

# Shape invariance approach to exact solutions of the Klein-Gordon equation

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## Abstract

Using the shape invariance property we obtain exact solutions of the  $(1 + 1)$  dimensional Klein-Gordon equation for certain types of scalar and vector potentials. We also discuss the possibility of obtaining real energy spectrum with non-Hermitian interaction within this framework.

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# 1 Introduction

In view of their importance in the study of various relativistic systems under the influence of strong potentials [1], there have been a growing interest in obtaining exact solutions of relativistic wave equations, particularly the Klein-Gordon (KG) equation [2, 3, 4]. In many of these of papers exact solutions were obtained for (1+1) dimensional or the  $s$  wave KG equation for different choices of the energy independent vector and the scalar potential. Furthermore the KG equation has also been considered in the context of non-Hermitian/ $\mathcal{PT}$  symmetric interactions [5]. Here our objective is to expand the class of exactly solvable KG equation for real (and unequal) as well as non-Hermitian vector and scalar interactions and supplement earlier results [2]. Also, instead of directly solving the relevant differential equation, we shall use algebraic techniques, namely, shape invariance [6] and intertwining property to obtain exact solutions.

## 2 Real scalar and vector potentials

To begin with let us briefly recall the notion of shape invariance [6]. We note that a supersymmetric pair of potentials  $U_{\pm}(x; a_1)(= W^2(x; a_1) \pm W'(x; a_1))$  are called shape invariant if

$$U_+(x; a_1) = U_-(x; a_2) + R(a_1) \quad (1)$$

where  $a_1$  is a set of parameters,  $a_2$  is a function of  $a_1$  and  $R(a_1)$  is independent of  $x$ . It can be shown that for such potentials the energy spectrum corresponding to  $U_-(x; a_1)$  is given by

$$E_n^- = \sum_{k=1}^n R(a_k) \quad (2)$$

while the corresponding wave function is given by

$$\psi_n^-(x; a_1) \propto A^\dagger(x; a_1)A^\dagger(x; a_2)\dots A^\dagger(x; a_n)\psi_0^-(x; a_{n+1}) \quad (3)$$

where  $A(x; a_1) = -\frac{d}{dx} + W(x; a_1)$  and  $\psi_0^-(x; a_{n+1})$  is the ground state wave function corresponding to  $U_-(x; a_{n+1})$ .

We shall now use the above results to obtain solutions of Klein-Gordon equation. Let us now consider stationary Klein-Gordon equation of the form

$$-\frac{d^2\psi}{dx^2} + [(m + S(x))^2 - (E - V(x))^2]\psi = 0 \quad (4)$$

where  $V(x)$  and  $S(x)$  denote the vector and the scalar potentials respectively. Written in full, Eq.(4) becomes

$$-\frac{d^2\psi}{dx^2} + U(x)\psi = \epsilon\psi \quad (5)$$

where

$$U(x) = S^2(x) - V^2(x) + 2[mS(x) + EV(x)] \quad , \quad \epsilon = E^2 - m^2 \quad (6)$$

To use the concept of shape invariance it is now necessary to identify (6) with one of  $U_{\pm}$ . To do this we choose the vector and scalar potentials to be of the form

$$V(x) = V_0f(x) \quad , \quad S(x) = S_0f(x) \quad (7)$$

Then from (6) it follows that

$$U(x) = (S_0^2 - V_0^2)f^2(x) + 2(mS_0 + EV_0)f(x) \quad (8)$$

Thus to put (8) in the form  $U_{\pm}$  it is necessary to choose  $f(x)$  such that  $f'(x)$  can be expressed in terms of  $f(x)$  and also impose suitable constraints on the coupling constants. Furthermore one has to be careful so that  $V(x), S(x)$  do not become energy dependent. We shall now obtain exact solutions of (5) for different choices of  $f(x)$ .

