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Nice surjections on spaces of operators

T S S R K RAO

Stat-Math Unit, Indian Statistical Institute, R.V. College P.O., Bangalore 560 059, India
E-mail: tss@isibang.ac.in

Abstract. A bounded linear operator is said to be nice if its adjoint preserves extreme points of the dual unit ball. Motivated by a description due to Labuschagne and Mascioni [9] of such maps for the space of compact operators on a Hilbert space, in this article we consider a description of nice surjections on $\mathcal{K}(X, Y)$ for Banach spaces X, Y. We give necessary and sufficient conditions when nice surjections are given by composition operators. Our results imply automatic continuity of these maps with respect to other topologies on spaces of operators. We also formulate the corresponding result for $\mathcal{L}(X, Y)$ thereby proving an analogue of the result from [9] for L^p $(1 < p \neq 2 < \infty)$ spaces. We also formulate results when nice operators are not of the canonical form, extending and correcting the results from [8].

Keywords. Nice surjections; isometries; spaces of operators.

1. Introduction

Let X, Y be Banach spaces. A linear map $T: X \to Y$ is said to be nice if $y^* \circ T$ is an extreme point of the unit ball of X^* for every extreme point y^* of the unit ball of Y^*. It is easy to see that such an operator is of norm one (see [9, 16]). Such operators between Banach spaces have been well studied in the literature (see [1, 15] and the references therein, and also the more recent article [11]). Also most often in the literature a standard technique for describing surjective isometries is the so-called extreme point method that use the simple fact that any surjective isometry is a nice operator (see [4]). In the case of a Hilbert space nice operators are precisely co-isometries (i.e., the adjoint is an isometry). Clearly the study of nice operators is of interest between spaces where the description of extreme points of the dual unit ball is known. For a Banach space X, let X_1 denote the unit ball and $\partial_e X_1$ denote the set of extreme points. In this paper we will be using the well-known result of Ruess, Stegall [21] and Tseitlin [22] (see [13] for the complex case) that identifies $\partial_e \mathcal{K}(X, Y)_1^*$ as $\{\Lambda \otimes y^* : \Lambda \in \partial_e X_1^{**}, y^* \in \partial_e Y_1^*\}$. Here $\Lambda \otimes y^*$ denotes the functional $(\Lambda \otimes y^*)(T) = \Lambda(T^*(y^*))$. The main aim of this paper is to formulate and prove an abstract analogue of part A of Theorem 16 of [9] that completely describes nice operators between spaces of compact operators on Hilbert spaces, for nice surjections. Our results are valid for a class of Banach spaces that include the L^p-spaces for $1 < p \neq 2 < \infty$.

A well-known theorem of Kadison [7] describes surjective isometries of $\mathcal{K}(H)$ in terms of composition operators involving unitaries or anti-unitaries (i.e., conjugate linear maps). It is known that in general surjective isometries on the space of compact operators need not be given by composition by isometries of the underlying spaces (see [18]). See also [12] where extreme point-preserving surjections were described on $\mathcal{L}(H)$ again as compositions by unitaries or anti-unitaries. In particular, it follows from Corollary 1 in [12] that any nice operator on $\mathcal{K}(H)^*$ whose adjoint is surjective, is weak*-continuous, a property which is well-known in the case of surjective isometries. Motivated by these results in the first two sections of this paper we study nice surjections on $\mathcal{K}(X, Y)$ that are of the form $T \to UTV$ for appropriate operators U and V. We shall call this the canonical form or composition operator. Such a representation has the additional advantage that it is continuous with respect to the strong operator topology. Nice operators that are surjections were classified for the space of affine continuous functions on a Choquet simplex in [18]. This paper is a part of a series where we have been trying to answer certain questions raised in [18], dealing with several aspects of 'Kadison type' theorems (see also [20]).

We first show that for an isometry V of X whose range is an ideal in the sense of [3], V^* is a nice operator and for a nice operator U with a right inverse on Y, $T \to UTV$ is a nice surjection. For reflexive spaces X and Y such that each is not isometric to a subspace of the dual of the other, under the assumption of metric approximation property and strict convexity of X and Y^*, we show that any nice surjection of $\mathcal{K}(X, Y)$ is given by the composition operator.

