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DEFORMATION OF LEIBNIZ ALGEBRA MORPHISMS OVER COMMUTATIVE LOCAL ALGEBRA BASE

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We study deformations of Leibniz algebra morphisms over a commutative local algebra base with* 1. *We construct the associated deformation cohomology that controls deformations using the cochain complex defining the Leibniz cohomology.

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1. INTRODUCTION

Leibniz algebras are one of the new algebras introduced by Loday [16–18] in connection with the study of periodicity phenomenon in algebraic K-theory. Leibniz algebras have been introduced as a non-antisymmetric analogue of Lie algebras. The other types of algebras, namely, dialgebra, Zinbiel, dendriform, etc., are studied in Loday [18]. Each of these is an algebra over a suitable operad. For instance, Leibniz algebras are precisely the algebras over the binary quadratic operad Leib. The aim of this article is to study an algebraic deformation theory of Leibniz algebra morphisms over a commutative local algebra base. The original deformation theory of algebraic structures dates back to the monumental work of Gerstenhaber [7–11]. Following Gerstenhaber's idea algebraic deformation theory of other algebraic structures have been studied. For instance the Lie algebra case is done in Nijenhuis and Richardson [22], and more recently deformation of dialgebras have been studied in Majumdar and Mukherjee [20]. As shown in Balavoine [4] formal one parameter deformation theory of all the above mentioned algebras are best studied in a unified way through formal deformation theory of algebras over quadratic operads. The relative version, the deformation of associative algebra morphisms, is studied in a series of articles by Gerstenhaber and Schack [12–14]. Deformations of Lie algebra morphisms have been studied in Nijenhuis and Richardson [23]. Recently, the relative version for Zinbiel algebras, dialgebras and Leibniz algebras have been studied in Yau [24, 25], and Mandal [21], respectively. A major part in each of the above articles [21, 24, 25] is a tedious computation to show that the obstructions to extending an infinitesimal deformation to a higher order deformation are 3-cocycles

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in the respective deformation complex. In the absolute case, this is usually done by using a pre-Lie system structure on the deformation complex as shown by Gerstenhaber. However, in the relative case, it is not yet clear how to define such a structure on the deformation complex. So either one has to resort to a direct computation, or else, use a version of Cohomology Comparison Theorem [12–14] in the respective cases, which are yet to be studied.

The aim of this article is to show that when viewed from a general perspective, namely, deformation over a commutative local algebra base (rather than a one parameter formal deformation) the picture becomes more transparent. This approach also gives a nice relationship between the Harrison cohomology of the base of the deformation and the deformation cohomology. Obstructions in this case, arise as a map from the Harrison cohomology of the base to the deformation cohomology in question. Although we explain this by considering deformation of Leibniz algebra morphisms, deformation theory of morphisms of other algebras mentioned above can be treated similarly.

The notion of deformation over a commutative local algebra base was considered in Fialowski and Fuchs [5] for constructing miniversal deformation of Lie algebras. Recently a similar construction for Leibniz algebras is given in Fialowski et al. [6].

The article is organized as follows. In Section 2, we recall the definitions of a Leibniz algebra and its cohomology. In Section 3, we introduce the relevant deformation cohomology which controls the deformation. In Section 4, we describe the notion of deformation of Leibniz algebra morphisms over a commutative local algebra base and discuss related concepts. In Section 5, we show that the obstructions in lifting a given deformation to a deformation over an extended base are realized as a map from the second Harrison cohomology of the base to the third deformation cohomology. Finally, in Section 6, we introduce formal deformations and obtain a sufficient criterion for existence of a formal deformation with a given differential and infinitesimal part.

2. LEIBNIZ ALGEBRA AND ITS COHOMOLOGY

In this section, we recall the definition of a Leibniz algebra and describe its cohomology. Let $\mathbb{K}$ be a fixed field. The tensor product over $\mathbb{K}$ will be denoted by $\otimes$.

Definition 2.1. A Leibniz algebra is a $\mathbb{K}$-module L, equipped with a bracket operation, which is $\mathbb{K}$-bilinear and satisfies the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \qquad \text{for } x, y, z \in L.$$

Any Lie algebra is automatically a Leibniz algebra, as in the presence of antisymmetry, the Jacobi identity is equivalent to the Leibniz identity.

Example 2.2. Let (L, d) be a differential Lie algebra with the Lie bracket $[,]$. Then L is a Leibniz algebra with the bracket operation $[x, y]_d := [x, dy]$. The new bracket on L is called the *derived bracket*.

More examples are given in [1–3, 19].

Let L be a Leibniz algebra and M a representation of L. By definition [19], M is a $\mathbb{K}$-module equipped with two actions (left and right) of L,

$$[-,-] : L \times M \longrightarrow M \quad \text{and} \quad [-,-] : M \times L \longrightarrow M \quad \text{such that}$$
$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

holds, whenever one of the variable is from M, and the others from L. In particular, L is a representation of itself with the obvious action given by the bracket in L.

Definition 2.3. Let L be a Leibniz algebra and M a representation of L. Let $CL^n(L;M) := \mathrm{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$, $n \geq 0$, and

$$\delta^n : CL^n(L;M) \longrightarrow CL^{n+1}(L;M)$$

be the $\mathbb{K}$-morphism given by

$$\delta^n f(x_1,\ldots,x_{n+1}) := [x_1, f(x_2,\ldots,x_{n+1})] + \sum_{i=2}^{n+1}(-1)^i\big[f(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}), x_i\big]$$
$$+ \sum_{1\leq i<j\leq n+1}(-1)^{j+1} f\big(x_1,\ldots,x_{i-1},[x_i,x_j],x_{i+1},\ldots,\hat{x}_j,\ldots,x_{n+1}\big).$$

Then $(CL^*(L;M),\delta)$ is a cochain complex [19]. The corresponding cohomology denoted by $HL^*(L;M)$, is called the *cohomology of the Leibniz algebra L with coefficients in the representation M*. When $M = L$ with the obvious action as mentioned above, we denote the cohomology by $HL^*(L;L)$.

3. DEFORMATION COMPLEX OF A LEIBNIZ ALGEBRA MORPHISM

In the present section, we recall from [21] the definition of the deformation complex of a Leibniz algebra morphism. Let L and M be Leibniz algebras over a field $\mathbb{K}$. To make our exposition simpler, we use the same notation $[-,-]$ for the brackets of L and M.

Definition 3.1. A $\mathbb{K}$-linear map $f : L \longrightarrow M$ is said to be a Leibniz algebra morphism if it preserves the brackets. In other words, $f([x,y]) = [f(x), f(y)]$ for $x, y \in L$.

Let $f : L \longrightarrow M$ be a Leibniz algebra morphism. We regard M as a representation of L via f, where the actions of L on M, again denoted by $[-,-]$, are $[-,-] : L \times M \longrightarrow M$, $[l,m] := [f(l),m]$ and $[-,-] : M \times L \longrightarrow M$, $[m,l] := [m, f(l)]$ for $l \in L$ and $m \in M$.

Define a cochain complex $(CL^*(f;f), d)$ as follows. Set $CL^0(f;f) := 0$. For $n \geq 1$, the module of n-cochains is

$$CL^n(f;f) := CL^n(L;L) \times CL^n(M;M) \times CL^{n-1}(L;M).$$

The coboundary $d^n : CL^n(f;f) \longrightarrow CL^{n+1}(f;f)$ is defined by the formula $d^n(u,v;w) := (\delta^n u, \delta^n v; fu - vf - \delta^{n-1}w)$ for $(u,v;w) \in CL^n(f;f)$. Here δ^n on the right-hand side are the coboundaries of the complexes defining Leibniz cohomology groups, the map $vf: L^{\otimes n} \longrightarrow M$ is the linear map defined by $vf(x_1, \ldots, x_n) = v(f(x_1), \ldots, f(x_n))$, and fu is the composition of maps. Observe that for $(u, v; w) \in CL^n(f; f)$, $\delta^{n+1}\delta^n u = 0 = \delta^{n+1}\delta^n v$, and

$$f\delta^n u - (\delta^n v)f - \delta^n(fu - vf - \delta^{n-1}w) = f\delta^n u - (\delta^n v)f - \delta^n fu + \delta^n vf = 0.$$

Thus $d^{n+1}d^n(u, v; w) = 0$ for $n \geq 0$. Hence we obtain the following proposition.

Proposition 3.2. *$(CL^*(f; f), d)$ is a cochain complex.*

The cochain complex $(CL^*(f; f), d)$ is called the *deformation complex* of f, and the corresponding cohomology modules are denoted by

$$HL^n(f; f) := H^n\big((CL^*(f; f), d)\big).$$

The proof of the following proposition, which relates $HL^*(f; f)$ to $HL^*(L; L)$, $HL^*(M; M)$, and $HL^*(L; M)$, is similar to that of Proposition 3.3 in [25].

