ON THE COLEMAN INDICES OF VOTING POWER*

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Abstract

Coleman (1971) suggested two indices of voting power, power to prevent an action and power to initiate an action. This paper rigorously demonstrates relationship between the two indices and shows that they satisfy several attractive properties. It is also shown that an attainable upper bound of the power to prevent an action can as well be regarded as a suitable voting power index.

Key words: voting game, voting power, Coleman's indices, properties.

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1. Introduction

One of the most important concepts of political science is power. While power is a multi-faceted phenomenon, in this paper we will deal with the power of an agent in a collective decision making procedure modeled as a voting game. That is, we are concerned with the power of a member of a voting body or board that makes yes-or-no decision about a proposed resolution (or bill) by votes according to some unambiguous criterion. An index of power of a member of a body, a voter, is a numerical measure of the voter's influence to bring about the passage or defeat of a proposed resolution. It should be based on the voter's importance in casting the deciding vote.

The most well known indices of voting power are the Shapley-Shubik (1954) and Banzhaf (1965) indices. Essential to the construction of the former index is the concept of swing or pivotal voter. Given an ordering of voters, the swing voter for this ordering is the person whose deletion from the coalition of voters of which he is the last member in the given order, transforms this contracting coalition from a winning to a losing one. (A coalition of voters is called winning if passage of a resolution is guaranteed by 'yea' votes from exactly the voters in that coalition. Coalitions that are not winning are called losing.) The Shapley-Shubik index for voter i is the fraction of orderings in which i is the swing voter. Strictly speaking, this index is an application of the well-known Shapley value (Shapley, 1953) to a voting game, which is a formulation of a voting system in a coalitional form game. The Banzhaf power index of a voter is based on the number of coalitions in which the voter is pivotal. More precisely, it depends on the number of possibilities in which a voter is in the critical position of being able to change the voting outcome by changing his vote.

Alternatives and variations of the Shapley-Shubik and Banzhaf indices were suggested, among others, by Deegan and Packel (1978), Johnston (1978) and Holler (1982). Johnston argued that the Banzhaf index, which is based on the idea of a critical defection of a voter from a winning coalition, does not take into account the total number of voters whose defections from a given coalition are critical. Clearly, if a voter is the only person whose defection from a coalition is critical, then this gives a stronger indication of power than in the case where all person's defections are critical. This is the central idea underlying the Johnston index. According to Deegan and Packel (1978) only minimal winning coalitions should be considered in determining the power of a voter. (A coalition is called minimal winning if none of its proper subset is winning.) They suggested an index under the assumptions that all minimal winning coalitions are equiprobable and any two voters belonging to the same minimal winning coalition should enjoy the same power. The Holler index is given by the number of minimal winning coalitions containing the voter divided by the sum of such numbers across all voters.

Coleman (1971) pointed out the notion of voting power quantified by the Shapley-Shubik index is not the power to affect the outcome of voting body in the usual sense, that is, whether a resolution is passed or blocked. Rather, it is the power of the voter to appropriate a share in the fixed prize of victory, available only to the winning camp. For the sake of convenience, the size of the prize is assumed to be 1 unit, so that the sum of the indices across voters is always 1. The indices suggested by Deegan and Packel (1978), Johnston (1978) and Holler (1982) also reflect this notion of power (see Felsenthal and Machover, 1998, chapter 6).

Coleman (1971) also suggested two indices of voting power. His first index, the power of voter i to prevent action, is given by the number of coalitions in which voter i is swing divided by the number of winning coalitions in the game. The idea is that given that the voting body makes a positive decision, it determines the conditional probability that voter i will be able to prevent the decision by changing side. Coleman's second index, the power of voter i to initiate act, is defined as the number of coalitions in which voter i is swing divided by the total number of losing coalitions in the game. Brams and Affuso (1976) pointed out that these two indices are proportional to the Banzhaf index and to each other. Dubey and Shapley (1979) showed that the harmonic mean of these two indices becomes the Banzhaf index. Felsenthal and Machover (1998) demonstrated by constructing examples that in going from one voting game to another, while the Banzhaf index of a voter may reduce slightly, his loss of power to initiate action may be very considerable. In contrast, he may gain a lot of power to prevent action. Thus, these two indices 'can give you information that you cannot get by looking at β' (the Banzhaf index) alone' (Felsenthal and Machover, 1998, p.51). However, these indices have not been discussed much in the literature. Therefore, studying relationship between them in different types of voting games and demonstrating their properties rigorously will be a worthwhile exercise.

Felsenthal and Machover (1998) proposed three postulates, namely, the transfers principle, the bloc principle and the dominance principle as the major desiderata for an index of power. The transfers principle requires that the power of voter j should not increase if he donates a part of his voting right to another voter i. According to the bloc principle, under a voluntary merger between two voters a and b, where b is capable of affecting the voting outcome, the power of the merged entity will be at least as large as that of a. The dominance principle demands that if voter *j*'s contribution to the victory of a resolution can be equaled or bettered by another voter i, then i should not possess lower power than *j* [1].

The objective of this paper is to examine the two Coleman indices in the light of different postulates suggested by Felsenthal and Machover (1998). It is shown rigorously that these indices satisfy the three principles. We also establish a formal relation between these two indices. We then show that an attainable upper bound of the power of a voter to prevent act, namely, the number of winning coalitions in which the voter is swing divided by the total number of winning coalitions containing the voter, can also be regarded as a suitable index of power in the sense that it satisfies the transfers, bloc and dominance principles. We refer to this upper bound as the transformed preventive index.

This paper is arranged in several sections. Section 2 deals with the background material. In section 3 we discuss the postulates that an index of voting power should satisfy. In section 4 we discuss the properties of the Coleman indices. The transformed preventive index has also been proposed in this section. Finally, section 5 concludes.

2. The Background

It is possible to model a voting situation as a coalitional form game, the hallmark of which is that any subgroup of players can make contractual agreements among its members independently of the remaining players. Let $N = \{1,2,...,n\}$ be a set of players. The power set of N, that is, the collection of all subsets of N, is denoted by 2^N. Any member of 2^N is called a coalition. A coalitional form game with the player set N is a pair $(N;V)$, where $V:2^N \to R_+$ such that $V(\phi)=0$, where R_+ is the nonnegative part of the real line. For any coalition S, the real number $V(S)$ is the worth of the coalition, that is, this is the amount that S can guarantee to its members. For any set S, $|S|$ will denote the number of elements of S.

We frame a voting system as a coalitional form game by assigning the value 1 to any coalition which can pass a bill and 0 to any coalition which cannot. In this context, a player is a voter and the set $N=\{1,2,...,n\}$ is called the set of voters. Throughout the paper we assume that voters are not allowed to abstain from voting. A coalition S will be called winning or losing depending on whether it can or cannot pass a resolution.