In §3 which has the main result, we deal with an analogue of part B of Theorem 16 [9]. We first formulate conditions similar to operators preserving ultra-weakly continuous extreme points on $\mathcal{L}(H)$ and show that for certain reflexive Banach spaces X, Y, weak*-continuous-extreme point-preserving surjections on $\mathcal{L}(X, Y)$ are given by composition operators. It is easy to see that the assumption of 'ultra weakly continuity' on the Ψ made in [9] is actually a consequence of the adjoint preserving such extreme points. As a consequence we have that such a map Ψ leaves the compacts invariant and is in fact the bi-transpose of a nice operator on $\mathcal{K}(X, Y)$.

In §4 we consider the situation when nice operators on $\mathcal{L}(X, Y)$ are not given by composition operators. We give examples where nice operators do not map compact operators to compact operators. We show that for any Banach space X, a nice operator on $\mathcal{L}(X, \ell^\infty)$ maps $\mathcal{K}(X, c_0)$ to itself. This extends the correct part of Theorem 2.1 of [8].

We refer to the monograph [2] for results from the tensor product theory that we will be using here while retaining the suffix ϵ and π to denote the injective and projective tensor products.

2. Nice surjections on $\mathcal{K}(X, Y)$

We first recall the full description of nice operators given by part A of Theorem 16 of [9] in the case of Hilbert spaces.

Theorem 1. *Let $\Phi: \mathcal{K}(H_1) \to \mathcal{K}(H_2)$ be a nice operator. Then either $\Phi(T) = U^*TV$ or $U^*c^*T^*cV$ where $U: H_2 \to H_1$, $V: H_2 \to H_1$ are injective partial isometries and $c: H_1 \to H_1$ is a anti-unitary or there exists a fixed unit vector $w \in H_1$ such that $\Phi(T) = JV(Tw)$ or $(JV(T^*)w))^*$, where J is the natural injection of the Hilbert–Schmidt class operators into compacts and V is a partial isometry of H_1 onto the Hilbert–Schmidt class on H_2.*

Remark 2. Note that if $\{T_\alpha\} \subset \mathcal{K}(H_1)$ is a net and $T_\alpha(w) \to 0$ then since the Hilbert–Schmidt norm $\|V(T_\alpha(w))\|_{\text{HS}} \to 0$, it follows that $\Phi(T_\alpha) \to 0$. Thus in this case Φ is (s. o. t.)-norm continuous.

As mentioned in the introduction we will only be considering nice surjections of the first kind. Also to avoid the case corresponding to anti-unitaries we assume that when X and Y

are reflexive, X is not isometric to a subspace of Y^* and Y is not isometric to a subspace of X^*. Among infinite dimensional classical Banach spaces, for $p \neq 2$ the L^p-spaces have this property.

Lemma 3. *Let $\Phi(T) = UTV$, where $U \in \mathcal{L}(Y)_1$ and $V \in \mathcal{L}(X)_1$, be a nice operator on $\mathcal{K}(X, Y)$. Then U and V^* are nice operators.*

Proof. Let $\Lambda \in \partial_e X_1^{**}$. Fix a $y^* \in \partial_e Y_1^*$. Since Φ^* preserves extreme points there exists $\Lambda_1 \in \partial_e X_1^{**}$ and $y_1^* \in \partial_e Y_1^*$ such that $\Phi^*(\Lambda \otimes y^*) = V^{**}(\Lambda) \otimes U^*(y^*) = \Lambda_1 \otimes y_1^*$. As $V^{**}(\Lambda) = \Lambda_1$ and $U^*(y^*) = y_1^*$ we have, V^* and U are nice operators.

Remark 4. When $Y = C(K)$ for a compact set K, it is well-known that the space $\mathcal{K}(X, Y)$ can be identified with the space of vector-valued functions $C(K, X^*)$. A description of nice isomorphisms of $C(K, X)$ was given in Theorem 2.9 of [4].

Next we would like to formulate a sufficient condition for $T \to UTV$ to be surjective. For this purpose we recall the notion of an ideal, from [3].

DEFINITION 5

A closed subspace $M \subset X$ is said to be an ideal if there is a projection $P \in \mathcal{L}(X^*)$ of norm one such that $\ker(P) = M^\perp$.