Proposition 3.3. *If $HL^n(L; L) = 0 = HL^n(M; M)$, and $HL^{n-1}(L; M) = 0$, then so is $HL^n(f; f)$.*

In this article, we shall need a more general version of the above deformation complex, which is described as follows. Let M_0 be a finite dimensional $\mathbb{K}$-module. For $n \geq 1$, we have isomorphisms:

$$CL^n(L; M_0 \otimes L) \cong M_0 \otimes CL^n(L; L), \;\; CL^n(M; M_0 \otimes M) \cong M_0 \otimes CL^n(M; M) \quad \text{and}$$
$$CL^n(L; M_0 \otimes M) \cong M_0 \otimes CL^n(L; M).$$

Define a cochain complex $(M_0 \otimes CL^*(f; f), d)$ by setting $M_0 \otimes CL^0(f; f) = 0$, and for $n \geq 1$,

$$M_0 \otimes CL^n(f; f) := CL^n(L; M_0 \otimes L) \times CL^n(M; M_0 \otimes M) \times CL^{n-1}(L; M_0 \otimes M),$$

where the coboundary $d^n : M_0 \otimes CL^n(f; f) \longrightarrow M_0 \otimes CL^{n+1}(f; f)$ is given by

$$d^n(m_1 \otimes u, m_2 \otimes v; m_3 \otimes w) := (m_1 \otimes \delta^n u, m_2 \otimes \delta^n v; m_1 \otimes fu - m_2 \otimes vf - m_3 \otimes \delta^{n-1} w),$$

for $(m_1 \otimes u, m_2 \otimes v; m_3 \otimes w) \in M_0 \otimes CL^n(f; f)$.

We shall denote the corresponding nth cohomology module by $M_0 \otimes HL^n(f; f)$. From now on we shall omit superscripts for coboundaries, it should be clear from the context which coboundary is being used.

4. DEFORMATIONS

Let $\mathbb{K}$ be a fixed field of characteristic zero and A be any commutative local algebra. Let A be any commutative local algebra with 1 over $\mathbb{K}$. Let $\mathfrak{M}$

be the maximal ideal of A and $\varepsilon : A \longrightarrow A/\mathfrak{M} \cong \mathbb{K}, \varepsilon(1) = 1$ be the canonical augmentation. In this section, we study the notion of deformation of Leibniz algebras and that of a Leibniz algebra morphism with base A.

Let L and M be Leibniz algebras over $\mathbb{K}$ and $f : L \longrightarrow M$ a Leibniz algebra morphism.

Definition 4.1. A deformation λ of L with base $(A, \mathfrak{M})$, or simply with base A, is an A-Leibniz algebra structure on the tensor product $A \otimes L$ with the bracket $[,]_\lambda$, such that

$$\varepsilon \otimes id : A \otimes L \to \mathbb{K} \otimes L,$$

is an A-Leibniz algebra morphism (where A-Leibniz algebra structure on $\mathbb{K} \otimes L$ is given via ε).

To define λ, it is enough to specify $[1 \otimes l_1, 1 \otimes l_2]_\lambda$ for $l_1, l_2 \in L$, by A-linearity of $[,]_\lambda$. Moreover, as $\varepsilon \otimes id : A \otimes L \to \mathbb{K} \otimes L$ is an A-Leibniz algebra morphism, we have

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_i m_i \otimes l_i'$$

where $l_i' \in L$ and $m_i \in \mathfrak{M}$ and $\sum_i$ is a finite sum.

Definition 4.2. A deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of f with base $(A, \mathfrak{M})$ (or simply with base A) is an A-Leibniz algebra morphism

$$f_{\lambda\mu} : (A \otimes L, \lambda) \longrightarrow (A \otimes M, \mu),$$

such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes L & \xrightarrow{\quad f_{\lambda\mu} \quad} & A \otimes M \\ {\scriptstyle \varepsilon \otimes id}\downarrow & & \downarrow{\scriptstyle \varepsilon \otimes id} \\ \mathbb{K} \otimes L = L & \xrightarrow[\quad f \quad]{} & M = \mathbb{K} \otimes M \end{array}$$

We shall also use the simpler notation $f_{\lambda\mu}$ to denote a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of f. By A-linearity, $f_{\lambda\mu}$ is determined by its value $f_{\lambda\mu}(1 \otimes l) \in A \otimes M$; for $l \in L$. The commutativity of the above diagram implies

$$f_{\lambda\mu}(1 \otimes l) = 1 \otimes f(l) + \sum_j m_j \otimes x_j; \qquad m_j \in \mathfrak{M} \text{ and } x_j \in M.$$

Definition 4.3. Any deformation λ of L or $f_{\lambda\mu}$ of f with base A is called an *infinitesimal deformation* if $\mathfrak{M}^2 = 0$.

Remark 4.4. If A is finite dimensional and $\{m_i\}_{1\leq i\leq r}$ is a basis of $\mathfrak{M}$, then a deformation λ of L and a deformation $f_{\lambda\mu}$ of $f : L \longrightarrow M$ can be written as

$$[1\otimes l_1, 1\otimes l_2]_\lambda = 1\otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes l_i' \qquad \text{and}$$

$$f_{\lambda\mu}(1\otimes l_1) = 1\otimes f(l_1) + \sum_{j=1}^{r} m_j \otimes x_j \qquad \text{for } l_1, l_2, l_i' \in L \text{ and } x_j \in M.$$

We define the notion of equivalence of deformations as follows.

Definition 4.5. Let λ and λ' be two deformations of L with base A. They are said to be *equivalent*, written as $\lambda \cong \lambda'$, if there exists a Leibniz algebra isomorphism

$$\Phi_{\lambda\lambda'} : (A\otimes L, [,]_\lambda) \to (A\otimes L, [,]_{\lambda'})$$

such that $(\varepsilon\otimes id)\circ\Phi_{\lambda\lambda'} = \varepsilon\otimes id$.

Definition 4.6. Any two deformations $(\lambda, \mu; f_{\lambda\mu})$ and $(\lambda', \mu'; f_{\lambda'\mu'})$ of f with base A are said to be *equivalent*, written as $(\lambda, \mu; f_{\lambda\mu}) \cong (\lambda', \mu'; f_{\lambda'\mu'})$, if there exist equivalences $\Phi_{\lambda\lambda'} : (A\otimes L, [,]_\lambda) \longrightarrow (A\otimes L, [,]'_\lambda)$ and $\Psi_{\mu\mu'} : (A\otimes L, [,]_\mu) \longrightarrow (A\otimes L, [,]'_\mu)$ such that $\Psi_{\mu\mu'}\circ f_{\lambda\mu} = f_{\lambda'\mu'}\circ\Phi_{\lambda\lambda'}$.

The Leibniz algebra structure $[,]_{\lambda_0}$ on $A\otimes L$ given by

$$[1\otimes l_1, 1\otimes l_2]_{\lambda_0} = 1\otimes[l_1, l_2], \qquad \text{for } l_1, l_2 \in L$$

is clearly a deformation of L and is denoted by λ_0.

Definition 4.7. Any deformation of the morphism f, which is equivalent to the deformation $(\lambda_0, \mu_0; f_{\lambda_0\mu_0})$ is said to be a *trivial deformation*. Here

$$f_{\lambda_0\mu_0}(1\otimes l) = 1\otimes f(l) \in A\otimes M.$$

Let A' be any other base with augmentation map ε'. Let $\phi : A \longrightarrow A'$ be an algebra morphism with $\phi(1) = 1$ and $\varepsilon'\circ\phi = \varepsilon$. Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be a given deformation of f with base A. The notion of push out of $\mathfrak{D}$ under ϕ is defined as follows. Observe that $A'\otimes L = (A'\otimes_A A)\otimes L = A'\otimes_A (A\otimes L)$, where A' is viewed as an A-module with the module structure $a'\cdot a = a'\phi(a)$.

Definition 4.8. The push out $\phi_*\mathfrak{D} = (\phi_*\lambda, \phi_*\mu; \phi_* f_{\lambda\mu})$ is a deformation of f with base A' where for any deformation λ of a Leibniz algebra L with base A, $\phi_*\lambda$ is given by

$$\left[a_1'\otimes_A (a_1\otimes l_1), a_2'\otimes_A (a_2\otimes l_2)\right]_{\phi_*\lambda} = a_1' a_2' \otimes_A [a_1\otimes l_1, a_2\otimes l_2]_\lambda$$

and $\phi_* f_{\lambda\mu} : (A'\otimes L, \phi_*\lambda) \longrightarrow (A'\otimes M, \phi_*\mu)$ is given by

$$\phi_* f_{\lambda\mu}(a_1'\otimes_A (a_1\otimes l_1)) = a_1'\otimes_A f_{\lambda\mu}(a_1\otimes l_1)$$

for $a_1', a_2' \in A'$, $a_1, a_2 \in A$ and $l_1, l_2 \in L$.

Next we focus our attention to infinitesimal deformations with finite dimensional base. Let $\mathscr{C}$ be the category of finite dimensional commutative local algebras with 1. Let λ be any deformation of L with base $A \in \mathscr{C}$. Let $\{m_i\}_{1\le i\le r}$ be a basis of $\mathfrak{M}$ and $\{\xi_i\}_{1\le i\le r}$ be the corresponding dual basis. Any element $\xi \in \mathfrak{M}'$ can be viewed as an element $\xi \in A'$ such that $\xi(1) = 0$.