Definition 1: Given a set of voters N, a voting game associated with N is a pair $(N;V)$, where $V: 2^N \to \{0,1\}$ satisfies the following conditions:

(i) $V(\phi)=0$.

(ii) $V(N)=1$.

(iii) If $S \subset T, S,T \in 2^N$, then $V(S) \le V(T)$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. It follows that the empty coalition ϕ is losing (condition (i)) and the grand coalition N is winning (condition (ii)). All other coalitions are either winning or losing. Condition (iii) can be regarded as a monotonicity principle. It ensures that if a coalition S can pass a bill, then any superset T of S can pass it as well. A voting game $G=(N;V)$ is called proper if $V(S)=V(T)=1$ implies that $S \cap T \ne \phi$. According to this condition two winning coalitions cannot be disjoint. On the other hand a voting game is called improper if there exists at least two winning coalitions which are disjoint. The collection of all voting games is denoted by **F.** For any $G=(N;V)$, we write $\mathbf{W}_G$ $(\mathbf{L}_G)$ for the set of all winning (losing) coalitions associated with G. Thus, for any $S \subseteq N$, $V(S)=1(0)$ is equivalent to the condition that $S \in \mathbf{W}_G(\mathbf{L}_G)$. For any game $G=(N;V)$ and for any voter $i \in N$, let $\mathbf{W}_G^i$ be the set of all winning coalitions that contain i.

Definition 2: A voting game $G=(N;V)$ with the voter set N, is called decisive if for all $S\in 2^N$, $V(S)+V(N-S)=1$.

Definition 3: Given a set of voters N, let $(N;V)$ be a voting game.

(i) For any coalition $S\in 2^N$, we say that $i\in N$ is swing in S if $V(S)=1$ but $V(S-\{i\})=0$.

(ii) For any coalition $S\in 2^N$, $i\in N$ is said to be swing outside S if $V(S)=0$ but $V(S\cup\{i\})=1$.

(iii) A coalition $S\in 2^N$, is said to be minimal winning if $V(S)=1$ but there does not exist $T\subset S$ such that $V(T)=1$.

Thus, voter i is swing, also called pivotal or key, in the winning coalition S if his deletion from S makes the resulting coalition $S-\{i\}$ losing. Similarly, voter i is swing outside the losing coalition S if his addition to S makes the resulting coalition $S\cup\{i\}$ winning. For any voter i, the number of winning coalitions in which he is swing is same as the number of losing coalitions outside which he is swing (Burgin and Shapley, 2001, Corollary 4.1). For any game $G=(N;V)\in\mathbf{F}$ and $i\in N$, we write $m_i(G)$ to denote this common number. Equivalently, $m_i(G)$ is the number of coalitions for which voter i is swing in G. It is often said that $m_i(G)$ is the number of swings of voter i.

Definition 4: For a set of voters N, let $(N;V)$ be a voting game. A voter $i\in N$ is called a dummy in $(N;V)$ if he is never swing in the game. A voter $i\in N$ is called a non-dummy in $(N;V)$ if he is not dummy in $(N;V)$.

Following Felsenthal and Machover (1998), we have

Definition 5: For a voting game $(N;V)$ with the set of voters N, a voter $i\in N$ is called a dictator if $\{i\}$ is the sole minimal winning coalition of the game.

A dictator in a game is unique. If a game has a dictator, then he is the only swing voter in the game.

Definition 6: Let $G = (N; V)$ be a voting game.

(i) A coalition S in G is a blocking coalition if its complement is losing, that is, $S \in 2^N$ is blocking if $N - S \in \mathbf{L}_G$.

(ii) A voter $i \in N$ is called a blocker in G if $\{i\}$ is blocking coalition.

A blocker can prevent any decision by leaving a winning coalition. We can characterize a blocker as a voter who belongs to every minimal winning coalition. A dictator in a game is the unique blocker in the game. However, a game may have several blockers. A game with two or more blockers does not have a dictator. In a decisive game winning and blocking are equivalent conditions.

A very important voting game is a weighted majority game.

Definition 7: For a set of voters $N = \{1, 2, ..., n\}$, a weighted majority game is a quadruplet $G = (N; V; \mathbf{w}; q)$, where $\mathbf{w} = (w_1, w_2, ..., w_n)$ is the vector of nonnegative weights of the $n = |N|$ voters in N, q is a nonnegative real number quota such that $q \leq \sum_{i=1}^{n} w_i$ and for any $S \in 2^N$,

$$V(S) = 1 \quad \text{if } \sum_{i \in S} w_i \geq q$$

$$= 0 \text{ otherwise.}$$

That is, the i^{th} voter casts w_i votes and q is the quota of votes needed to pass a bill. Note that a weighted majority game satisfies condition (i) - (iii) of definition 1. A weighted majority game $G = (N; V; \mathbf{w}; q)$ will be proper if $\sum_{i=1}^{n} w_i < 2q$. For an improper game we have $\sum_{i=1}^{n} w_i \geq 2q$.

Definition 8: Given a non-empty set X, a $t-$partition of X is a collection of coalitions $\mathbf{X}=(X_1, X_2, ..., X_t)$, where

1. $X_1, X_2, ..., X_t \subseteq X$
2. $X_i \cap X_j = \phi$, $i, j = 1,2,...,t$; $i \neq j$
3. $X_1 \cup X_2 \cup ... \cup X_t = X$.

When $t = 2$, we say that the set X has been bipartitioned.

Definition 9: Given $G=(N;V) \in \mathbf{F}$, a yes-no bipartition B is a map from N to $\{-1,1\}$. A player is assigned the value 1 if he votes 'yes' and –1 if he votes 'no'. The 'yes' voting camp is referred to as B^+, and the 'no' voting camp is denoted by B^-.

Definition 10: Given $G=(N;V) \in \mathbf{F}$, a voter $i \in N$ is said to agree with the outcome of a yes-no bipartition B in the game G, if either of the following two conditions hold:

1. $B(i)=1$ and $B^+ \in \mathbf{W}_G$.
2. $B(i)=-1$ and $B^+ \notin \mathbf{W}_G$.

The statement that i agrees with the outcome of a bipartition means that the decision goes i's way: i votes 'yes' and the bill is passed or i votes 'no' and the bill is rejected.

Definition 11: Given $G=(N;V) \in \mathbf{F}$, we say that $j \in N$ dominates $i \in N$ if whenever S is a coalition such that $j \notin S$ and $S \cup \{i\} \in \mathbf{W}_G$, then $S \cup \{j\} \in \mathbf{W}_G$.

3. Properties for an Index of Voting Power

Following Felsenthal and Machover (1995, 1998), we argue that an index of power $H_i : \mathbf{F} \to R_+$, which gives us an idea of the influence of a voter i over the outcome of the voting procedure, should satisfy the following postulates:

1. **Vanishing Just For Dummies** (**VJD**): For any $G=(N;V)\in\mathbf{F}$, $H_i(G)=0$ if and only if $i\in N$ is a dummy.