**Case 1.** We first consider

$$f(x) = \tanh x \quad (9)$$

In this case Eq.(5) can be written as

$$-\frac{d^2\psi}{dx^2} + U_1(x)\psi = \epsilon_1\psi \quad (10)$$

where

$$U_1(x) = -(S_0^2 - V_0^2)\operatorname{sech}^2 x + 2(mS_0 + EV_0)\tanh x, \quad \epsilon_1 = E^2 - m^2 - S_0^2 + V_0^2 \quad (11)$$

It is readily seen that if  $S_0^2 > V_0^2$  then  $U_1(x)$  can be treated as a shape invariant potential admitting discrete eigenvalues [6]. In this case we have

$$W(x; a_1) = A \tanh x + \frac{B}{A}, \quad a_i = (A - i + 1), \quad (12)$$

$$R(a_i) = [(A - i + 1)^2 - (A - i + 2)^2] + B^2 \left[ \frac{1}{(A - i + 1)^2} - \frac{1}{(A - i + 2)^2} \right]$$

where

$$A = \frac{-1 + \sqrt{1 + 4(S_0^2 - V_0^2)}}{2}, \quad B = mS_0 + E_n V_0 \quad (13)$$

Then from (2) it follows that

$$\epsilon_{1n} = \sum_{i=1}^n R(a_i) = -(A - n)^2 - \frac{B^2}{(A - n)^2}, \quad n = 0, 1, 2, \dots < (A - \sqrt{|B|}) \quad (14)$$

We note that in comparison to nonrelativistic Schrödinger equation, in the present case the number of discrete energy levels depends on the energy  $E_n$  rather than on  $n$ . Now from (14) and (13) the energy eigenvalues are found to be

$$E_n^{\pm} = \frac{-Q \pm \sqrt{Q^2 - 4P_n R_n}}{2P_n} \quad (15)$$

where

$$\begin{aligned} P_n &= (A - n)^2 + V_0^2 \\ Q &= 2mS_0 V_0 \\ R_n &= (A - n)^4 + m^2 S_0^2 - (A - n)^2 (m^2 + S_0^2 - V_0^2) \end{aligned} \quad (16)$$

Now using (3) and noting that the ground state wave function can be obtained from the relation  $A(x; a_1)\psi_0(x; a_1) = 0$ , the wave functions corresponding to (15) can be found to be [6, 7]

$$\psi_n(x; a_1) \propto (1 - \tanh x)^{s_1^{\pm}/2} (1 + \tanh x)^{s_2^{\pm}/2} P_n^{(s_1^{\pm}, s_2^{\pm})}(\tanh x) \quad (17)$$

where the( $\pm$ ) sign correspond to positive and negative energy,  $s_1^\pm = A - n + \frac{B}{A - n}$ ,  $s_2^\pm = A - n - \frac{B}{A - n}$  and  $P_n^{(a,b)}$  denotes Jacobi polynomials.

It is important to note that not all the energy levels in (15) are acceptable. To determine the acceptable levels we need to examine the behaviour of the wave functions. From (17) it is seen that the wave functions are acceptable if  $s_1^\pm, s_2^\pm > 0$ . But both  $s_1^\pm$  and  $s_2^\pm$  depend on  $A$  and  $B$  with  $B$  depending on  $E_n^\pm$ . However the resulting dependence of  $s_1^\pm$  and  $s_2^\pm$  on  $E_n^\pm$  is quite complicated and the above conditions has to be checked numerically for each level. We have computed the energy values and the parameters  $s_{1,2}^\pm$  for different values of the input parameters  $m, S_0, V_0$  and the results are given in tables 1 and 2.

**Case 2.** We shall now choose

$$f(x) = -e^{-x} \quad (18)$$

This case was treated in ref[2] by solving the KG equation. Here we rederive the results via the shape invariance property and examine the acceptable energy levels. For the choice (18) we obtain from (6)

$$U_2(x) = (S_0^2 - V_0^2)e^{-2x} - 2(mS_0 + EV_0)e^{-x} \quad (19)$$

In this case it can be shown that

$$\begin{aligned} W(x; a_1) &= A - Be^{-x}, \quad a_i = (A - i + 1), \\ R(a_i) &= (A - i + 1)^2 - (A - i)^2 \end{aligned} \quad (20)$$

where

$$A^\pm = \frac{mS_0 + EV_0}{\sqrt{S_0^2 - V_0^2}} - \frac{1}{2}, \quad B = \sqrt{S_0^2 - V_0^2} \quad (21)$$