Define a cochain $\alpha_{\lambda,\xi} \in CL^2(L; L)$ by $\alpha_{\lambda,\xi}(l_1, l_2) = (\xi \otimes id)([1 \otimes l_1, 1 \otimes l_2]_\lambda)$ for $l_1, l_2 \in L$. Similarly, in the case of a deformation $f_{\lambda\mu}$ of f, we have a cochain $f_{\lambda\mu,\xi} \in CL^1(L; M)$ defined by $f_{\lambda\mu,\xi}(l) = (\xi \otimes id) \circ f_{\lambda\mu}(1 \otimes l)$ for $l \in L$. In particular, for the basis elements ξ_j, $1 \le j \le r$, if we set $\psi_j^\lambda = \alpha_{\lambda,\xi_j}$ and $f_j = f_{\lambda\mu,\xi_j}$, then by Remark 4.4, the deformations λ and $f_{\lambda\mu}$ can be written as

$$\begin{aligned} &[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{j=1}^r m_j \otimes \psi_j^\lambda(l_1, l_2) \quad \text{and} \\ &f_{\lambda\mu}(1 \otimes l_1) = 1 \otimes f(l_1) + \sum_{j=1}^r m_j \otimes f_j(l_1), \ \text{respectively.} \end{aligned} \tag{1}$$

Thus we have a linear map $\alpha_{\mathfrak{D}} : \mathfrak{M}' \longrightarrow CL^2(f; f)$ given by

$$\alpha_{\mathfrak{D}}(\xi) = (\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_{\lambda\mu,\xi}) \qquad \text{for } \xi \in \mathfrak{M}'.$$

Theorem 4.9. *For any infinitesimal deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of f with base $A \in \mathscr{C}$, $\alpha_{\mathfrak{D}}$ takes values in cocycles.*

Proof. By the definition of the coboundary in $CL^*(f; f)$, we have to show that $\delta\alpha_{\lambda,\xi} = 0 = \delta\alpha_{\mu,\xi}$ and $f\alpha_{\lambda,\xi} - \alpha_{\mu,\xi}f = \delta f_{\lambda\mu,\xi}$ for any $\xi \in \mathfrak{M}'$. By definition,

$$\begin{aligned} \delta\alpha_{\lambda,\xi}(l_1, l_2, l_3) = {} & [l_1, \alpha_{\lambda,\xi}(l_2, l_3)] + [\alpha_{\lambda,\xi}(l_1, l_3), l_2] - [\alpha_{\lambda,\xi}(l_1, l_2), l_3] \\ & - \alpha_{\lambda,\xi}([l_1, l_2], l_3) + \alpha_{\lambda,\xi}([l_1, l_3], l_2) + \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) \end{aligned}$$

for $l_1, l_2, l_3 \in L$. Now observe that

$$\begin{aligned} & (\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda) \\ & = (\xi \otimes id)\left([1 \otimes l_1, 1 \otimes [l_2, l_3]]_\lambda + \left[1 \otimes l_1, \sum_{j=1}^r m_j \otimes \psi_j^\lambda(l_2, l_3)\right]_\lambda\right) \qquad \text{(using (1))} \\ & = \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) + \sum_{j=1}^r (\xi \otimes id)[1 \otimes l_1, m_j \otimes \psi_j^\lambda(l_2, l_3)]_\lambda. \end{aligned}$$

Now

$$\begin{aligned} & (\xi \otimes id)[1 \otimes l_1, m_j \otimes \psi_j^\lambda(l_2, l_3)]_\lambda \\ & \quad = (\xi \otimes id)m_j[1 \otimes l_1, 1 \otimes \psi_j^\lambda(l_2, l_3)]_\lambda \\ & \quad = (\xi \otimes id)m_j\left(1 \otimes [l_1, \psi_j^\lambda(l_2, l_3)] + \sum_{k=1}^r m_k \otimes \psi_k^\lambda(l_1, \psi_j^\lambda(l_2, l_3))\right) \\ & \quad = (\xi \otimes id)(m_j \otimes [l_1, \psi_j^\lambda(l_2, l_3)]) \qquad (\text{since } \mathfrak{M}^2 = 0) \\ & \quad = [l_1, (\xi \otimes id)(m_j \otimes \psi_j^\lambda(l_2, l_3))]. \end{aligned}$$

Therefore,

$$\begin{aligned}
&(\xi \otimes id)\big([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda\big) \\
&\quad = \alpha_{\lambda,\xi}\big(l_1, [l_2, l_3]\big) + \left[l_1, (\xi \otimes id)\sum_{j=1}^{r} m_j \otimes \psi_j^\lambda(l_2, l_3)\right] \\
&\quad = \alpha_{\lambda,\xi}\big(l_1, [l_2, l_3]\big) + \big[l_1, (\xi \otimes id)\big([1 \otimes l_2, 1 \otimes l_3]_\lambda - 1 \otimes [l_2, l_3]\big)\big] \qquad \text{(by using (1))} \\
&\quad = \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) + [l_1, \alpha_{\lambda,\xi}(l_2, l_3)] \qquad (\text{since } \xi(1) = 0).
\end{aligned}$$

Similarly,

$$\xi \otimes id\big([[1 \otimes l_1, 1 \otimes l_2]_\lambda, 1 \otimes l_3]_\lambda\big) = \alpha_{\lambda,\xi}\big([l_1, l_2], l_3\big) + [\alpha_{\lambda,\xi}(l_1, l_2), l_3]$$

and

$$\xi \otimes id\big([[1 \otimes l_1, 1 \otimes l_3]_\lambda, 1 \otimes l_2]_\lambda\big) = \alpha_{\lambda,\xi}([l_1, l_3], l_2) + [\alpha_{\lambda,\xi}(l_1, l_3), l_2].$$

Hence,

$$\begin{aligned}
\delta\alpha_{\lambda,\xi}(l_1, l_2, l_3) &= \xi \otimes id\big([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda - [[1 \otimes l_1, 1 \otimes l_2]_\lambda, 1 \otimes l_3]_\lambda \\
&\qquad + [[1 \otimes l_1, 1 \otimes l_3]_\lambda, 1 \otimes l_2]_\lambda\big) \\
&= 0 \qquad (\text{since } [\,,\,]_\lambda \text{ satisfies the Leibniz relation on } A \otimes L).
\end{aligned}$$

Similarly, $\delta\alpha_{\mu,\xi} = 0$. To complete the proof it is enough to show that

$$f\psi_i^\lambda - \psi_i^\mu f - \delta f_i = f\alpha_{\lambda,\xi_i} - \alpha_{\mu,\xi_i} f - \delta f_{\lambda\mu,\xi_i} = 0, \qquad 1 \le i \le r.$$

We know that $f_{\lambda\mu} : A \otimes L \longrightarrow A \otimes M$ is a Leibniz algebra morphism, that is, $f_{\lambda\mu}[1 \otimes l_1, 1 \otimes l_2]_\lambda - [f_{\lambda\mu}(1 \otimes l_1), f_{\lambda\mu}(1 \otimes l_2)]_\mu = 0$ for $l_1, l_2 \in L$. We have from (1)

$$\begin{aligned}
&f_{\lambda\mu}[1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&\quad = 1 \otimes f\big([l_1, l_2]\big) + \sum_{i=1}^{r} m_i \otimes f_i\big([l_1, l_2]\big) + f_{\lambda\mu}\left(\sum_{i=1}^{r} m_i \otimes \psi_i^\lambda(l_1, l_2)\right) \\
&\quad = 1 \otimes f\big([l_1, l_2]\big) + \sum_{i=1}^{r} m_i \otimes f_i\big([l_1, l_2]\big) + \sum_{i=1}^{r} m_i \otimes f\psi_i^\lambda(l_1, l_2) \qquad (\text{since } \mathfrak{M}^2 = 0).
\end{aligned}$$

Also,

$$\begin{aligned}
&\big[f_{\lambda\mu}(1 \otimes l_1), f_{\lambda\mu}(1 \otimes l_2)\big]_\mu \\
&\quad = \big[1 \otimes f(l_1), 1 \otimes f(l_2)\big]_\mu + \sum_{j=1}^{r} m_j\big[1 \otimes f(l_1), 1 \otimes f_j(l_2)\big]_\mu \\
&\qquad + \sum_{j=1}^{r} m_j\big[1 \otimes f_j(l_1), 1 \otimes f(l_2)\big]_\mu + \sum_{i,j=1}^{r} m_j m_i\big[1 \otimes f_i(l_1), 1 \otimes f_j(l_2)\big]_\mu
\end{aligned}$$

$$= 1 \otimes \big[f(l_1), f(l_2)\big] + \sum_{j=1}^{r} m_j \otimes \psi_j^{\mu}\big(f(l_1), f(l_2)\big) + \sum_{j=1}^{r} m_i \otimes \big[f(l_1), f_j(l_2)\big]$$

$$+ \sum_{j=1}^{r} m_j \otimes \big[f_j(l_1), f(l_2)\big] \qquad \text{(by using the fact that } \mathfrak{M}^2 = 0).$$

Now observe that

$$\begin{aligned}(f\alpha_{\lambda,\xi_i} - \alpha_{\mu,\xi_i} f - \delta f_{\lambda\mu,\xi_i})(l_1, l_2) \\ = (\xi_i \otimes id)(f_{\lambda\mu}[1 \otimes l_1, 1 \otimes l_2]_\lambda - [f_{\lambda\mu}(1 \otimes l_1), f_{\lambda\mu}(1 \otimes l_2)]_\mu) \\ = 0.\end{aligned}$$

□

Theorem 4.10. *Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f'_{\lambda'\mu'})$ be two infinitesimal deformations of f with base $A \in \mathscr{C}$. Then $\alpha_{\mathfrak{D}}(\xi)$ and $\alpha_{\mathfrak{D}'}(\xi)$ represent the same cohomology class for $\xi \in \mathfrak{M}'$, if and only if $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent deformations.*

Proof. Suppose $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f'_{\lambda'\mu'})$ are two equivalent infinitesimal deformations of f with base A.