2. **Iso-invariance** (**INV**): Let $G=(N;V)$ and $G^{'}=(N^{'};V')\in\mathbf{F}$ be two isomorphic games, that is, there exists a bijection h of N onto $N^{'}$ such that for all $S\subseteq N$, $V(S)=1$ if and only if $V(h(S))=1$, where $h(S)=\{h(x):x\in S\}$. Then $H_i(G)=H_i(G')$.

3. **Ignoring Dummies** (**IGD**): For any $G=(N;\ V)\in\mathbf{F}$ and for any dummy $d\in N$, $H_i(G)=H_i(G_{-d})$ for all $i\in N-\{d\}$, where G_{-d} is the game obtained from G by excluding d. Similarly, $H_i(G)=H_i(G_{+d})$, where G_{+d} is the game obtained from G by including $d\notin N$ as a dummy.

4. **Dominance** (**DOM**): For any $G=(N;\ V)\in\mathbf{F}$ if j dominates i, then $H_j(G)\geq H_i(G)$.

5. Transfers Principle (TRP): Let G and G' be two voting games with the same voter set N and let i and j be two distinct voters such that the following three conditions hold:

T1. Whenever i and j are on the same side of a yes-no bipartition B, the outcome of B is identical in G and G'.

T2. Whenever i and j are on opposite sides of a yes-no bipartition B and i agrees with the outcome of B in G then i also agrees with the outcome of B in G'.

T3. There exists at least one yes-no bipartition B such that i agrees with the outcome of B in G' but not in G.

Then we shall say that G' arises from G by a transfer from j to i.

We say that an index H satisfies the *transfer postulate,* if whenever the above conditions hold, $H_i(G') \geq H_i(G)$. Likewise, $H_j(G') \leq H_j(G)$.

6. **Bloc Postulate** (**BOP**): Given $G = (N;V)$, assume that the voters $i, j \in N$ are amalgamated into one voter ij. Then the post-merger voting game is $G' = (N';V')$, where $N' = N - \{i, j\} \cup \{ij\}$, and $V'(S) = V(S)$ if $S \subseteq N' - \{ij\}$ and $V'(S) = V((S - \{ij\} \cup \{i, j\})$ if $ij \in S$. The bloc postulate requires that for any voter $k \in \{i, j\}$, $H_{ij}(G') \geq H_k(G)$ provided that $l \in \{i, j\} - \{k\}$ is non-dummy.

By definition, a power index is always nonnegative. **VJD** shows that the necessary and sufficient condition that the power index attains its lower bound, zero, is that the concerned voter is a dummy. If a voter is a dummy, then he has no influence over the final outcome of the voting procedure. In no situation can he change the outcome by changing his vote. Since the essence of power of a voter lies in his capability of being a pivotal voter, a voter's power should be minimal (zero) if he is a dummy (see also Dubey, 1975, Dubey and Shapley, 1979, Taylor, 1995 and Burgin and Shapley, 2001). A similar argument applies from the reverse direction.

INV is an anonymity condition. It says that any reordering of the voters does not change the power enjoyed by a voter. Influence of a voter over the outcome does not depend on the irrelevant characteristics of the voter, like his name or place of residence etc. Even if those characteristics change (e.g. he swaps his place of residence with another voter), his influence remains unaltered.

Since a dummy can never affect the outcome of voting it is natural to expect that if a dummy is excluded from a voting game, the powers of the remaining voters remain unaltered. Likewise, inclusion of a dummy in the game will not change the powers of the existing voters. This is essentially what **IGD** says.

According to **DOM**, if j dominates i then any contribution that i can make to the victory of a coalition should not be higher than that of j. A special case of this is a monotonicity principle, which demands that in a weighted majority game, a voter with a larger voting weight cannot have less power than a voter with a smaller weight.

TRP demands that power of voter j should not increase under a transfer of a part of his voting right to another voter i. Likewise the power of voter i should not reduce under the transfer. In the case of weighted majority games **TRP** means that voter j cannot gain power by distributing some of his voting weight to another voter i. Similarly, voting power of i cannot reduce when he receives some voting weight from another voter j.

BOP can be interpreted as follows. When a voter k acquires the voting power of a nondummy voter l, then voting power of the bloc consisting of these two voters should not be lower than that of k. In other words, k is not losing power by swallowing the power of a nondummy voter l. This is quite reasonable intuitively. A person will not join a bloc if the voting right of the bloc is lower than his own voting right.

Note that **TRP** is formulated in terms of power of either the donor or the recipient of the voting right. We can also have a relative version of **TRP,** which involves powers of both j and i, the donor and the recipient of the voting right.

Relative Transfers Principle (**RTP**): Let G and G' be the games as given in **TRP**. Then

$$\frac{H_j(G')}{H_i(G')} \leq \frac{H_j(G)}{H_i(G)},$$

where H_i's are assumed to be positive.

Clearly, **TRP** implies **RTP**. But the converse is not true. For instance, the normalized Banzhaf index $\frac{m_i}{\sum_{j=1}^{|N|} m_j}$ satisfies **RTP** but not **TRP**.

We can also formulate a relative version of **IGD**.

Relative Dummy Ignoring Principle (RDP): Let G and G_{-d} be the games as given in **IGD**. Then for any $i, j \in N-\{d\}$, $\frac{H_i(G_{-d})}{H_j(G_{-d})} = \frac{H_i(G)}{H_j(G)}$,

where H_j's are assumed to be positive.

RDP says that the power of voter i relative to another voter j remains unaltered if a dummy is excluded from the game. Obviously, we can analogously formulate a relative dummy inclusion principle. Clearly, all indices that satisfy **IGD** will satisfy **RDP** but the converse is not true. For instance, the index $m_i(G)$ satisfies **RDP** but not **IGD**.

4. The Coleman Indices

The power of an individual member of a voting body, when power is interpreted as 'influence' over the outcome of the voting procedure, can be exercised in two ways: the member can initiate an action or can stall an action from being taken. The difference between these two becomes obvious if one considers the case of a 'vetoer' or a bloc voter. By the definition of a bloc voter, his 'yes' vote is necessary but not sufficient to obtain the passage of a bill. So while the blocker can stall the passage of a bill by individual action (without reference to how others vote), he cannot pass a bill by individual action. For this he needs to consider how others vote.

To capture these two aspects of power, Coleman (1971) suggested two different power indices for individual voters, namely, preventive power index and initiative power index, which give us a measure of an individual's power to prevent action and to initiate action respectively.