Thus we have

$$\epsilon_{2n} = \sum_{i=1}^n R(a_i) = -(A - n)^2, \quad n = 0, 1, 2, \dots < A \quad (22)$$

and consequently

$$E_n^\pm = \frac{-Q_n \pm \sqrt{Q_n^2 - 4PR_n}}{2P} \quad (23)$$

where

$$\begin{aligned} P &= S_0^2 \\ Q_n &= -2(n + \frac{1}{2})V_0\sqrt{S_0^2 - V_0^2} + 2mV_0S_0 \\ R_n &= (n + \frac{1}{2})^2(S_0^2 - V_0^2) + m^2V_0^2 - 2(n + \frac{1}{2})mS_0\sqrt{S_0^2 - V_0^2} \end{aligned} \quad (24)$$

The wave functions corresponding to (23) are given by [6, 8]

$$\psi_n(x) \propto (2Be^{-x})^{A^\pm - n} e^{-Be^{-x}} L_n^{2A^\pm - 2n}(2Be^{-x}) \quad (25)$$

where  $L_n^\alpha(x)$  denotes generalised Laguerre polynomials. From (25) it follows that normalisability of the wave functions requires  $B > 0$  and  $A^\pm > 0$  ( $A^\pm$  denotes the values of  $A$  corresponding to positive and negative energy). The first of these conditions can be ensured by choosing  $S_0^2 > V_0^2$  and since  $A^\pm$  depends on  $E_n^\pm$ , the second condition has to be checked for each individual quantum number  $n$ . Furthermore, the number of discrete energy states depends on the values of  $A^\pm$ . We have numerically evaluated the energy

values and the parameter  $A^\pm$  for some value of the input parameters  $m, S_0$  and  $V_0$  and the results are presented in table 3.

**Case 3.** Finally we consider

$$f(x) = x/2 \quad (26)$$

In this case we obtain

$$U_3(x) = \frac{1}{4}(S_0^2 - V_0^2)x^2 + (mS_0 + EV_0)x \quad (27)$$

Clearly  $U_3(x)$  is a shape invariant potential with

$$W(x; a_1) = Ax + B \quad , \quad a_i = A \quad , \quad R(a_i) = 2A \quad (28)$$

where

$$A^2 = \frac{S_0^2 - V_0^2}{4} \quad , \quad B = \frac{mS_0 + EV_0}{\sqrt{S_0^2 - V_0^2}} \quad (29)$$

Thus we get

$$\epsilon_{3n} = (2n + 1)A - B^2 \quad (30)$$

so that

$$E_n^\pm = \frac{-Q_n \pm \sqrt{Q_n^2 + 4PR_n}}{2P_n} \quad (31)$$

where

$$\begin{aligned} P &= S_0^2 \\ Q &= 2mS_0V_0 \\ R_n &= (S_0^2 - V_0^2)^{3/2}(n + \frac{1}{2}) - m^2V_0^2 \end{aligned} \quad (32)$$

The corresponding wave functions are given by

$$\psi_n(x) \propto e^{-y^2/2} H_n(y) \quad , \quad y = \sqrt{A}(x + \frac{B}{A}) \quad (33)$$

It follows from (27) that for discrete energy states the condition  $S_0^2 > V_0^2$  should hold and in this case all the energy levels exist. In table 4 we present some energy values for certain values of the parameters  $m, V_0$  and  $S_0$ .

### 3 Non-Hermitian scalar and vector potentials.

Here we shall consider non-Hermitian vector and scalar potentials yielding real energy. Generally there are two ways to generate non-Hermitian interaction. The first one involves considering complex coupling constants [9] while in the second approach one considers a complex coordinate shift [10]. Although the spectrum for potentials complexified via the first method can be obtained easily for Schrödinger equation, it requires extensive numerical computation in the case of KG equation. Consequently here we shall follow the second approach which is considerably simpler. Thus we consider vector and scalar potentials to be of the form

$$V(x) = V_0 f(x - ic) \quad , \quad S(x) = S_0 f(x - ic) \quad (34)$$

where  $c$  is a real constant. As an illustration let us consider  $f(x) = \tanh(x)$ . So from (11) we get

$$U_1^{NH}(x - ic) = -(S_0^2 - V_0^2) \operatorname{sech}^2(x - ic) + 2(mS_0 + EV_0) \tanh(x - ic) \quad , \quad \epsilon_1 = E^2 - m^2 - S_0^2 + V_0^2 \quad (35)$$

Comparing with (11) it is easy to see that the non-Hermitian potential  $U_1^{nh}(x)$  is shape invariant. Also the effective coupling constants do not contain any imaginary term and since the shape invariance condition (1) is coordinate independent, the (real) energy spectrum is given by (14).