Let $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_{\lambda\mu,\xi})$ and $(\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_{\lambda'\mu',\xi})$ be the associated 2-cocycles in $CL^2(f; f)$ determined by $\mathfrak{D}$ and $\mathfrak{D}'$, respectively.

Let $\Phi_{\lambda\lambda'} : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda')$ and $\Psi_{\mu\mu'} : (A \otimes M, \mu) \longrightarrow (A \otimes M, \mu')$ be as in Definition 4.6 so that

$$\Psi_{\mu\mu'} \circ f_{\lambda\mu} = f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}. \tag{2}$$

Since λ and μ are equivalent to λ' and μ', respectively, it follows from the claim (a) in the proof of Proposition 4.4 [6] that $\alpha_{\lambda,\xi}$ and $\alpha_{\mu,\xi}$ determine the same cohomology class as $\alpha_{\lambda',\xi}$ and $\alpha_{\mu',\xi}$, respectively. In fact, as shown in the Proposition 4.4 [6], the A-Leibniz algebra isomorphisms $\Phi_{\lambda\lambda'}$ and $\Psi_{\mu\mu'}$ are determined by some linear maps $b_\phi : \mathfrak{M}' \longrightarrow \mathrm{Hom}(L; L)$ and $b_\psi : \mathfrak{M}' \longrightarrow \mathrm{Hom}(M; M)$ so that for $\xi \in$ and $l \in L$, $x \in M$, we have

$$\Phi_{\lambda\lambda'}(1 \otimes l) = 1 \otimes l + \sum_{i=1}^{r} m_i \otimes b_\phi(\xi_i)(l)$$

$$\Psi_{\mu\mu'}(1 \otimes x) = 1 \otimes x + \sum_{i=1}^{r} m_i \otimes b_\psi(\xi_i)(x)$$

where $\alpha_{\lambda,\xi} - \alpha_{\lambda',\xi} = \delta b_\phi(\xi)$ and $\alpha_{\mu,\xi} - \alpha_{\mu',\xi} = \delta b_\psi(\xi)$.

Now if we denote $f_j = f_{\lambda\mu,\xi_j}$ and $f'_j = f_{\lambda'\mu',\xi_j}$ we get,

$$\Psi_{\mu\mu'} \circ f_{\lambda\mu}(1 \otimes l) = \Psi_{\mu\mu'}\left(1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f_j(l)\right)$$

$$= 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes b_\psi(\xi_i)(f(l)) + \sum_{j=1}^{r} m_j \otimes f_j(l)$$
$$+ \sum_{1\le i,j\le r} m_j m_i \otimes b_\psi(\xi_i)(f_j(l))$$
$$= 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes b_\psi(\xi_i)(f(l)) + \sum_{i=1}^{r} m_i \otimes f_i(l)$$

and

$$f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}(1 \otimes l) = f'_{\lambda'\mu'}\left(1 \otimes l + \sum_{i=1}^{r} m_i \otimes b_\phi(\xi_i)(l)\right)$$
$$= 1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f'_j(l) + \sum_{i=1}^{r} m_i \otimes f b_\phi(\xi_i)(l)$$
$$+ \sum_{1\le i,j\le r} m_i m_j \otimes f'_j\big(b_\phi(\xi_i)(l)\big)$$
$$= 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes f'_i(l) + \sum_{i=1}^{r} m_i \otimes f b_\phi(\xi_i)(l) \qquad (\text{since } \mathfrak{M}^2 = 0).$$

It follows from the above expressions

$$(\xi_i \otimes id) \circ \Psi_{\mu\mu'} \circ f_{\lambda\mu}(1 \otimes l) = b_\psi(\xi_i)(f(l)) + f_i(l)$$

and

$$(\xi_i \otimes id) \circ f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}(1 \otimes l) = f b_\phi(\xi_i)(l) + f'_i(l).$$

Hence by (2) we get $f b_\phi(\xi_i) - b_\psi(\xi_i) f = f_i - f'_i$ for $1 \le i \le r$. Thus it follows that $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(b_\phi(\xi), b_\psi(\xi); 0)$ for $\xi \in \mathfrak{M}'$.

Conversely, suppose $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f'_{\lambda'\mu'})$ are two infinitesimal deformations of f with base A such that for $\xi \in \mathfrak{M}'$, $\alpha_{\mathfrak{D}}(\xi)$ and $\alpha_{\mathfrak{D}'}(\xi)$ represent the same cohomology class. Let

$$(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(u, v; w)$$

for some 1-cochain $(u, v; w) \in CL^1(f; f)$.

In particular, we can take $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(u, v; 0)$ as $d(u, v; w) = d(u, v + \delta w; 0)$. For $\xi = \xi_i$, let $(u_i, v_i; 0) \in CL^1(f; f)$ be such that $(\alpha_{\lambda,\xi_i} - \alpha_{\lambda',\xi_i}, \alpha_{\mu,\xi_i} - \alpha_{\mu',\xi_i}; f_i - f'_i) = d(u, v; 0) = (\delta u_i, \delta v_i; f u_i - v_i f)$ for $1 \le i \le r$. Define A-linear maps

$$\Phi_{\lambda\lambda'} : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda') \text{ by } \Phi_{\lambda\lambda'}(1 \otimes l) = 1 \otimes l + \sum_{i=1}^{r} m_i \otimes u_i(l) \qquad \text{and}$$

$$\Psi_{\mu\mu'} : (A \otimes M, \mu) \longrightarrow (A \otimes M, \mu') \text{ by } \Psi_{\mu\mu'}(1 \otimes x) = 1 \otimes l + \sum_{i=1}^{r} m_i \otimes v_i(x)$$

for $l \in L$ and $x \in M$.

Then $\Phi_{\lambda\lambda'}$ and $\Psi_{\mu\mu'}$ define equivalences $\lambda \cong \lambda'$ and $\mu \cong \mu'$, respectively. Using the above definitions of $\Phi_{\lambda\lambda'}$ and $\Psi_{\mu\mu'}$ and the fact that $\mathfrak{M}^2 = 0$, we get

$$\Psi_{\mu\mu'} \circ f_{\lambda\mu}(1 \otimes l) = 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes v_i f(l) + \sum_{i=1}^{r} m_i \otimes f_i(l) \quad \text{and}$$

$$f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}(1 \otimes l) = 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes fu_i(l) + \sum_{i=1}^{r} m_i \otimes f'_i(l).$$

Thus

$$(\Psi_{\mu\mu'} \circ f_{\lambda\mu})(1 \otimes l) - (f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'})(1 \otimes l) = \sum_{i=1}^{r} m_i \otimes \big(fu_i(l) - v_i f(l) - f_i(l) + f'_i(l)\big) = 0.$$

This gives $\Psi_{\mu\mu'} \circ f_{\lambda\mu} = f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}$. Consequently, $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent infinitesimal deformations of f with base A. □

Suppose $A \in \mathscr{C}$ and $\mathfrak{M}$ is the unique maximal ideal of A. The algebra $A/\mathfrak{M}^2$ is obviously local with maximal ideal $\mathfrak{M}/\mathfrak{M}^2$ and having the additional property $(\mathfrak{M}/\mathfrak{M}^2)^2 = 0$. Let $p_2 : A \longrightarrow A/\mathfrak{M}^2$ be the obvious quotient map. If $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ is any deformation of f with base A, then we get the induced deformation $p_{2*}\mathfrak{D} = (p_{2*}\lambda, p_{2*}\mu; f_{p_{2*}\lambda\ p_{2*}\mu})$ with base $A/\mathfrak{M}^2$, which is obviously infinitesimal. As a consequence, $\alpha_{p_{2*}\mathfrak{D}}$ takes vales in cocycles, and hence we have a map

$$a_{p_{2*}\mathfrak{D}} : (\mathfrak{M}/\mathfrak{M}^2)' \to HL^2(f; f) \text{ defined by } a_{p_{2*}\mathfrak{D}}(\xi) = [\alpha_{p_{2*}\mathfrak{D}}],$$

where $[\alpha_{p_{2*}\mathfrak{D}}]$ denotes the cohomology class represented by $\alpha_{p_{2*}\mathfrak{D}}$.

Definition 4.11. The linear maps $\alpha_{p_{2*}\mathfrak{D}}$ and $a_{p_{2*}\mathfrak{D}}$ are, respectively, called the *infinitesimal* and the *differential* of $\mathfrak{D}$. The deformation $p_{2*}\mathfrak{D}$ may be called the *infinitesimal part* of $\mathfrak{D}$.