Preventive Index

The Coleman preventive power index for voter i is defined as the number of winning coalitions in which i is decisive, divided by the total number of winning coalitions in the game. Formally, in a game G, where $m_i(G)$ is the number of winning coalitions in which i is critical, voter i's power to bloc action is calculated as

$$P_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \in S}} \left[V(S) - V(S \setminus \{i\})\right]}{\sum_{S \subseteq N} V(S)} = \frac{m_i(G)}{\left|\mathbf{W}_G\right|}. \qquad (1)$$

The index can be interpreted as voter i 's probability to bloc a bill. $\left|\mathbf{W}_G\right|$ is the number of possible situations which lead to the bill being passed. Since voter i 's 'yes' vote is pivotal in m_i of these situations, given that other voters do not change their vote, i can bloc the bill by changing his vote to 'no' only in these situations. So the probability that voter i can bloc a bill is $\frac{m_i(G)}{\left|\mathbf{W}_G\right|}$. More clearly, $P_i(G)$ gives voter i 's probability of being decisive (or swing), conditional to the proposal being accepted if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $\frac{1}{2}$ for each and all the voters vote independently (Laruelle and Valenciano, 2002a).

Initiative Index

The Coleman initiative power index for voter i is defined as the number of losing coalitions outside which i is critical divided by the number of losing coalitions in the game. Formally, voter i 's power to initiate action is calculated as

$$I_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \notin S}} \left[V(S \cup \{i\}) - V(S)\right]}{\sum_{S \subseteq N} [1 - V(S)]} = \frac{m_i(G)}{\left|\mathbf{L}_G\right|} = \frac{m_i(G)}{2^{|N|} - \left|\mathbf{W}_G\right|}. \qquad (2)$$

The index can be interpreted as voter i 's probability to initiate action. While in Coleman's preventive power index, swings of a voter i are regarded as measuring his ability to destroy a winning coalition, in Coleman's initiative power index, swings are thought of as measuring a voter's ability to turn an otherwise losing coalition into a wining one. Laruelle and Valenciano (2002a) provided an interpretation of this index in terms of voter i 's being decisive conditional to the proposal being rejected.

Dubey and Shapley (1979) showed that the non-normalized Banzhaf index $\beta_i' = \frac{m_i}{2^{|N|-1}}$, which is a measure of power of voter i, is the harmonic mean of these two indices. More precisely,

$$\frac{1}{\beta_i'} = \frac{1}{2}\left(\frac{1}{P_i} + \frac{1}{I_i}\right).$$

Having defined both the preventive and initiative power indices, we are now in a position to compare their relative strengths for an individual voter. We know that a voting game G satisfying the conditions in definition 1 can either be proper or improper. The following theorem compares I_i and P_i for proper games.

<u>Theorem 1:</u> If $G = (N;V)$ is a proper game, then an individual's power to initiate action I_i is always less than or equal to the power to prevent action P_i.

<u>Proof:</u> Since the number of all possible coalitions of the set of players in a game $G = (N;V)$ is $2^{|N|}$,

$$|\mathbf{W}_G| + |\mathbf{L}_G| = 2^{|N|}. \qquad (3)$$

Since the game is proper, we must have $|\mathbf{W}_G| \le 2^{|N|-1}$ (see Burgin and Shapley, 2001 and Barua, Chakravarty and Roy, 2003). It therefore follows from (3) that

$$|\mathbf{W}_G| \le |\mathbf{L}_G|.$$

Or, $\frac{1}{|\mathbf{L}_G|} \le \frac{1}{|\mathbf{W}_G|}$.

This implies that $\frac{m_i(G)}{|\mathbf{L}_G|} \le \frac{m_i(G)}{|\mathbf{W}_G|}$.

Hence $I_i \le P_i$. □

The demonstration of the relationship between I_i and P_i for improper games relies on lemma 2, whose proof requires lemma 1.

<u>Lemma 1:</u> Let $G = (N; V; \mathbf{w}; q)$ be an improper weighted majority game. Then there cannot exist any bipartition of the set of players N, such that both the coalitions are losing.

<u>Proof:</u> We prove this result by contradiction.

Suppose a bipartition of N exists such that both the coalitions are losing. Let (N_1, N_2) be such a bipartition. Then $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \phi$. Since N_1 and N_2 are both losing coalitions, we have the following set of inequalities:

$$\sum_{i \in N_1} w_i < q$$

and $\sum_{i \in N_2} w_i < q$.

Adding both sides we get $\sum_{i \in N_1} w_i + \sum_{i \in N_2} w_i < 2q$.

Or, $\sum_{i \in N} w_i < 2q$.

Or, $\sum_{i=1}^{|N|} w_i < 2q$.

This contradicts the improperness of $G = (N; V; \mathbf{w}; q)$. Hence the proof of lemma 1. □

Lemma 1 underlines an important distinction between proper and weighted improper games. While in proper games, there can be no bipartition of the set of players such that both the coalitions are winning, in weighted improper games, we can have no bipartition of the player set such that both the coalitions are losing.

We are now ready to prove lemma 2.

<u>Lemma 2:</u> Let $G = (N; V; \mathbf{w}; q)$ be a weighted majority game. Then G is improper if and only if $|\mathbf{L}_G| < |\mathbf{W}_G|$.

<u>Proof:</u> Suppose G is improper. By lemma 1, if $S \in \mathbf{L}_G$, then $N - S \in \mathbf{W}_G$. Then $S \to N - S$ defines a one to one map from $\mathbf{L}_G$ into $\mathbf{W}_G$. Hence $|\mathbf{L}_G| \leq |\mathbf{W}_G|$. Now, since G is improper, $\exists S^* \subseteq N$ such that both S^* and $N - S^*$ are in $\mathbf{W}_G$. Hence, S^* is not the image of any coalition $S \in \mathbf{L}_G$ under this map. Hence $|\mathbf{L}_G| < |\mathbf{W}_G|$.

Conversely, suppose that $|\mathbf{L}_G| < |\mathbf{W}_G|$. Let the game G be a proper game. Then we cannot obtain any bipartition of the player set such that both the coalitions are winning. So, if a coalition $S \in \mathbf{W}_G$, then $N - S \in \mathbf{L}_G$. Thus, with every coalition S in $\mathbf{W}_G$, we can associate a unique element $N - S$ in $\mathbf{L}_G$. Therefore, $|\mathbf{W}_G| \le |\mathbf{L}_G|$. This is a contradiction. Therefore, G is an improper game.

Hence the proof of lemma 2. □

The following result drops out as an interesting corollary to lemma 2.

Corollary 1: A weighted majority game G is proper if and only if $|\mathbf{W}_G| \le |\mathbf{L}_G|$.

We can now compare I_i and P_i for improper games.

<u>Theorem 2</u>: Let $G = (N;V)$ be an improper game.

(i) Then if G can be represented as a weighted majority game, we have $|\mathbf{W}_G| > 2^{|N|-1}$. Consequently, a non-dummy individual's power to initiate action I_i is always greater than the power to prevent action P_i.

(ii) However, if G is not representable by a weighted majority game, then I_i need not be greater than P_i.