Next to examine the behaviour of the system with respect to  $\mathcal{PT}$  symmetry, we note that a Hamiltonian  $H$  is said to be  $\mathcal{PT}$  invariant if [9]

$$\mathcal{P}\mathcal{T}H = H\mathcal{P}\mathcal{T} \quad (36)$$

where the space inversion ( $\mathcal{P}$ ) and time reversal ( $\mathcal{T}$ ) operators are defined by their actions on position, momentum and identity operators as [9]

$$\mathcal{P}x\mathcal{P} = -x, \mathcal{P}p\mathcal{P} = \mathcal{T}p\mathcal{T} = -p, \mathcal{T}(i.1)\mathcal{T} = -i.1 \quad (37)$$

Now using (37) it is easily seen that the non-Hermitian potential (35) is not  $\mathcal{PT}$  symmetric i.e,  $U_1^{NH}(x) \neq U_1^{NH*}(-x)$ . Thus (35) is a non-Hermitian and non  $\mathcal{PT}$  symmetric potential which has real spectrum. Clearly the complex coordinate shift method can be extended to the other cases considered here to obtain complex non  $\mathcal{PT}$  symmetric potentials with real spectrum.

Finally, we explain the equivalence of a Hermitian and non-Hermitian system in a different way. First let us consider the operator [11]

$$\eta = e^{c\mathcal{P}}, \quad c \text{ real} \quad (38)$$

with the properties

$$\eta V(x)\eta^{-1} = V(x - ic), \quad \eta p\eta^{-1} = p, \quad \eta\psi(x) = \psi(x - ic) \quad (39)$$

Then it follows that

$$\eta U_1(x)\eta^{-1} = U_1^{NH}(x - ic) \quad (40)$$

Then using (39) and (40) it follows that if  $E$  is an eigenvalue of the potential  $U_1(x)$  with eigenfunction  $\psi(x)$ , then  $E$  is also an eigenfunction of the potential  $U_1^{NH}(x - ic)$  with eigenfunction  $\eta\psi(x) = \psi(x - ic)$ . Thus the two Hamiltonians are intertwined and share the same real spectrum.

## 4 Conclusion

Here we have obtained exact solutions for KG equation for a class of (energy independent) vector and scalar potentials using the shape invariance technique. It has also been shown that the KG equation admits real eigenvalues in the presence of non Hermitian vector and scalar potentials. It may also be noted that here we have considered shape invariance based on translation of parameters and it would be interesting to examine the possibility of obtaining exact solutions of KG equation for self similar potentials.

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$n$	$E_n^+$	$s_1^+$	$s_2^+$	$E_n^-$	$s_1^-$	$s_2^-$
0	1.83314	3.98281	3.049	-1.88921	3.61226	3.41955
1	2.99136	3.32952	1.70229	-3.09985	2.48214	2.54967
2	3.39932	2.96043	0.071382	-3.68852	1.32395	1.70786

**Table 1.**  $s_1^\pm, s_2^\pm$  and  $E_n^\pm$  for  $m = 0.25$ ,  $S_0 = 4$  and  $V_0 = 0.35$

$n$	$E_n^+$	$s_1^+$	$s_2^+$	$E_n^-$	$s_1^-$	$s_2^-$
0	1.791	4.26304	2.76877	-1.90315	3.8953	3.13652
1	2.8921	3.71318	1.31863	-3.10908	2.87833	2.15348

**Table 2.**  $s_1^\pm, s_2^\pm$  and  $E_n^\pm$  for  $m = 0.5$ ,  $S_0 = 4$  and  $V_0 = 0.35$

$n$	$E_n^+$	$A^+$	$E_n^-$	$A^-$
0	1.08989	1.17139	-1.22751	1.02626
1	1.58713	1.20252	-1.6	1.00294

**Table 3.**  $E_n^\pm$  and  $A^\pm$  for  $m = 1.6$ ,  $S_0 = 4$  and  $V_0 = 0.25$

$n$	$E_n^+$	$E_n^-$
0	1.36234	-1.44984
1	2.39166	-2.47916
2	3.10035	-3.18785

**Table 4.**  $E_n^\pm$  for  $m = 0.5$ ,  $S_0 = 4$  and  $V_0 = 0.35$