Corollary 4.12. *Two infinitesimal deformations $\mathfrak{D}$ and $\mathfrak{D}'$ with base $A \in \mathscr{C}$ are equivalent if and only if they have the same differential.*

Corollary 4.13. *Suppose $\mathfrak{D}$ and $\mathfrak{D}'$ are two equivalent deformations of f with base A, then they have the same differential.*

5. OBSTRUCTIONS

Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be a deformation of $f : L \longrightarrow M$ with base $A \in \mathscr{C}$. Let $\varepsilon : A \longrightarrow \mathbb{K}$ be the augmentation. It is well known [15] that the isomorphism classes of 1-dimensional extensions

$$0 \longrightarrow \mathbb{K} \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} A \longrightarrow 0$$

of A are in one to one correspondence with elements of the second Harrison cohomology $H^2_{\text{Harr}}(A; \mathbb{K})$. Let $[\psi] \in H^2_{\text{Harr}}(A; \mathbb{K})$. Suppose

$$0 \longrightarrow \mathbb{K} \overset{i}{\longrightarrow} B \overset{p}{\longrightarrow} A \longrightarrow 0$$

is a representative of the class of 1-dimensional extensions of A, corresponding to the cohomology class $[\psi]$.

Let

$$I = (i \otimes id) : L \cong \mathbb{K} \otimes L \longrightarrow B \otimes L, \qquad I_1 = (i \otimes id) : M \cong \mathbb{K} \otimes M \longrightarrow B \otimes M$$
$$P = (p \otimes id) : B \otimes L \longrightarrow A \otimes L, \qquad P_1 = (p \otimes id) : B \otimes M \longrightarrow A \otimes M$$
$$E = (\hat{\varepsilon} \otimes id) : B \otimes L \longrightarrow \mathbb{K} \otimes L \cong L, \qquad E_1 = (\hat{\varepsilon} \otimes id) : B \otimes M \longrightarrow \mathbb{K} \otimes M \cong M,$$

where $\hat{\varepsilon} = \varepsilon \circ p$ is the augmentation for B corresponding to the augmentation ε of A. Fix a section $q : A \longrightarrow B$ of p in the above extension, then

$$b \longmapsto \big(p(b), i^{-1}(b - q \circ p(b))\big)$$

is a $\mathbb{K}$-module isomorphism $B \longrightarrow (A \oplus \mathbb{K})$. Let us denote by $(a, k)_q \in B$, the inverse of $(a, k) \in (A \oplus \mathbb{K})$ under the above isomorphism. The algebra structure of B is determined by ψ and is given by

$$(a_1, k_1)_q \circ (a_2, k_2)_q = \big(a_1 a_2, a_1 k_2 + a_2 k_1 + \psi(a_1, a_2)\big)_q.$$

Suppose $\dim(A) = r + 1$ and $\{m_i\}_{1 \le i \le r}$ is a basis of the maximal ideal $\mathfrak{M}_A$ of A. Then $\{n_i\}_{1 \le i \le r+1}$ is a basis of the maximal ideal $\mathfrak{M}_B = p^{-1}(\mathfrak{M}_A)$ of B, where $n_j = (m_j, 0)_q$ for $1 \le j \le r$ and $n_{r+1} = (0, 1)_q$. Take the dual basis $\{\xi_i\}_{1 \le i \le r}$ of $(\mathfrak{M}_A)'$. As in Section 4, let $\psi_i^\lambda = \alpha_{\lambda, \xi_i} \in CL^2(L; L)$, $\psi_i^\mu = \alpha_{\mu, \xi_i} \in CL^2(M; M)$ and $f_i = f_{\lambda\mu, \xi_i} \in CL^1(L; M)$ for $1 \le i \le r$. Then by (1), the brackets $[,]_\lambda$ and $[,]_\mu$ can be written as

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^\lambda(l_1, l_2) \qquad \text{for } l_1, l_2 \in L \qquad \text{and}$$

$$[1 \otimes x_1, 1 \otimes x_2]_\mu = 1 \otimes [x_1, x_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^\mu(x_1, x_2) \qquad \text{for } x_1, x_2 \in M.$$

Also,

$$f_{\lambda\mu}(1 \otimes l) = 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes f_i(l).$$

We consider the problem of lifting the deformation $\mathfrak{D}$ from the base A to the base B.

Using an arbitrary element $(\psi_L, \psi_M) \in CL^2(L; L) \times CL^2(M; M)$ we may define B-bilinear operations,

$$\{,\}_L : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes L \qquad \text{and} \qquad \{,\}_M : (B \otimes M)^{\otimes 2} \longrightarrow B \otimes M$$

as follows:

$$\{b_1 \otimes l_1, b_2 \otimes l_2\}_L = b_1 b_2 \otimes [l_1, l_2] + \sum_{j=1}^{r} b_1 b_2 n_j \otimes \alpha_{\lambda, \xi_j}(l_1, l_2) + b_1 b_2 n_{r+1} \otimes \psi_L(l_1, l_2) \quad \text{and}$$

$$\{b_1 \otimes x_1, b_2 \otimes x_2\}_M = b_1 b_2 \otimes [x_1, x_2] + \sum_{j=1}^{r} b_1 b_2 n_j \otimes \alpha_{\mu, \xi_j}(x_1, x_2) + b_1 b_2 n_{r+1} \otimes \psi_M(x_1, x_2).$$

We may also define a B-linear map $\tilde{f} : B \otimes L \longrightarrow B \otimes M$ by

$$\tilde{f}(b \otimes l) = b \otimes f(l) + \sum_{j=1}^{r} b n_j \otimes f_j(l).$$

It is straightforward to verify the following identities:

(i) $P\{l_1, l_2\}_L = [P(l_1), P(l_2)]_\lambda$ and $P_1\{x_1, x_2\}_M = [P_1(x_1), P_1(x_2)]_\mu$ for $l_1, l_2 \in B \otimes L$, and $x_1, x_2 \in B \otimes M$;

(ii) $\{I(l), l_1\}_L = I[l, E(l_1)]$ for $l \in L, l_1 \in B \otimes L$ and

$$\{I_1(x), x_1\}_M = I_1[x, E_1(x_1)] \qquad \text{for } x \in M,\ x_1 \in B \otimes M; \tag{3}$$

(iii) $(\hat{\varepsilon} \otimes id) \circ \tilde{f} = f \circ (\hat{\varepsilon} \otimes id)$;

(iv) $f_{\lambda\mu} \circ P = P_1 \circ \tilde{f}$.

Thus the Leibniz algebra structures λ on $A \otimes L$ and μ on $A \otimes M$ can be lifted to B-bilinear operations $\{,\}_L$ on $B \otimes L$ and $\{,\}_M$ on $B \otimes M$, and the A-Leibniz algebra morphism $f_{\lambda\mu}$ can be lifted to a B-linear map $\tilde{f} : B \otimes L \longrightarrow B \otimes M$ so that the triple $(\{,\}_L, \{,\}_M; \tilde{f})$ satisfies the conditions in (3). Consider the maps

$$\phi_L : (B \otimes L)^{\otimes 3} \longrightarrow B \otimes L, \qquad \phi_M : (B \otimes M)^{\otimes 3} \longrightarrow B \otimes M, \qquad \text{and}$$

$\phi_{f_{\lambda\mu}} : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes M$ given, respectively, by

$$\phi_L(l_1, l_2, l_3) = \{l_1, \{l_2, l_3\}_L\}_L - \{\{l_1, l_2\}_L, l_3\}_L + \{\{l_1, l_3\}_L, l_2\}_L$$

$$\phi_M(x_1, x_2, x_3) = \{x_1, \{x_2, x_3\}_M\}_M - \{\{x_1, x_2\}_M, x_3\}_M + \{\{x_1, x_3\}_M, x_2\}_M, \qquad \text{and}$$

$$\phi_{f_{\lambda\mu}}(l_1, l_2) = \tilde{f}\{l_1, l_2\}_L - \{\tilde{f}(l_1), \tilde{f}(l_2)\}_M,$$

where

$$l_1, l_2, l_3 \in B \otimes L \text{ and } x_1, x_2, x_3 \in B \otimes M. \tag{4}$$

It is clear that $\phi_L = 0$ if and only if $\{,\}_L$ is a Leibniz bracket on $B \otimes L$, $\phi_M = 0$ if and only if $\{,\}_M$ is a Leibniz bracket on $B \otimes M$, and $\phi_{f_{\lambda\mu}} = 0$ if and only if $\tilde{f}$ preserves the B-bilinear operations $\{,\}_L$ and $\{,\}_M$.

