduction, the NC phase space that we have obtained, turns out to be a generalized version of κ-Minkowski algebra and in fact reduces to the latter for the choice $m = \kappa$. This choice of m, the rest mass, is the upper limit for which the MS relation becomes non-relativistic, $p_0 = (\vec{p})^2/(2\kappa) + \kappa/2$.

Section II: Symmetry Properties

It is well known that *individually* both Snyder and κ-Minkowski extensions of the Poincare algebra, keep the Lorentz subalgebra sector unchanged and only transformation properties under boost are modified [15]. It is reassuring to find that this feature holds for the present interpolating algebra (12) as well, which happens to be a combination of Snyder and κ-Minkowski algebra.

Let us consider the conventional form of Lorentz generators $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and using (12) we find that the Lorentz algebra remains intact,

$$\{L^{\mu\nu}, L^{\alpha\beta}\} = g^{\mu\beta}L^{\nu\alpha} + g^{\mu\alpha}L^{\beta\nu} + g^{\nu\beta}L^{\alpha\mu} + g^{\nu\alpha}L^{\mu\beta}. \tag{16}$$

However, x^μ and p^μ transform in the following non-trivial way:

$$\{L^{\mu\nu}, x^\rho\} = g^{\nu\rho}x^\mu - g^{\mu\rho}x^\nu - \frac{1}{\kappa}(\eta^\mu L^{\rho\nu} - \eta^\nu L^{\rho\mu}) \; ; \; \{L^{\mu\nu}, p^\rho\} = g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu - \frac{1}{\kappa}(\eta^\nu p^\mu - \eta^\mu p^\nu)p^\rho. \tag{17}$$

Notice that only the boost relations are modified [8] (see also Kowalski-Glikman in [2]),

$$\{L^{0i}, x^\mu\} = x^0 g^{i\mu} - x^i g^{0\mu} - \frac{1}{\kappa}L^{\mu i} \; ; \; \{L^{0i}, p^\mu\} = p^0 g^{i\mu} - p^i g^{0\mu} + \frac{1}{\kappa}p^i p^\mu. \tag{18}$$

We note that the mass-shell condition (5) is Lorentz invariant on-shell:

$$\{L^{\mu\nu}, p^2 - m^2(1 - \frac{(\eta p)}{\kappa})^2\} = -\frac{2}{\kappa}(p^\mu\eta^\nu - p^\nu\eta^\mu)(p^2 - m^2(1 - \frac{(\eta p)}{\kappa})^2). \tag{19}$$

Section III: Coordinate space Lagrangian

Our objective is now to recover the coordinate space geometric Lagrangian (2), stated at the beginning. We follow the procedure described in a recent work [13]. Since the relativistic particle model is reparameterization invariant, it has a vanishing Hamiltonian and the action will consist of the normal kinetic term and constraints:

$$S = \int d\tau [(p\dot{x}) - \frac{\lambda_1}{2}\{p^2 - m^2(1 - \frac{(\eta p)}{\kappa})^2\} - \lambda_2(xp)] \equiv \int d\tau [(p\dot{x}) - \frac{\lambda_1}{2}\psi_1 - \lambda_2\psi_2]. \qquad (20)$$

The idea is to exploit the equation of motion along with the constraint conditions to express the multipliers λ_1 and λ_2 and p_μ in terms of coordinate (x_μ) and velocity $(\dot{x}_\mu)$ variables. The equation of motion is,

$$\dot{x}_\mu - \lambda_1\{p_\mu + \frac{m^2}{\kappa}(1 - \frac{\eta p)}{\kappa})\eta_\mu\} - \lambda_2 x_\mu = 0. \qquad (21)$$

The above is rewritten as,

$$p_\mu = \frac{1}{\lambda_1}[\dot{x}_\mu - \lambda_2 x_\mu + \frac{m^2}{\kappa^2 - m^2}\{(\eta\dot{x}) - \kappa\lambda_1 - (\eta x)\lambda_2\}\eta_\mu]. \qquad (22)$$

The constraint $(xp) = 0$ yields a relation between λ_1 and λ_2. Subsequently we substitute this relation together with (22) in the MS mass-shell constraint relation $\psi_1 = 0$ and obtain an algebraic quadratic equation for λ_2. For λ_2 we take the solution with the positive root of the discriminant and then in turn we compute λ_1. The expressions are,

$$\lambda_2 = \frac{1}{D}[\{\frac{(\eta\dot{x})}{(\eta x)} + \frac{\kappa^2 - m^2}{m^2}\frac{(x\dot{x})}{(\eta x)^2}\} + \frac{1}{(\eta x)}\sqrt{B}], \qquad (23)$$

$$\lambda_1 = \frac{1}{\kappa}\sqrt{B}. \qquad (24)$$

Finally we are able to get p_μ in terms of x_μ and $\dot{x}_\mu$ only by using (22). This yields the cherished expression for the Lagrangian that was provided at the beginning in (2).