It is easy to see that the range of a norm one projection on X is an ideal. Thus every closed subspace of a Hilbert space is an ideal. Also in reflexive spaces ideals are precisely ranges of projections of norm one.

Theorem 6. *Let $U \in \mathcal{L}(Y)$ be a nice operator with a right inverse U'. Suppose $V\colon X \to X$ is an into isometry, V^* is a nice operator and $V(X)$ is an ideal in X. Then $\Phi(T) = UTV$ is a nice surjection.*

Proof. It follows from the arguments given during the proof of lemma that the hypothesis implies that Φ is a nice operator.

Let $S \in \mathcal{K}(X, Y)$. Since $V(X)$ is an ideal, let P^* be a projection in $\mathcal{L}(X^{**})$ such that range $(P^*) = V(X)^{\perp\perp}$. Let $S' = U'SV^{-1}\colon V(X) \to Y$. Since S' is a compact operator, we have $S'^{**}\colon V(X)^{**} \to Y$. We now identify $V(X)^{**}$ with $V(X)^{\perp\perp}$. Let $R = S'^{**} P^*|X \in \mathcal{K}(X, Y)$.

Now for $x \in X$, $\Phi(R)(x) = U(R(V(x)) = U(S'(V(x)) = U(U'(S(x)) = S(x)$. Thus Φ is onto. $\square$

In the following proposition which is of independent interest, we exhibit a large class of Banach spaces for which the range of every isometry is an ideal.

PROPOSITION 7

Let X be any Banach space such that X^* is isometric to $L^p(\mu)$ for $1 \leq p \leq \infty$. The range of any isometry of X is an ideal.

Proof. Let V be an isometry of X. Clearly when $1 < p < \infty$, $X = L^q(\mu)$. Since V is an isometry, it follows from Theorem 3 on page 162 of [10] that $V(X)$ is the range of a projection of norm one and hence an ideal.

When $p = \infty$, from general isometric theory we have that X as well as $V(X)$ are L^1-spaces. Thus from the same theorem again we have that $V(X)$ is an ideal.

When $p = 1$, X, $V(X)$ are the so-called L^1-predual spaces. It follows from Proposition 1 in [17] that $V(X)$ is an ideal in X. □

Remark 8. When X is a reflexive and strictly convex space for any isometry V of X, since V maps extreme points to extreme points, V^* is a nice operator . Thus for $1 < p < \infty$, $X = L^p(\mu)$ satisfies both the conditions imposed in the Theorem on V.

We now give a partial answer to the necessary condition for nice surjections. The formulation and its proof are based on the proof of Theorem 1.1 in [8]. This result implies automatic weak*-continuity of extreme point-preserving maps on certain domains.

Theorem 9. *Let X and Y be reflexive Banach spaces with X and Y^* strictly convex. Assume that X is not isometric to a subspace of Y^* and Y^* is not isometric to a subspace of X^*. Suppose one of X^* or Y has the metric approximation property. Let $\Psi\colon X \otimes_\pi Y^* \to X \otimes_\pi Y^*$ be a bounded one-to-one linear operator mapping extreme points of the unit ball to extreme points. Then $\Psi(T) = VTU^*$ for $U \in \mathcal{L}(Y)$ such that U^* is an into isometry and an into isometry $V \in \mathcal{L}(X)$. Moreover Ψ is weak*-continuous with respect to the weak*-topology induced by $X^* \otimes_\epsilon Y = \mathcal{K}(X, Y)$ and hence it is the adjoint of a nice surjection. In particular, any nice surjection of $\mathcal{K}(X, Y)$ is of the form $T \to UTV$ for nice surjections U and V^*.*