By the property (i) in (3) and the definitions of ϕ_L and ϕ_M it follows that

$$P \circ \phi_L(l_1, l_2, l_3) = 0 \quad \text{and} \quad P_1 \circ \phi_M(x_1, x_2, x_3) = 0$$
$$\text{for } l_1, l_2, l_3 \in B \otimes L \text{ and } x_1, x_2, x_3 \in B \otimes M.$$

Moreover, by the properties (i) and (iv) in (3) and the definition of $\phi_{f_{\lambda\mu}}$, we get

$$\begin{aligned} P_1 \circ \phi_{f_{\lambda\mu}}(l_1, l_2) &= P_1 \circ \tilde{f}\{l_1, l_2\}_L - P_1 \circ \{\tilde{f}(l_1), \tilde{f}(l_2)\}_M \\ &= f_{\lambda\mu} \circ P\{l_1, l_2\}_L - \left[P_1 \circ \tilde{f}(l_1), P_1 \circ \tilde{f}(l_2)\right]_\mu \\ &= f_{\lambda\mu}[P(l_1), P(l_2)]_\lambda - \left[f_{\lambda\mu} \circ P(l_1), f_{\lambda\mu} \circ P(l_2)\right]_\mu \\ &= 0 \qquad (\text{as } f_{\lambda\mu} \text{ is a Leibniz algebra morphism}), \end{aligned}$$

for $l_1, l_2 \in B \otimes L$.

Therefore, ϕ_L and ϕ_M take values in $\ker(P)$ and $\ker(P_1)$, respectively, and $\phi_{f_{\lambda\mu}}$ takes values in $\ker(P_1)$. Note that

$$\ker(P) = \operatorname{im}(i) \otimes L = \mathbb{K}n_{r+1} \otimes L, \qquad \ker(P_1) = \operatorname{im}(i) \otimes M = \mathbb{K}n_{r+1} \otimes M,$$

and $n_j n_{r+1} = 0$ for $1 \le j \le r+1$. From this, one can show that $\phi_L(l_1, l_2, l_3) = 0$ whenever one of the arguments is in $\ker(E)$ and $\phi_M(x_1, x_2, x_3) = 0$ whenever one of the arguments is in $\ker(E_1)$ (see Section 5, [6]). Moreover, $\phi_{f_{\lambda\mu}} = 0$ whenever one of the arguments is in $\ker(E)$. For suppose $l_1 = b \otimes l \in \ker(E) \subseteq B \otimes L$. Since $\ker(E) = \ker(\hat{\varepsilon}) \otimes L = p^{-1}(\ker(\varepsilon)) \otimes L = \mathfrak{M}_B \otimes L$, we can write $l_1 = \sum_{j=1}^{r+1} n_j \otimes l'_j$ with $l'_j \in L$; $1 \le j \le r+1$. Then for $l_2 \in B \otimes L$, we get

$$\phi_{f_{\lambda\mu}}(l_1, l_2) = \phi_{f_{\lambda\mu}}\left(\sum_{j=1}^{r+1} n_j \otimes l'_j, l_2\right) = \sum_{j=1}^{r+1} n_j \phi_{f_{\lambda\mu}}(l'_j, l_2) = 0.$$

This is because $\phi_{f_{\lambda\mu}}(l'_j, l_2) \in \ker(P_1) = \operatorname{im}(I_1) = \operatorname{im}(i) \otimes M = i(\mathbb{K}) \otimes M$ and for any element $k \in \mathbb{K}$ and $x \in M$,

$$n_j i(k) \otimes x = i(p(n_j)k) \otimes x = i(m_j k) \otimes x = i(\varepsilon(m_j)k) \otimes x = 0 \qquad \text{for } 1 \le j \le r$$

and $n_{r+1} i(k) \otimes x = k n_{r+1}^2 \otimes x = 0$ ($m_j \in \mathfrak{M} \subset A$ and $m_j k = \varepsilon(m_j)k$). A similar argument shows that $\phi_{f_{\lambda\mu}} = 0$ whenever $l_2 \in \ker(E)$.

Thus we have induced linear maps

$$\begin{aligned} &\tilde{\phi}_L : \left(\frac{B \otimes L}{\ker(E)}\right)^{\otimes 3} \longrightarrow \ker(P), \qquad \tilde{\phi}_M : \left(\frac{B \otimes M}{\ker(E_1)}\right)^{\otimes 3} \longrightarrow \ker(P_1), \\ &\text{and} \quad \tilde{\phi}_{f_{\lambda\mu}} : \left(\frac{B \otimes L}{\ker(E)}\right)^{\otimes 2} \longrightarrow \ker(P_1) \end{aligned} \tag{5}$$

determined by the values of ϕ_L, ϕ_M and $\phi_{f_{\lambda\mu}}$ on the coset representatives, respectively. Observe that we have isomorphisms

$$\frac{B \otimes L}{\ker(E)} \cong L \quad \text{and} \quad \frac{B \otimes M}{\ker(E_1)} \cong M$$

induced by the linear maps E and E_1, respectively. Moreover, $\ker(P) = \mathrm{im}(I) = \mathbb{K}i(1) \otimes L \cong L$ and $\ker(P_1) = \mathrm{im}(I_1) = \mathbb{K}i(1) \otimes M \cong M$.

Explicitly, the isomorphisms $\beta : \ker(P) \longrightarrow L$ and $\beta_1 : \ker(P_1) \longrightarrow M$ are given by $\beta(kn_{r+1} \otimes l) = kl$ and $\beta_1(kn_{r+1} \otimes x) = kx$ for $l \in L$ and $x \in M$.

We use these isomorphisms and the linear maps $\tilde{\phi}_L, \tilde{\phi}_M, \tilde{\phi}_{f_{\lambda\mu}}$ to get cochains $\bar{\phi}_L \in CL^3(L; L)$, $\bar{\phi}_M \in CL^3(M; M)$, and $\bar{\phi}_{f_{\lambda\mu}} \in CL^2(L; M)$ where for $l_1, l_2, l_3 \in L$ and $x_1, x_2, x_3 \in M$,

$$\begin{aligned} n_{r+1} \otimes \bar{\phi}_L(l_1, l_2, l_3) &= \phi_L(1 \otimes l_1, 1 \otimes l_2, 1 \otimes l_3) \\ n_{r+1} \otimes \bar{\phi}_M(x_1, x_2, x_3) &= \phi_M(1 \otimes x_1, 1 \otimes x_2, 1 \otimes x_3), \qquad \text{and} \\ n_{r+1} \otimes \bar{\phi}_{f_{\lambda\mu}}(l_1, l_2) &= \phi_{f_{\lambda\mu}}(1 \otimes l_1, 1 \otimes l_2). \end{aligned} \tag{6}$$

The resulting 3-cochain $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) \in CL^3(f; f)$ is called the *obstruction cochain* for extending the deformation $\mathfrak{D}$ of f with base A to the base B. The next result shows that the obstruction cochains are cocycles.

Proposition 5.1. *The obstruction cochain $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ is a 3-cocycle in $CL^3(f; f)$.*

Proof. By the definition of the coboundary d, we have

$$d(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) = (\delta\bar{\phi}_L, \delta\bar{\phi}_M; f\bar{\phi}_L - \bar{\phi}_M f - \delta\bar{\phi}_{f_{\lambda\mu}}).$$

Thus it is enough to show that $\delta\bar{\phi}_L = 0 = \delta\bar{\phi}_M$ and $\delta\bar{\phi}_{f_{\lambda\mu}} = f\bar{\phi}_L - \bar{\phi}_M f$. The result will follow if we show that

$$\beta^{-1} \circ \delta\bar{\phi}_L = 0 = {\beta_1}^{-1} \circ \delta\bar{\phi}_M \quad \text{and} \quad {\beta_1}^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}} = {\beta_1}^{-1} \circ (f\bar{\phi}_L - \bar{\phi}_M f).$$

We give a proof of the last equality. For proofs of the other two equalities we refer Section 5 of [6].

First observe that for $l_i, l_j \in L$,

$$\{1 \otimes l_i, 1 \otimes l_j\}_L = 1 \otimes [l_i, l_j] + Y_{i,j} \quad \text{where } Y_{i,j} \in \ker(E). \tag{7}$$

For,

$$\begin{aligned} E(\{1 \otimes l_i, 1 \otimes l_j\}_L) &= (\varepsilon \otimes id) \circ P\{1 \otimes l_i, 1 \otimes l_j\}_L \\ &= (\varepsilon \otimes id)\big[P(1 \otimes l_1), P(1 \otimes l_2)\big]_\lambda \qquad \text{(by (i) of (3))} \\ &= 1 \otimes [l_1, l_2] \\ &= E(1 \otimes [l_1, l_2]). \end{aligned}$$

Also using the equalities (iii) and (iv) of (3), one gets

$$\tilde{f}(1 \otimes l_i) = 1 \otimes f(l_i) + X_i \qquad \text{for } X_i \in \ker(E_1) \qquad \text{and} \tag{8}$$

$$\tilde{f}\{1 \otimes l_i, 1 \otimes l_j\}_L = \{\tilde{f}(1 \otimes l_i), \tilde{f}(1 \otimes l_j)\}_M + Z_{i,j} \qquad \text{for } Z_{i,j} \in \ker(P_1). \tag{9}$$

Now

$$\begin{aligned} \beta_1^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3) = {} & \beta_1^{-1}\Big[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\Big] + \beta_1^{-1}\Big[\bar{\phi}_{f_{\lambda\mu}}(l_1, l_3), f(l_2)\Big] \\ & - \beta_1^{-1}\Big[\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2), f(l_3)\Big] - \beta_1^{-1}\Big(\bar{\phi}_{f_{\lambda\mu}}([l_1, l_2], l_3)\Big) \\ & + \beta_1^{-1}\Big(\bar{\phi}_{f_{\lambda\mu}}([l_1, l_3], l_2)\Big) + \beta_1^{-1}\Big(\bar{\phi}_{f_{\lambda\mu}}(l_1, [l_2, l_3])\Big). \end{aligned} \tag{10}$$