<u>Proof:</u>

(i) We know that $|\mathbf{W}_G| + |\mathbf{L}_G| = 2^{|N|}$. By lemma 2 for an improper weighted majority game, $|\mathbf{L}_G| < |\mathbf{W}_G|$. Hence, we can say that $|\mathbf{W}_G| > 2^{|N|-1}$. It then follows that

$$\frac{m_i(G)}{|\mathbf{L}_G|} > \frac{m_i(G)}{|\mathbf{W}_G|}, \text{ if } i \text{ is non-dummy.}$$

Therefore, $I_i > P_i$.

(ii) To show this part of the theorem we consider the following example.

$N = \{a,b,c,d\}$ and $\mathbf{W}_G = \{\{a,b\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$.

Clearly, it is an improper game. Suppose that the game is a weighted majority game. Hence by lemma 2, $|\mathbf{L}_G| < |\mathbf{W}_G|$. But in this game $|\mathbf{W}_G| = 7 < 2^{4-1} = 8$, $|\mathbf{L}_G| = 9 > |\mathbf{W}_G|$ =7, which contradicts the preceding inequality . Therefore, the game is not representable as a weighted majority game. Thus $I_i < P_i$ $(i = a,b,c,d)$.

Hence the proof of theorem 2. □

We are now in a position to study the Coleman indices in the light of the properties discussed in section 3.

Theorem 3 below discusses the preventive power index P_i, defined in (1), in terms of the postulates laid down by Felsenthal and Machover (1995, 1998).

<u>Theorem 3:</u>

(a) The Coleman index of the power to prevent act, P_i, satisfies VJD, INV, IGD, DOM, TRP and BOP for all voting games.

(b) The index P_i achieves its upper bound of 1 if and only if voter i is a bloc voter.

(c) If $\hat{G} = (\hat{N}, \hat{V}) \in \mathbf{F}$ is obtained from $G = (N,V) \in \mathbf{F}$ by adding $b \notin N$ as a blocker in $\hat{G}$, then for any two non-dummy voters $i, j \in N$, we have

$$\frac{P_i(G)}{P_j(G)} = \frac{P_i(\hat{G})}{P_j(\hat{G})}.$$

<u>Proof:</u>

(a) Given that $i \in N$ is a dummy in the game $G = (N;V) \in \mathbf{F}$, we have $m_i(G) = 0$ which in turn shows that $P_i = 0$. Conversely, $P_i = 0$ implies that $m_i(G) = 0$, that is, i is a dummy. Hence P_i fulfils the **VJD**.

A permutation of the voter set does not alter $m_i(G)$ and $\mathbf{W}(G)$, hence $P_i(G)$ satisfies **INV**.

To check verification of the **IGD**, note that we can write $\mathbf{W}(G)$ as

$$\mathbf{W}(G) = \mathbf{W}^1(G) \cup \mathbf{W}^2(G),$$

where $\mathbf{W}^1(G) = \{S \in \mathbf{W}(G) : d \in S\}$ and $\mathbf{W}^2(G) = \{S \in \mathbf{W}(G) : d \notin S\}$.

Clearly, $\mathbf{W}^2(G)$ coincides with $\mathbf{W}(G_{-d})$, the collection of all winning coalitions of the game G_{-d}. Since $S \subseteq N$ is winning in G if and only if $S - \{d\}$ is winning in G_{-d}, it follows that the mapping $S \to S - \{d\}$ is a bijection of $\mathbf{W}^1(G)$ onto $\mathbf{W}^2(G)$. Hence

$$\begin{aligned} |\mathbf{W}(G)| &= |\mathbf{W}^1(G)| + |\mathbf{W}^2(G)| \\ &= 2|\mathbf{W}^2(G)| \\ &= 2|\mathbf{W}(G_{-d})|. \end{aligned}$$

Therefore, $\left|\mathbf{W}_{G_{-d}}\right| = \frac{|\mathbf{W}_G|}{2}$. By a similar argument $m_i(G_{-d}) = \frac{m_i(G)}{2}$. Hence $P_i(G) = P_i(G_{-d})$. We can establish analogously that $P_i(G) = P_i(G_{+d})$. Thus P_i satisfies the **IGD**.

We will now demonstrate that P_i satisfies **TRP**.

To understand the conditions **T1**- **T3**, we first define certain sets.

Let G and G' be two voting games with the same voter set N. Let i and j be two

Condition **T1**:

This condition means that if i and j vote *together* in favour of the bill or against the bill, the outcome of the voting process in G' is same as in G. Consider a coalition $S \subseteq N$. Let $i, j \in S$.

Suppose in the yes-no bipartition B, S votes 'yes' and $N - S$ votes 'no'. Condition **T1** says that if the outcome of B in the game G is positive (the bill is passed), then the outcome of B in the game G' will also be positive. However if the outcome in G is negative (the bill is rejected), it will also be negative in G'. That is, if S is winning (losing) in G, it will also be winning (losing) in the game G' and vice versa. That is, $\mathbf{W}_G^{i,j} = \mathbf{W}_{G'}^{i,j}$.

Suppose in B, S votes 'no' and $N - S$ votes 'yes'. Condition **T1** says that if the outcome of B in the game G is positive (the bill is passed), then the outcome of B in the game G' will also be positive. However if the outcome in G is negative (the bill is rejected), it will also be negative in G'. That is, if $N - S$ is winning (losing) in G, it will also be winning (losing) in the game G' and vice versa. That is, $\mathbf{W}_G^{\sim(i,j)} = \mathbf{W}_{G'}^{\sim(i,j)}$.

So, condition **T1** can be summarized as

$$\mathbf{W}_G^{i,j} = \mathbf{W}_{G'}^{i,j}$$

$$\mathbf{W}_G^{\sim(i,j)} = \mathbf{W}_{G'}^{\sim(i,j)}$$

Condition **T2**:

Consider a coalition $S \subseteq N$. Let $i \in S$. Then $j \in N - S$.

Then, condition **T2** says that if S is winning in G, it will also be winning in G', and if $N - S$ is losing in G, then it will also be losing in G'. This means $\mathbf{W}_G^i \subseteq \mathbf{W}_{G'}^i$ and $\mathbf{L}_G^j \subseteq \mathbf{L}_{G'}^j$.

So, condition **T2** can be summarized as

$$\mathbf{W}_G^i \subseteq \mathbf{W}_{G'}^i \Rightarrow \left|\mathbf{W}_G^i\right| \leq \left|\mathbf{W}_{G'}^i\right|$$

$$\mathbf{L}_G^j \subseteq \mathbf{L}_{G'}^j \Rightarrow \left|\mathbf{W}_G^j\right| \geq \left|\mathbf{W}_{G'}^j\right|$$

Condition **T3**:

Consider a coalition $S \subseteq N$. Let $i \in S$. Then $j \in N - S$.