Proof. We proceed as in the proof of Step II of Theorem 1.1 in [8]. Since X and Y^* are strictly convex and Ψ is one-one, we see that for any $y^* \in Y^*$, $\Psi(X \otimes_\pi \text{span}\{y^*\}) \subset X \otimes_\pi \text{span}\{g\}$ for some $g \in Y^*$. Note that we do not get the equality of the sets here since we are not assuming that Ψ is onto. We also note that our assumption about the spaces not being isometric to the subspace of the dual of the other ensures that as in Case (ii) of the proof of Theorem 1.1 in [8] the only possible action of Ψ is that $\Psi(X \otimes_\pi \text{span}\{y^*\}) \subset X \otimes_\pi \text{span}\{g\}$. Thus we can define operators $U \in \mathcal{L}(Y)$ and $V \in \mathcal{L}(X)$ such that $\Psi(x \otimes y^*) = V(x) \otimes U^*(y^*)$. Since left or right composition by an operator is a weak*-continuous map on $X \otimes_\pi Y^*$, we get that Ψ is weak*-continuous. The other properties of U and V follow from the assumptions of reflexivity and strict convexity.

Further if Φ is a nice surjection, applying the above argument to $\Psi = \Phi^*$, it is easy to see that U and V^* are surjections and $\Phi(T) = UTV$. □

We next give an example which shows that among other things strict convexity cannot be omitted from the hypothesis of the above theorem. As our example is a surjective isometry, in particular it shows that Theorem 1.1 of [8] also fails if the hypothesis of strict convexity is omitted (Remark 1.3 of [8] is incorrect). See [19] for related results.

Example 10. Let X be any Banach space with two linearly independent isometries U_1 and U_2 and let $Y = c_0$. Let $\{e_n\}$ denote the canonical basis of ℓ^1. Define $\Phi\colon \mathcal{K}(X, c_0) \to \mathcal{K}(X, c_0)$ by $\Phi(T)(x)(e_n) = T(U_n(x))(e_n)$ for $n = 1, 2$ and as identity elsewhere. It is easy to see that Φ is an isometry. It is well-known that isometries of c_0 are given by permutation of the coordinates along with multiplication by scalars of absolute value one. Since U_1 and U_2 are linearly independent, clearly Φ is not of the canonical form. It is also easy to see that even though it is not given by composition, Φ is continuous with respect to the s. o. t.

3. 'Nice' surjections on $\mathcal{L}(X, Y)$

In this section we consider nice surjections that are similar to part B of Theorem 16 in [9] (Theorem 11 below) that describes operators that preserve ultra-weakly continuous extreme points of $\mathcal{L}(H)$. Since in general there is no description of $\partial_e \mathcal{L}(X, Y)_1^*$ available, one can only talk about preserving a subclass of the set of extreme points.

Theorem 11. *Let $\Psi: \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ be a ultra-weakly continuous linear map such that $\rho \circ \Psi \in \partial_e \mathcal{L}(H_1)_1^*$ whenever $\rho \in \partial_e \mathcal{L}(H_2)_1^*$ is ultra-weakly continuous. Then either $\Psi(T) = U^* T V$ or $U^* c^* T^* c V$ where $U: H_2 \to H_1$, $V: H_2 \to H_1$ are injective partial isometries and $c: H_1 \to H_1$ is a anti-unitary or there exists a fixed unit vector $w \in H_1$ such that $\Psi(T) = JV(Tw)$ or $(JV(T^*)w)^*$, where J is the natural injection of the Hilbert–Schmidt class operators into $\mathcal{L}(H_2)$ and V is a partial isometry of H_1 onto the Hilbert–Schmidt class on H_2.*

Remark 12. We note that an important consequence of the above theorem is that Ψ maps compact operators to compact operators. Also Ψ is the bi-transpose of a nice operator from $\mathcal{K}(H_1) \to \mathcal{K}(H_2)$. Further, in the cases where Ψ is not a composition operator, the range of Ψ consists of the compact operators.

In order to understand the hypothesis of the above theorem in a general set-up, let $T: X \to X$ be a nice operator. Consider $T^{**}: X^{**} \to X^{**}$. Let $\tau \in \partial_e X_1^{***}$ be a weak*-continuous map. Then $\tau = x^* \in \partial_e X_1^*$ and as T is a nice operator, $T^{***}(x^*) = T^*(x^*) \in \partial_e X_1^*$. Thus T^{***} maps extreme points of X_1^{***} that are weak*-continuous to extreme points of X_1^*. It should be noted that in general $\partial_e X_1^*$ is not contained in $\partial_e X_1^{***}$.

However there are several classes of Banach spaces where weak*-continuous extreme points of X_1^{***} are precisely extreme points of X_1^*. Though this point is not essential to our analysis we mention these examples below.