Let us compute the first term on the right-hand side of (10).

$$\begin{aligned} & \beta_1^{-1}\Big[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\Big] \\ & = n_{r+1} \otimes \Big[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\Big] \\ & = I_1\Big[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\Big] \qquad (i(1) = n_{r+1}) \\ & = \Big\{I_1 f(l_1), 1 \otimes \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\Big\}_M \qquad \text{(by (ii) of (3))} \\ & = \big\{1 \otimes f(l_1), \phi_{f_{\lambda\mu}}(1 \otimes l_2, 1 \otimes l_3)\big\}_M \qquad \text{(using B-bilinearity of } \{,\}_M \text{ and by (6))} \\ & = \big\{1 \otimes f(l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L - \{\tilde{f}(1 \otimes l_2), \tilde{f}(1 \otimes l_3)\}_M\big\}_M \qquad \text{(by (4))} \\ & = \big\{\tilde{f}(1 \otimes l_1) - X_1, \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M - \big\{1 \otimes f(l_1), \{\tilde{f}(1 \otimes l_2), \tilde{f}(1 \otimes l_3)\}_M\big\}_M \\ & \quad \text{(by (8))} \\ & = \big\{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M - \big\{X_1, \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M \\ & \quad - \big\{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\big\}_M \\ & = \big\{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M - \big\{X_1, \{\tilde{f}(1 \otimes l_2), \tilde{f}(1 \otimes l_3)\}_M + Z_{2,3}\}_M \\ & \quad - \big\{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\big\}_M \qquad \text{(using (9))} \\ & = \big\{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M - \big\{X_1, \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\big\}_M \\ & \quad - \{X_1, Z_{2,3}\}_M - \big\{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\big\}_M \\ & = \big\{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\big\}_M - \big\{X_1, \{1 \otimes f(l_2), 1 \otimes f(l_3)\}_M\big\}_M \\ & \quad - \big\{X_1, \{1 \otimes f(l_2), X_3\}_M\big\}_M - \big\{X_1, \{X_2, 1 \otimes f(l_3)\}_M\big\}_M - \big\{X_1, \{X_2, X_3\}_M\big\}_M \\ & \quad - \{X_1, Z_{2,3}\}_M - \big\{1 \otimes f(l_1), \{1 \otimes f(l_2), 1 \otimes f(l_3)\}_M\big\}_M \\ & \quad - \big\{1 \otimes f(l_1), \{X_2, 1 \otimes f(l_3)\}_M\big\}_M - \big\{1 \otimes f(l_1), \{1 \otimes f(l_2), X_3\}_M\big\}_M \\ & \quad - \big\{1 \otimes f(l_1), \{X_2, X_3\}_M\big\}_M. \end{aligned}$$

Similarly, computing the other terms on the right hand side of (10) using (7)–(9), and substituting, we get

$$\begin{aligned}
&\beta_1^{-1}\delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3)\\
&\quad = -\phi_M\big(X_1, 1\otimes f(l_2), 1\otimes f(l_3)\big) - \phi_M\big(X_1, 1\otimes f(l_2), X_3\big) - \phi_M\big(X_1, X_2, 1\otimes f(l_3)\big)\\
&\qquad - \phi_M(X_1, X_2, X_3) - \phi_M\big(1\otimes f(l_1), X_2, 1\otimes f(l_3)\big) - \phi_M\big(1\otimes f(l_1), 1\otimes f(l_2), X_3\big)\\
&\qquad - \phi_M\big(1\otimes f(l_1), X_2, X_3\big) + \phi_{f_{\lambda\mu}}(Y_{1,2}, 1\otimes l_3) - \phi_{f_{\lambda\mu}}(Y_{1,3}, 1\otimes l_2)\\
&\qquad - \phi_{f_{\lambda\mu}}(1\otimes l_1, Y_{2,3}) - \{X_1, Z_{2,3}\}_M - \{Z_{1,3}, X_2\}_M + \{Z_{1,2}, X_3\}_M\\
&\qquad + \tilde{f}\phi_L(1\otimes l_1, 1\otimes l_2, 1\otimes l_3) - \phi_M\big(1\otimes f(l_1), 1\otimes f(l_2), 1\otimes f(l_3)\big).
\end{aligned}$$

Now recall that $\phi_M(l_1, l_2, l_3) = 0$, whenever one of the argument is in $\ker(E_1)$ and $\phi_{f_{\lambda\mu}}(l_1, l_2) = 0$, whenever one of the argument is in $\ker(E)$. Moreover, note that $\{X_1, Z_{2,3}\}_M = 0$ as $\{,\}_M$ is B-bilinear and $n_j \cdot n_{r+1} = 0$ for $1 \le j \le r$. Similarly, $\{Z_{1,3}, X_2\}_M = 0 = \{Z_{1,2}, X_3\}_M$. Therefore, $\beta_1^{-1}\delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3) = \tilde{f}\phi_L(1\otimes l_1, 1\otimes l_2, 1\otimes l_3) - \phi_M\big(1\otimes f(l_1), 1\otimes f(l_2), 1\otimes f(l_3)\big) = \beta_1^{-1}(f\bar{\phi}_L - \bar{\phi}_M f)(l_1, l_2, l_3)$. □

Remark 5.2. It is straightforward to show that the cohomology class of the obstruction cocycle $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ depends only on the cohomology class of $[\psi]$. In other words, suppose $(\{\,,\,\}'_L, \{\,,\,\}'_M; \tilde{f}')$ is another triple satisfying conditions (3) determined by some other choice of $[\psi]$, and let $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ be the corresponding cocycle. Then $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ represents the same cohomology class as $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$.

Thus we have a map

$$\Theta_{\mathfrak{D}} : H^2_{\mathrm{Harr}}(A; \mathbb{K}) \longrightarrow HL^3(f; f) \text{ given by } \Theta_{\mathfrak{D}}([\psi]) = [(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})],$$

where $[(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})]$ denotes the cohomology class of $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$. This map is called the *obstruction map*.

Proposition 5.3. *The deformation $\mathfrak{D}$ of f with base A can be extended to a deformation of f with base B if and only if $\Theta_{\mathfrak{D}}([\psi]) = 0$, for any 1-dimensional extension*

$$0 \longrightarrow \mathbb{K} \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} A \longrightarrow 0$$

representing $[\psi]$.

Proof. Suppose $\Theta_{\mathfrak{D}}([\psi]) = 0$. Let $(\{\,,\,\}_L, \{\,,\,\}_M; \tilde{f})$ be a triple satisfying conditions in (3). Let ϕ_L, ϕ_M and $\phi_{f_{\lambda\mu}}$ be maps as defined in (4). Let $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ be the associated cocycle.

Since $\Theta_{\mathfrak{D}}([\psi]) = [(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})] = 0$, there is a 2-cochain $(u, v; w) \in CL^2(f; f)$ such that $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) = d(u, v; w)$. Therefore, $\delta u = \bar{\phi}_L, \delta v = \bar{\phi}_M$, and $fu - vf - \delta w = \bar{\phi}_{f_{\lambda\mu}}$. Take $\rho'_L = -u$, $\rho'_M = -v$, and $\rho'_f = -w$. Define the linear maps

$$\{,\}'_L : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes L \text{ by } \{l_1, l_2\}'_L = \{l_1, l_2\} - \mathrm{Iu}\big(E(l_1), E(l_2)\big),$$

$$\{,\}'_M : (B \otimes M)^{\otimes 2} \longrightarrow B \otimes M \text{ by } \{x_1, x_2\}'_M = \{x_1, x_2\} - I_1 v\big(E_1(x_1), E_1(x_2)\big),$$

and

$$\tilde{f}' : B \otimes L \longrightarrow B \otimes M \text{ by } \tilde{f}'(l_1) = \tilde{f}(l_1) - I_1 w(E(l_1))$$

for $l_1, l_2 \in B \otimes L$ and $x_1, x_2 \in B \otimes M$.

We claim that $(\{,\}'_L, \{,\}'_M; \tilde{f}')$ is a deformation of f with base B lifting the deformation $\mathfrak{D}$. Let ϕ'_L, ϕ'_M, and $\phi'_{f_{\lambda\mu}}$ be the associated maps as defined in (4). Let $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ be corresponding 3-cocycle. Then it is easy to see that $\bar{\phi}_L - \bar{\phi}'_L = \delta u$, $\bar{\phi}_M - \bar{\phi}'_M = \delta v$, and $\bar{\phi}_{f_{\lambda\mu}} - \bar{\phi}'_{f_{\lambda\mu}} = fu - vf - \delta w$. Thus $\bar{\phi}'_L = 0$, $\bar{\phi}'_M = 0$, and $\bar{\phi}'_{f_{\lambda\mu}} = 0$. It follows from (6) that $\phi'_L = 0 = \phi'_M$ and $\phi'_{f_{\lambda\mu}} = 0$. Consequently, $(\{,\}'_L, \{,\}'_M; \tilde{f}')$ is a deformation of f with base B extending $\mathfrak{D}$.