Condition **T3** says that there must exist at least one yes-no bipartition B such that

(i) If S votes 'yes' in B, the bill is passed in G' but not in G. That is, there must exist at least one coalition S, $i \in S$, $j \notin S$, such that S is losing in G but becomes a winning coalition in G'.

Or

(ii) If S votes 'no' in B, the bill is passed in G but not in G'. That is, there must exist at least one coalition S, $i \in S$, $j \notin S$, such that $N - S$ is winning in G but becomes a losing coalition in G'.

Let $A_1 = \{S \subseteq N : i \in S, j \notin S; S \notin \mathbf{W}_G^i, S \in \mathbf{W}_{G'}^i\}$ and

$A_2 = \{S \subseteq N : j \in S, i \notin S;\ S \in \mathbf{W}_G^j$ but $S \notin \mathbf{W}_{G'}^j\}$.

Condition **T3** can be summarized as below:

If $|A_1| = \alpha_1$ and

$|A_2| = \alpha_2$

Then $\alpha_1 + \alpha_2 \geq 1$

That means,

$\left|\mathbf{W}_{G'}^i\right| > \left|\mathbf{W}_G^i\right|$ or $\left|\mathbf{W}_{G'}^j\right| < \left|\mathbf{W}_G^j\right|$

It is easy to note that

$$\Delta\left|\mathbf{W}_G^i\right| = \left|\mathbf{W}_{G'}^i\right| - \left|\mathbf{W}_G^i\right| = \alpha_1$$

$$\Delta\left|\mathbf{W}_G^j\right| = \left|\mathbf{W}_{G'}^j\right| - \left|\mathbf{W}_G^j\right| = -\alpha_2$$

Note that $\Delta\left|\mathbf{W}_G\right| = \Delta\left|\mathbf{W}_G^i\right| + \Delta\left|\mathbf{W}_G^j\right| + \Delta\left|\mathbf{W}_G^{i,j}\right| + \Delta\left|\mathbf{W}_G^{\sim(i,j)}\right| = \alpha_1 + (-\alpha_2) + 0 + 0$.

Consider an element S in the set A_1. Then $S \notin \mathbf{W}_G^i$ but $S \in \mathbf{W}_{G'}^i$. Now $S \notin \mathbf{W}_G^i$ implies that $S \setminus \{i\}$ is also losing in G. Since condition **T1** says that $\mathbf{W}_G^{\sim(i,j)} = \mathbf{W}_{G'}^{\sim(i,j)}$, this means that $S \setminus \{i\}$ is losing in G' as well. But since $(S \setminus \{i\}) \cup \{i\}$ is winning in G', it implies that i is a critical member of these coalitions in the game G' but not in G.

Now consider an element S in the set A_2. Then $S \in \mathbf{W}_G^j$ but $S \notin \mathbf{W}_{G'}^j$. Now $S \notin \mathbf{W}_{G'}^j$ implies that $S \setminus \{j\}$ is losing in G'. By condition **T1** we know that $S \setminus \{j\}$ must be losing in G as well. But $(S \setminus \{j\}) \cup \{j\}$ is winning in G. So j is a critical member of these coalitions in the game G but not in G'.

Though the set $\mathbf{W}_G^{i,j}$ is the same as the set $\mathbf{W}_{G'}^{i,j}$, i might become a critical player in some of these coalitions, in which he was previously non-critical. Let the number of these coalitions be α_3. Formally, let $A_3 = \{S \subseteq N:\ i, j \in S;\ S \in \mathbf{W}_G^{i,j}$ but $S \setminus \{i\} \in \mathbf{W}_G^j$, and $S \in \mathbf{W}_{G'}^{i,j}$ but $S \setminus \{i\} \notin \mathbf{W}_{G'}^j\}$. Then $|A_3| = \alpha_3$.

Again there might be some winning coalitions containing both i and j in which j is critical in the game G, but ceases to be critical in G'. Let the number of these coalitions be α_4. Formally, let $A_4 = \{S \subseteq N: i, j \in S;\ S \in \mathbf{W}_G^{i,j}$ but $S \setminus \{j\} \notin \mathbf{W}_G^i$, and $S \in \mathbf{W}_{G'}^{i,j}$ but $S \setminus \{j\} \in \mathbf{W}_{G'}^i\}$. Then $|A_4| = \alpha_4$.

It is easy to note that there cannot exist any winning coalition S containing both i and j such that i is a critical member of S in G but not in G'. To see this, suppose that such a coalition $S \in \mathbf{W}_G^{i,j}$ exists. Then $S \setminus \{i\} \notin \mathbf{W}_G^j$ and $S \setminus \{i\} \in \mathbf{W}_{G'}^j$. This violates condition **T2**, which says that if a coalition containing j and not i is losing in G, then it must be losing in G' as well.

By a similar reasoning we can note that there cannot exist any winning coalition S containing both i and j such that j is a non-critical member of S in the game G, but a critical member in the game G'.

Therefore, we have the following:

$\Delta m_i = m_i(G') - m_i(G) = \alpha_1 + \alpha_3$.

$\Delta m_j = m_j(G') - m_j(G) = -(\alpha_2 + \alpha_4)$

$\Delta|\mathbf{W}_G| = |\mathbf{W}_{G'}| - |\mathbf{W}_G| = \alpha_1 - \alpha_2$.

Note that α_1, α_2, α_3, $\alpha_4 \geq 0$.

It is obvious that $\Delta|\mathbf{W}_G|$ could be non-negative or non-positive. Accordingly $\Delta|\mathbf{L}_G|$ could be non-positive or non-negative.

Case 1: $\Delta|\mathbf{W}_G| \leq 0$.

(i) Then since, $\Delta m_i \geq 0$, P_i will rise. That is, the power to prevent action of the recipient will not fall.

(ii) From the above discussion it is clear that $\Delta m_j \leq 0$. Since α_1, α_2, α_3, $\alpha_4 \geq 0$, it is obvious that $\alpha_2 + \alpha_4 \geq \alpha_2 - \alpha_1$. That is, $-\Delta m_j \geq -\Delta|\mathbf{W}_G|$. Also $|\mathbf{W}_G| \geq$ $m_j(G)$. Therefore, we have

$$-\Delta m_j |\mathbf{W}_G| \geq -m_j(G).\Delta|\mathbf{W}_G|. \qquad (4)$$

Therefore,

$$P_j(G') - P_j(G) = \frac{m_j(G) + \Delta m_j}{|\mathbf{W}_G| + \Delta|\mathbf{W}_G|} - \frac{m_j(G)}{|\mathbf{W}_G|}$$

$$= \frac{\Delta m_j |\mathbf{W}_G| - m_j(G)\Delta|\mathbf{W}_G|}{(|\mathbf{W}_G| + \Delta|\mathbf{W}_G|)|\mathbf{W}_G|} \leq 0 \text{ (Using (4).)}$$

That is,

$P_j(G') \leq P_j(G)$.