We recall from chapter III of [5] that X is said to be a M-ideal in its bi-dual (a M-embedded space) if under the canonical embedding of X in X^{**} there exists a projection $P \in \mathcal{L}(X^{***})$ with $\ker(P) = X^\perp$ such that $\|\tau\| = \|P(\tau)\| + \|\tau - P(\tau)\|$ for all $\tau \in X^{***}$. In this set-up, X^{***} is the ℓ^1-direct sum of X^* and $X^\perp$. It is well-known that $\mathcal{K}(H)$ is a M-ideal in its bi-dual $\mathcal{L}(H)$. See chapters III and VI of [5] for more information and examples of these spaces from among function spaces and spaces of operators. For any such X its dual has the Radon–Nikodým property (Theorem III.3.1 of [5]).

Since $\partial_e X_1^{***} = \partial_e X_1^* \cup \partial_e X_1^\perp$, weak*-continuous extreme points of X_1^{***} are precisely the extreme points of X_1^*.

Now let X be any Banach space and let $S: X^{**} \to X^{**}$ be a linear map such that $x^* \circ S \in \partial_e X_1^*$ for all $x^* \in \partial_e X_1^*$. Suppose now X_1^* is the norm closed convex hull of its extreme points (this for example happens when X^* has the Radon–Nikodým property and also for the class of M-embedded spaces mentioned above (see [5], chapter III)). For $\Lambda \in X^{**}$, $\|S(\Lambda)\| = \sup\{|S(\Lambda)(x^*)| = |(x^* \circ S)(\Lambda)| : x^* \in \partial_e X_1^*\} \leq \|\Lambda\|$. Thus $\|S\| = 1$ and $S^*|X^*: X^* \to X^*$. This in particular means that S is weak*–weak* continuous. Note that unlike in the above theorem where the Ψ is ultra-weakly and hence weak*-continuous, we have not assumed weak*-continuity of S. These operators are the correct analogues of the ones described in the above theorem.

Now let X be a reflexive Banach space and let Y be a M-embedded space. If one of X^* or Y has the metric approximation property then as before, using tensor product theory ([2], Chapter VIII) we have that $\mathcal{K}(X, Y)^* = X \otimes_\pi Y^*$ and thus $\mathcal{K}(X, Y)^{**} = \mathcal{L}(X, Y^{**})$.

As noted earlier the L^p spaces ($1 < p \neq 2 < \infty$) satisfy the hypothesis of the following theorem and the range of V is an ideal.

Theorem 13. *Let X, Y be reflexive, with X, Y^* strictly convex such that each is not isometric to a subspace of the dual of the other.*

Assume further that X or Y^* has the metric approximation property. Let $\Phi : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y)$ be a linear surjective map with Φ^*-preserving weak*-continuous extreme points. Then it is a composition operator and is the bi-transpose of a nice surjection on $\mathcal{K}(X, Y)$.

Proof. Since by our assumptions $\mathcal{K}(X, Y)^* = (X^* \otimes_\epsilon Y)^* = X \otimes_\pi Y^*$ has the Radon–Nikodým property (see chapter VIII, §4 of [2]), from the remarks made above it follows that Φ^* maps $\mathcal{K}(X, Y)^* = X \otimes_\pi Y^*$ to itself and preserves the extreme points. Therefore from Theorem 9 we know that $\Phi^*|\mathcal{K}(X, Y)^*$ is a weak*-continuous one-to-one map that preserves extreme points. Thus it is the adjoint of a nice surjection on $\mathcal{K}(X, Y)$. In particular, there exist nice operators $U \in \mathcal{L}(Y)$ and $V^* \in \mathcal{L}(X^*)$ such that if $\Psi : \mathcal{K}(X, Y) \to \mathcal{K}(X, Y)$ is such that $\Psi(T) = UTV$, then $\Psi^* = \Phi^*|\mathcal{K}(X, Y)^*$. Now $\Psi^{***} = \Phi^*$ on $\mathcal{K}(X, Y)^*$. Since $\mathcal{K}(X, Y)^*$ is weak*-dense in its bi-dual $\mathcal{L}(X, Y)^*$ we get that $\Phi(T) = \Psi^{**}(T) = UTV$. □

Remark 14. We do not know if the above theorem remains true if one merely assumes that Y is a M-embedded space with a strictly convex dual. It follows from Proposition III.2.2 of [5] that for such a Y any onto isometry is weak*-continuous. To complete the proof as above one would require a into isometry of Y^* to be weak*-continuous.