The converse is clear. □

So far we were concerned with the lifting problem for 1-dimensional extension of the base $A \in \mathscr{C}$ of a deformation. An analogous consideration holds for any finite dimensional extension of the algebra A by an A-module.

Recall that an extension B of an algebra A by an A-module M is a $\mathbb{K}$-algebra B together with an exact sequence of $\mathbb{K}$-modules

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0,$$

where p is a $\mathbb{K}$-algebra homomorphism, and the B-module structure on $i(M)$ is given by the A-module structure of M by $i(m) \cdot b = i(mp(b))$. In particular, if we identify M with its image $i(M)$, then M is an ideal in B satisfying $M^2 = 0$.

Let M_0 be a finite dimensional A-module satisfying $\mathfrak{M}M_0 = 0$. It is well known [15] that $H^2_{\mathrm{Harr}}(A; M_0)$ is in one to one correspondence with the isomorphism classes of extensions

$$0 \longrightarrow M_0 \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0. \tag{11}$$

Let $[\psi] \in H^2_{\mathrm{Harr}}(A; M_0)$ correspond to the class of extensions represented by the extension in (11). If we proceed with the above extension as in the case of 1-dimensional extension, we obtain a triple $(\{,\}_L, \{,\}_M; \tilde{f})$ and the maps ϕ_L, ϕ_M and $\phi_{f_{\lambda\mu}}$ as determined in (4). We define $\tilde{\phi}_L, \tilde{\phi}_M$, and $\tilde{\phi}_{f_{\lambda\mu}}$ using ϕ_L, ϕ_M, and $\phi_{f_{\lambda\mu}}$, respectively, as in (5). As before we have isomorphisms $B \otimes L/\ker(E) \cong L$ and $B \otimes M/\ker(E_1) \cong M$. Moreover, in this general case, we have isomorphisms

$\ker(P) \cong M_0 \otimes L$ and $\ker(P_1) \cong M_0 \otimes M$. We use these isomorphisms to obtain cochains

$$\bar{\phi}_L \in CL^3(L; M_0 \otimes L) \cong M_0 \otimes CL^3(L; L),$$
$$\bar{\phi}_M \in CL^3(M; M_0 \otimes M) \cong M_0 \otimes CL^3(M; M) \quad \text{and}$$
$$\bar{\phi}_{f_{\lambda\mu}} \in CL^2(L; M_0 \otimes M) \cong M_0 \otimes CL^2(L; M).$$

An argument analogous to the proof of Proposition 5.1 shows that $(\bar{\phi}_L, \bar{\phi}_M, \bar{\phi}_{f_{\lambda\mu}})$ is a cocycle in $M_0 \otimes CL^3(f; f)$.

As a consequence, we have the *obstruction map*

$$\Theta_{\mathfrak{D}} : H^2_{\text{Harr}}(A; M_0) \longrightarrow M_0 \otimes HL^3(f; f).$$

As in Proposition 5.3, we have the following proposition.

Proposition 5.4. *A deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of f with base $A \in \mathscr{C}$ can be lifted to a deformation of f with base B if and only if $\Theta_{\mathfrak{D}}([\psi]) = 0$ for any extension* (11) *of A by the A-module M_0 representing $[\psi]$.*

6. FORMAL DEFORMATIONS

Let A be any local algebra with maximal ideal $\mathfrak{M}$. Throughout this section, we assume that $dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) < \infty$ for each $k \geq 1$. The local algebra A is said to be a *complete local algebra* if $A = \lim_{k\to\infty}(A/\mathfrak{M}^k)$. Note that $A/\mathfrak{M}^k \in \mathscr{C}$ for $k \geq 1$.

A deformation λ of L with base A is said to be a *formal deformation* if λ is obtained as the projective limit of deformations λ_k of L with base $A/\mathfrak{M}^k$. In other words, $p_{k*}\lambda = \lambda_k$ for each k, where $p_k : A \longrightarrow A/\mathfrak{M}^k$ is the quotient map.

Definition 6.1. A formal deformation of a Leibniz algebra morphism $f : L \longrightarrow M$ with base A is a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ which is obtained as a projective limit of deformations $\mathfrak{D}_k = (\lambda_k, \mu_k; f_{\lambda_k\mu_k})$ with base $A/\mathfrak{M}^k$.

Definition 6.2. For a formal deformation $\mathfrak{D}$ of f with base A, $p_{2*}\mathfrak{D}$ is called the *infinitesimal part* of $\mathfrak{D}$ and $a_{p_{2*}\mathfrak{D}}$ is the differential of $\mathfrak{D}$.

Example 6.3. A formal 1-parameter deformation of a Leibniz algebra morphism as developed in [21] is a formal deformation in above sense where $A = \mathbb{K}[[t]]$.

Example 6.4. Let A be a complete local algebra with maximal ideal $\mathfrak{M}$. Then $A/\mathfrak{M}^2$ is a finite dimensional local algebra with the maximal ideal $\mathfrak{M}/\mathfrak{M}^2$. Note that any deformation with base $A/\mathfrak{M}^2$ is infinitesimal. Let $\{\bar{m}_i\}_{1\leq i\leq r}$ be a basis of $\mathfrak{M}/\mathfrak{M}^2$ with $\{\bar{\xi}_i\}_{1\leq i\leq r}$ be the corresponding dual basis of $(\mathfrak{M}/\mathfrak{M}^2)'$. Consider any linear map

$$\alpha : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow CL^2(f; f)$$

such that $\alpha(\bar{\xi}_i) = (\psi_i^\lambda, \psi_i^\mu; f_i)$ is a 2-cocycle for $1 \leq i \leq r$. We define a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of f with base $A/\mathfrak{M}^2$ as follows:

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^\lambda(l_1, l_2),$$

$$[1 \otimes x_1, 1 \otimes x_2]_\mu = 1 \otimes [x_1, x_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^\mu(x_1, x_2),$$

$$f_{\lambda\mu}(1 \otimes l_1) = 1 \otimes f(l_1) + \sum_{i=1}^{r} m_i \otimes f_i(l_1) \qquad \text{for } l_1, l_2 \in L \text{ and } x_1, x_2 \in M.$$

Since α takes values in cocycles, it is straightforward to check that $\mathfrak{D}$ is indeed a deformation (cf. Theorem 4.9). Thus $\mathfrak{D}$ is an infinitesimal deformation of f with base $A/\mathfrak{M}^2$ and $\alpha_{\mathfrak{D}} = \alpha$. Moreover, it follows from Theorem 4.10 that the equivalence class of $\mathfrak{D}$ is determined by the map $a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f; f)$, where $a(\xi) = [\alpha(\xi)]$ for $\xi \in (\mathfrak{M}/\mathfrak{M}^2)'$. Clearly, $a = a_{\mathfrak{D}}$ is the differential of $\mathfrak{D}$ (cf. Definition 4.11).

Given any linear map $a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f; f)$, let $\mathfrak{D}$ be an infinitesimal deformation of f with base $A/\mathfrak{M}^2$ determined by a. It is natural to ask whether $\mathfrak{D}$ can be lifted to a formal deformation with base A with a as its differential. This is the integrability question in the present context. To answer this question one has to start with the infinitesimal deformation $\mathfrak{D}$ with base $A/\mathfrak{M}^2$ and study the obstruction maps of Section 5 in lifting $\mathfrak{D}$ to higher order deformations. More precisely, we proceed as follows.

Suppose A is a complete local algebra with the maximal ideal $\mathfrak{M}$. Let $a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f; f)$ be any linear map and $\mathfrak{D}$ be an infinitesimal deformation with base $A/\mathfrak{M}^2$ and $a_{\mathfrak{D}} = a$. Suppose the deformation $\mathfrak{D}$ can be lifted to a deformation $\mathfrak{D}_k$ with base $A/\mathfrak{M}^k$ for $k \geq 2$. Consider the extension

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \xrightarrow{i_k^{k+1}} A/\mathfrak{M}^{k+1} \xrightarrow{p_k^{k+1}} A/\mathfrak{M}^k \longrightarrow 0$$

representing a cohomology class $[\psi_k] \in H^2_{\text{Harr}}(A/\mathfrak{M}^k; M_k)$, where $M_k = \mathfrak{M}^k/\mathfrak{M}^{k+1}$.

Let $\theta_k = \Theta_{\mathfrak{D}_k}([\psi_k]) \in M_k \otimes HL^3(f; f)$. Then by Proposition 5.4 we obtain the following proposition.

Proposition 6.5. *Let A be a complete local algebra with the maximal ideal $\mathfrak{M}$. Let $a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f; f)$ be a given linear map. Let $\mathfrak{D}$ be any infinitesimal deformation with base $A/\mathfrak{M}^2$ and $a_{\mathfrak{D}} = a$. Then there exists a formal deformation of f with base A and with the given map a as its differential if and only if $\theta_k = 0$ for all $k \geq 2$.*

Corollary 6.6. *If $HL^3(f; f) = 0$, then every linear map*

$$a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f; f)$$

is the differential of some formal deformation $\mathfrak{D}$ of f with base A.

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