Thus, the power to prevent action of the donor will not rise.

Case 2: $\Delta\left|\mathbf{W}_G\right| > 0$

(i) First note that since $\Delta m_i \geq \Delta\left|\mathbf{W}_G\right|$ and $m_i(G) \leq \left|\mathbf{W}_G\right|$,

$$\Delta m_i \left|\mathbf{W}_G\right| \geq m_i(G)\Delta\left|\mathbf{W}_G\right|. \tag{5}$$

Therefore,

$$P_i(G') - P_i(G) = \frac{m_i(G) + \Delta m_i}{\left|\mathbf{W}_G\right| + \Delta\left|\mathbf{W}_G\right|} - \frac{m_i(G)}{\left|\mathbf{W}_G\right|}$$

$$= \frac{m_i(G)\left|\mathbf{W}_G\right| + \Delta m_i\left|\mathbf{W}_G\right| - m_i(G)\left|\mathbf{W}_G\right| - m_i(G)\Delta\left|\mathbf{W}_G\right|}{\left(\left|\mathbf{W}_G\right| + \Delta\left|\mathbf{W}_G\right|\right)\left|\mathbf{W}_G\right|}$$

$$= \frac{\Delta m_i\left|\mathbf{W}_G\right| - m_i(G)\Delta\left|\mathbf{W}_G\right|}{\left(\left|\mathbf{W}_G\right| + \Delta\left|\mathbf{W}_G\right|\right)\left|\mathbf{W}_G\right|}$$

≥ 0 (Using (5).)

Therefore,

$P_i(G') \geq P_i(G)$.

Thus, the power to prevent action of the recipient does not fall after the transfer.

(ii) For the donor j, the proof is straightforward. Since $\Delta m_j \leq 0$ and $\Delta\left|\mathbf{W}_G\right| \geq 0$, the power to prevent action of the donor can never rise.

Since P_i satisfies **INV** and **TRP**, by theorem 7.11 of Felsenthal and Machover (1995) we can conclude that it satisfies **DOM**. Satisfaction of **BOP** by P_i follows from the fact that it satisfies **TRP**, **INV** and **IGD** (Felsenthal and Machover, 1995, theorem 7.10).

(b) If voter i is a blocker, then by definition he can stall a bill by voting 'no', irrespective of how others vote. This means that he is a pivotal voter in every winning coalition in the game. Therefore, $m_i = \left|\mathbf{W}_G\right|$, or, P_i=1. Conversely, if P_i=1, it means that $m_i = \left|\mathbf{W}_G\right|$. That is, i is critical in every winning coalition of the game. So i has to be a blocker.

(c) Let us denote the set of winning coalitions in the game $\hat{G}$ by $\mathbf{W}_{\hat{G}}$. Now $\mathbf{W}_{\hat{G}} = \left\{S \in 2^{\hat{N}} : b \in S, S \setminus \{b\} \in \mathbf{W}_G\right\}$. Therefore, $\left|\mathbf{W}_{\hat{G}}\right| = \left|\mathbf{W}_G\right|$. Also it is obvious from the definition of $\mathbf{W}_{\hat{G}}$ that the number of coalitions in which any player i is critical remains unaltered in the new game. Therefore, power to prevent action of any player remains unchanged and hence the desired result follows.

Hence the proof of theorem.□

The fact that Coleman's preventive power index P_i satisfies **VJD, INV, IGD, DOM, TRP and BOP** implies that it can be used as a measure of the extent of influence that voter i enjoys over the outcome of the voting process. Felsenthal and Machover (1998) suggested property **(c)** of theorem 3 as a desirable postulate for an index of power as a prize (e.g., the Shapley-Shubik index) and referred to it as the *added blocker postulate* (**ABP**). Although P_i is regarded as an index of power of influence, we note its satisfaction of the **ABP**. It also follows from theorem 3 that P_i is bounded between zero and one, where the lower and upper bounds are achieved when voter i is a dummy and a blocker respectively. Finally, since P_i satisfies **TRP** and **IGD**, it satisfies **RTP** and **RDP** as well.

The following theorem discusses the Coleman initiative power index I_i, defined in (2), in terms of the postulates laid down in section 3.

<u>Theorem 4:</u>

(a) The Coleman index of the power to initiate act, I_i, satisfies VJD, INV, IGD, DOM, TRP and BOP for all voting games.

(b) In a proper voting game, the index I_i achieves its upper bound of 1 if and only if the voter is a dictator.

(c) If $\hat{G} = \left(\hat{N}, \hat{V}\right) \in \mathbf{F}$ is obtained from $G = \left(N, V\right) \in \mathbf{F}$ by adding $b \notin N$ as a blocker in $\hat{G}$, then for any two non-dummy voters $i, j \in N$, we have

$$\frac{I_i(G)}{I_j(G)} = \frac{I_i(\hat{G})}{I_j(\hat{G})}.$$

That is, I_i satisfies ABP.

The proof of this theorem relies on the following lemmas.

Lemma 3: Consider the voting games G and G', and the voters j and i as described in **TRP**. Let $\mathbf{L}_G$ $(\mathbf{L}_{G'})$ be the set of losing coalitions in G (G'). Then assuming that $\Delta|\mathbf{L}_G| = |\mathbf{L}_{G'}| - |\mathbf{L}_G| \le 0$, we have $|\Delta m_j| \ge |\Delta|\mathbf{L}_G||$.

Proof:

We have noted in the proof of theorem 3 that $\Delta m_j = -(\alpha_2 + \alpha_4)$. Also $\Delta|\mathbf{W}_G| = \alpha_1 - \alpha_2$, which means that $\Delta|\mathbf{L}_G| = \alpha_2 - \alpha_1 \le 0$, by hypothesis. Suppose, contrary to what the lemma says, we have $|\Delta|\mathbf{L}_G|| > |\Delta m_j|$. Then we must have

$\alpha_1 - \alpha_2 > \alpha_2 + \alpha_4$,

or, $\alpha_1 - \alpha_4 > 2\alpha_2$ (6)

Let us recall the definition of the sets A_1 and A_4 which have been considered in the proof of theorem 3.

$A_1 = \{S \subseteq N : i \in S, j \notin S; S \notin \mathbf{W}_G^i, S \in \mathbf{W}_{G'}^i\}$, $|A_1| = \alpha_1$.

$A_4 = \{S \subseteq N : i, j \in S;\ S \in \mathbf{W}_G^{i,j}$ but $S \setminus \{j\} \notin \mathbf{W}_G^i$, and $S \in \mathbf{W}_{G'}^{i,j}$ but $S \setminus \{j\} \in \mathbf{W}_{G'}^i\}$, $|A_4| = \alpha_4$.