Section IV: Summary and Future Outlook

In the present work, for the first time a fully dynamical model of a particle having the (Magueijo-Smolin) DSR dispersion relation has been provided in the *Lagrangian framework*. The connection between DSR and NC phase space is also attained in a direct way since the model enjoys an inherent NC phase space. The NC algebra is in itself novel and interesting since it interpolates between the normal Snyder algebra and κ-Minkowski algebra. There is also an improvement (in an esthetic way) compared to previous works [16, 12] since *no gauge fixing is necessary*.

We list some of the problems that we intend to study in near future:
(i) It is very important to study the behavior of the DSR particle in presence of interaction. So far this has not been done mainly because a suitable Lagrangian formulation of the model was lacking. Indeed, in our approach, an external interaction can be easily included, the simplest being a minimal coupling to a $U(1)$ gauge field and we end up with the interacting system,

$$L_I = \dot{x}_\mu p^\mu + \mathcal{A}_\mu(x)\dot{x}^\mu - \frac{\lambda_1}{2}\psi_1 - \lambda_2\psi_2, \qquad (25)$$

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where $\mathcal{A}_\mu(x)$ is the external *physical* gauge field.

(ii) It will be interesting to construct, in the present framework, point particle models of other DSR structures. It has been shown that these different DSR bases are related through non-linear transformations of variables and explicit rules are provided in [15]. However, whether this (mathematical) equivalence generates physically equivalent models can be tested by considering dynamical models of the present form, where the particle behavior can be studied in a direct way.

References

[1] C.Rovelli and L.Smolin, Nucl.Phys. B442 (1995)593; Erratum: *ibid* 456 (1995)734; For a recent review on quantum gravity phenomenology, see G.Amelino-Camelia, Mod.Phys.Lett. A17 (2002) 899 (gr-qc/0204051).

[2] G.Amelino-Camelia, Nature 418 34(2002); Phys.Lett.B 510 (2001)255; Int.J.Mod.Phys. A11 (2002)35; For a review on DSR, see J.Kowalski-Glikman, hep-th/0405273.

[3] G.Amelino-Camelia and T.Piran, Phys.Rev. D64 (2001)036005 (astro-ph/0008107).

[4] H.S.Snyder, Phys.Rev. 71 (1947)68. This is the first instance of an NC phase space and it plays an important role in our work.

[5] S.Majid and H.Ruegg, Phys.Lett. B334 (1994)348; J.Lukierski, H.Ruegg, W.J.Zakrzewski, Annals Phys. 243 (1995)90.

[6] J.Lukierski, A.Nowicki, H.Ruegg and V.N.Tolstoy, Phys.Lett. B264 (1991)331.

[7] P.Kosinski, J.Lukierski, P.Maslanka and J.Sobczyk, Mod.Phys.Lett. A10 (1995)2599.

[8] J.Magueijo and L.Smolin, Phys.Rev.Lett. 88 (2002)190403; Phys.Rev. D67 (2003)044017.

[9] J.Kowalski-Glikman and S.Nowak, Phys.Lett.B 539 (2002)126 (hep-th/0203040); Class.Quant.Grav. 20 (2003)4799 hep-th/0304101.

[10] A.Blaut, M.Daszkiewicz, J.Kowalski-Glikman and S.Nowak, Phys.Lett. B582 (2004) 82 (hep-th/0312045).

[11] P.A.M.Dirac, *Lectures on Quantum Mechanics*, Yeshiva University Press, New York, 1964.

[12] S.Ghosh, Phys.Rev. D74 084019 (hep-th/0608206).

[13] R.Banerjee, S.Kulkarni and S.Samanta, JHEP 0605 (2006) 077 (hep-th/0602151).

[14] J.Kowalski-Glikman and S.Nowak, Int.J.Mod.Phys. D12 (2003) 299 (hep-th/0204245); A.Agostini, G.Amelino-Camelia and F.D'Andrea, Int.J.Mod.Phys. A19 (2004) 5187 (hep-th/0306013).

[15] J.Kowalski-Glikman, Phys.Lett. B547 (2002)291 (hep-th/0207279); J.Kowalski-Glikman and S.Nowak, hep-th/0411154.

[16] S.Ghosh, Phys.Lett.B 623 (2005)251; S.Ghosh and P.Pal, Phys.Lett.B 618 (2005)243; F.Girelli, T.Konopka, J.Kowalski-Glikman and E.R.Livine, Phys.Rev. D73 (2006)045008 (hep-th/0512107). .