The proof of the following corollary is immediate from the proof of the above theorem.

COROLLARY 15

Let X and Y be as in the above theorem. Let $\Phi : \mathcal{K}(X, Y) \to \mathcal{L}(X, Y)$ be a linear surjection such that for every weak*-continuous extreme point $\tau \in \partial_e \mathcal{L}(X, Y)_1^*$, $\tau \circ \Phi \in \partial_e \mathcal{K}(X, Y)_1^*$. Then Φ is given by composition operators. Hence the range of Φ consists of compact operators, also Φ has a natural extension to $\mathcal{L}(X, Y)$.

Remark 16. See [20] for another interpretation of 'niceness' and for questions related to uniqueness of extension from the space of compact operators to the space of bounded operators.

4. Nice operators on $\mathcal{L}(X, Y)$

In this section we consider nice operators on $\mathcal{L}(X, Y)$ that are not given by composition operators. Our first result shows that even a surjective isometry need not be of the canonical form given by the composition operator of surjective isometries of X, Y.

We recall that a Banach space X is said to be a Grothendieck space, if weak* and weak sequential convergence coincide in X^* (see [2], page 179). The well-known non-reflexive examples include $L^\infty(\mu)$ for a σ-finite measure and more generally any von Neumann algebra [14].

Theorem 17. *Let K be an infinite first countable compact Hausdorff space. Let X be a Grothendieck space such that there is an isometry U of X^* that is not weak*-continuous. Then there is an isometry of $\mathcal{L}(X, C(K))$ that is not continuous with respect to the strong operator topology. Hence isometries of $\mathcal{L}(X, C(K))$ are not of the canonical form.*

Proof. Let $W^*C(K, X^*)$ denote the space of X^*-valued functions on K that are continuous when X^* has the weak*-topology, equipped with the supremum norm. We use the well-known identification of $\mathcal{L}(X, C(K))$ with this space via the map $T \to T^* \circ \delta$ (where δ is the Dirac map). Define $\Phi\colon W^*C(K, X^*) \to W^*C(K, X^*)$ by $\Phi(F) = U \circ F$. Since for any sequence $k_n \to k$ in K, $F(k_n) \to F(k)$ weakly in X^*, we get that $U(F(k_n)) \to U(F(k))$ weakly and hence in the weak*-topology. Thus Φ is a well-defined map. It is easy to see that Φ is an isometry.

Let $x_\alpha^* \to 0$ be a weak*-convergent net such that $\{U(x_\alpha^*)\}$ does not converge to 0 in the weak*-topology. Define $T_\alpha\colon X \to C(K)$ by $T_\alpha(x)(k) = x_\alpha^*(x)$. It is easy to see that $T_\alpha \to 0$ in the s. o. t. However since $\Phi(T_\alpha)(x) = U(x_\alpha^*)(x)$ we see that $\{\Phi(T_\alpha)\}$ does not converge to 0 in the s. o. t. Therefore Φ is not of the canonical form.

Remark 18. By taking a measurable unimodular function on the Stone space K of $L^\infty(\mu)$ that is not continuous it is easy to generate an isometry of $L^\infty(\mu)^* = C(K)^*$ that is not weak*-continuous. Such examples can also be generated in $\mathcal{L}(H)^*$ when H is infinite dimensional or more generally on duals of atomic σ-finite von Neumann algebras.

It is well-known, for example by identifying $\mathcal{K}(X, c_0)$ with the c_0 direct sum $\bigoplus_{c_0} X^*$ and $\mathcal{L}(X, \ell^\infty)$ with $\bigoplus_\infty X^*$, that $\mathcal{L}(X^{**}, \ell^\infty)$ is the bi-dual of $\mathcal{K}(X, c_0)$. In particular for a reflexive Banach space X, $\mathcal{L}(X, \ell^\infty)$ is the bi-dual of $\mathcal{K}(X, c_0)$. In Theorem 2.1 of [8] the authors claim that surjective isometries of $\mathcal{L}(c_0)$ are of the canonical form and hence leave the space of compact operators invariant. Our Example 10 shows that the isometries are not of the canonical form. However it is still true that surjective isometries of $\mathcal{L}(c_0)$ leave the space of compact operators invariant. The following result extends Theorem 2.1 from [8].

Theorem 19. *Let X be any Banach space and let $\Phi\colon \mathcal{L}(X, \ell^\infty) \to \mathcal{L}(X, \ell^\infty)$ be a nice operator. Then for any $T \in \mathcal{K}(X, c_0)$, $\Phi(T) \in \mathcal{K}(X, c_0)$.*

Proof. Let $T \in \mathcal{K}(X, c_0)$. We recall that $\|T\| = \|\Phi(T)\| = \|\Phi(T)^*\| = \sup\{|\Phi(T)^*(e_n)|: n \geq 1\}$.

Fix n such that $\Phi(T)^*(e_n) \neq 0$. Let $\tau \in \partial_e X_1^{**}$ be such that $\tau(\Phi(T)^*(e_n)) = \|\Phi(T)^*(e_n)\|$. It is easy to see that the functional $\tau \otimes e_n\colon \mathcal{L}(X, \ell^\infty) \to \mathcal{L}(X, \ell^\infty)$ defined by $(\tau \otimes e_n)(S) = \tau(S^*(e_n))$ is an extreme point of the dual unit ball. Thus by hypothesis, $\Phi^*(\tau \otimes e_n) \in \partial_e \mathcal{L}(X, \ell^\infty)_1^*$.

Now using the identification of $\mathcal{K}(X, c_0)$ with the c_0 direct sum $\bigoplus_{c_0} X^*$ and of $\mathcal{L}(X, \ell^\infty)$ with $\bigoplus_\infty X^*$, we see that $\mathcal{L}(X, \ell^\infty)^* = \mathcal{K}(X, c_0)^* \bigoplus_1 \mathcal{K}(X, c_0)^\perp$ (see arguments from [6], page 129 that also work for the vector-valued case). Since $\Phi^*(\tau \otimes e_n)(T) \neq 0$ we have $\Phi^*(\tau \otimes e_n) \in \partial_e \mathcal{K}(X, c_0)_1^*$. Therefore by the identification mentioned before, $\Phi^*(\tau \otimes e_n)$ and $\tau' \otimes e_{n_0}$ for some $\tau' \in \partial_e X_1^{**}$ and n_0. Now $\|\Phi^*(T^*(e_n))\| = \Phi^*(\tau \otimes e_n)(T) = \tau'(T^*(e_{n_0})) \leq \|T^*(e_{n_0})\|$. As $\|T^*(e_n)\| \to 0$ we get that $\Phi(T) \in \mathcal{K}(X, c_0)$. □

The following corollary can be proved using arguments identical to the ones given above and the fact that for any Banach space X, $\mathcal{K}(X, c_0)$ is a M-ideal in $\mathcal{L}(X, c_0)$ (see Example VI.4.1 in [5]).

COROLLARY 20

For any Banach space X every isometry of $\mathcal{L}(X, c_0)$ leaves $\mathcal{K}(X, c_0)$ invariant.

The following is an example where an isometry does not preserve compact operators. It also shows that c_0 cannot be replaced by c, the space of convergent sequences in the above result.

Example 21. Let $X = \ell^2$ and let U_n denote the unitary that interchanges the first and the nth coordinate. We denote by e_n the coordinate vectors in either space. Define $\Phi\colon \mathcal{L}(\ell^2, \ell^\infty) \to \mathcal{L}(\ell^2, \ell^\infty)$ such that $\Phi(T)^*(e_k) = U_k(T^*(e_k))$. It is easy to see that Φ is an isometry. The operator $T_0^*(e_k) \equiv e_1$ for all k, being 'constant-valued' is clearly compact. But since $\Phi(T_0)^*(e_k) = U_k(T_0^*(e_k)) = U_k(e_1) = e_k$ for all k, $\Phi(T_0)^*$ and hence $\Phi(T_0)$ is not a compact operator.

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