Now, consider an element $S \in A_1$. $S \in \mathbf{W}_{G'}^i$ implies that $S \cup \{j\} \in \mathbf{W}_{G'}^{i,j}$ (since a super set of a winning coalition is also winning). By condition **T1** we know that $S \cup \{j\} \in \mathbf{W}_G^{i,j}$. This combined with the fact that $S \notin \mathbf{W}_G^i$ (since $S \in A_1$) tells us that player j is a critical member of the coalition $S \cup \{j\}$ in the game G but not in G'. That is, $S \cup \{j\} \in A_4$. Thus, for every coalition $S \in A_1$, we get a unique coalition $S \cup \{j\} \in A_4$.

Conversely, it can be checked that with every coalition $S \in A_4$, we can associate a unique coalition $S \setminus \{j\} \in A_1$. In other words, with every element in A_1 we can associate a unique element in A_4 and vice versa. This implies $|A_1| = |A_4|$, or, $\alpha_1 = \alpha_4$.

Using the above equality and (6) we know that if $|\Delta|\mathbf{L}_G|| > |\Delta m_j|$, then $0 > \alpha_2$, which is a contradiction, since α_2 is the cardinality of a set and hence cannot be negative. So $|\Delta m_j| \geq |\Delta|\mathbf{L}_G||$. Hence the proof. □

Lemma 4: Consider the voting games G and G', and the voters j and i as described in **TRP**. Let $\mathbf{L}_G$ $(\mathbf{L}_{G'})$ be the set of losing coalitions in G (G'). Then assuming that $\Delta|\mathbf{L}_G| = |\mathbf{L}_{G'}| - |\mathbf{L}_G| \geq 0$, we have $\Delta m_i = m_i(G') - m_i(G) \geq \Delta|\mathbf{L}_G|$.

Proof:

We have noted in the proof of theorem 3 that $\Delta m_i = \alpha_1 + \alpha_3$. Also $\Delta|\mathbf{W}_G| = \alpha_1 - \alpha_2$, which means that $\Delta|\mathbf{L}_G| = \alpha_2 - \alpha_1$. Suppose, contrary to what the lemma says, we have $\Delta|\mathbf{L}_G| > \Delta m_i$. Then we must have

$\alpha_2 - \alpha_1 > \alpha_1 + \alpha_3$,

or $\alpha_2 - \alpha_3 > 2\alpha_1$. (7)

(Note that by assumption $\Delta|\mathbf{L}_G| \geq 0$.)

Recall the definitions of the sets A_2 and A_3 in the proof of theorem 3. Consider an element $S \in A_2$. It is easy to note that $S \cup \{i\} \in \mathbf{W}_G^{i,j}$ and hence by condition **T1** $S \cup \{i\} \in \mathbf{W}_{G'}^{i,j}$. This combined with the fact $S \in \mathbf{W}_G^j$ and $S \notin \mathbf{W}_{G'}^j$ (since $S \in A_2$) says that i is non-critical member of the winning coalition $S \cup \{i\}$ in the game G but becomes critical in the game G'. Therefore $S \cup \{i\} \in A_3$. Thus, for each $S \in A_2$, $S \cup \{i\} \in A_3$. Conversely it can be checked that for each $S' \in A_3$, $S' - \{i\} \in A_2$. Hence the correspondence $S \rightarrow S \cup \{i\}$ is a bijection from A_2 onto A_3.

In other words, with every element in A_2 we can associate a unique element in A_3 and vice versa. Therefore, $\alpha_2 = \alpha_3$

Using the above equality and (7) we know that if $\Delta|\mathbf{L}_G| > \Delta m_i$, then $0 > \alpha_1$, which is a contradiction, since α_1 is the cardinality of a set and hence cannot be negative. So $\Delta m_i \geq \Delta|\mathbf{L}_G|$. Hence the proof. □

<u>Proof of theorem 4</u>:

(a) The proof that I_i satisfies **VJD**, **INV**, **IGD** is similar to the proof in theorem 3. We now prove that I_i satisfies **TRP**. We have already noted above that $\Delta|\mathbf{W}_G|$ could be non-negative or non-positive, and that $\Delta m_i \geq 0$ and $\Delta m_j \leq 0$.

Case 1: $\Delta|\mathbf{W}_G| \geq 0$.

This means $\Delta|\mathbf{L}_G| \leq 0$. That is,

(i) $|\mathbf{L}_{G'}| \leq |\mathbf{L}_G|$

$$\therefore \frac{1}{|\mathbf{L}_{G'}|} \geq \frac{1}{|\mathbf{L}_G|}$$

or, $\dfrac{m_i(G)+\Delta m_i}{|\mathbf{L}_{G'}|} \geq \dfrac{m_i(G)}{|\mathbf{L}_G|}$ (since $\Delta m_i \geq 0$).

Thus, I_i or the power to initiate action of the recipient does not fall after the transfer.

(ii) To show that the power to initiate of the donor does not rise after the transfer, we first note that by lemma 3, $|\Delta m_j| \geq |\Delta|\mathbf{L}_G||$. Also we know that $|\mathbf{L}_G| \geq m_j(G)$. So

$\Delta m_j.|\mathbf{L}_G| \leq m_j(G).\Delta|\mathbf{L}_G|$ (since $\Delta m_j \leq 0$ and $\Delta|\mathbf{L}_G| \leq 0$) (8)

$$I_j(G') - I_j(G) = \frac{m_j(G)+\Delta m_j}{|\mathbf{L}_G| + \Delta|\mathbf{L}_G|} - \frac{m_i(G)}{|\mathbf{L}_G|}$$

$$= \frac{\Delta m_j.|\mathbf{L}_G| - m_i(G).\Delta|\mathbf{L}_G|}{(|\mathbf{L}_G| + \Delta|\mathbf{L}_G|)|\mathbf{L}_G|} \leq 0. \text{ (Using (8))}$$

Thus, the power of the donor cannot rise after the transfer.

Case 2: $\Delta|\mathbf{W}_G| \leq 0$.

This means $\Delta|\mathbf{L}_G| \geq 0$.

(i) From lemma 4 and using the fact that $|\mathbf{L}_G| \geq m_i(G)$, we can say that

$$\Delta m_i|\mathbf{L}_G| \geq m_i(G).\Delta|\mathbf{L}_G|. \qquad (9)$$

Let us now evaluate the expression $I_i(G') - I_i(G)$.

$I_i(G') - I_i(G)$

$$= \frac{m_i(G')}{|\mathbf{L}_{G'}|} - \frac{m_i(G)}{|\mathbf{L}_G|}$$

$$= \frac{m_i(G') + \Delta m_i}{(|\mathbf{L}_G| + \Delta|\mathbf{L}_G|)} - \frac{m_i(G)}{|\mathbf{L}_G|}$$

$$= \frac{m_i(G).|\mathbf{L}_G| + \Delta m_i|\mathbf{L}_G| - m_i(G)|\mathbf{L}_G| - m_i(G)\Delta|\mathbf{L}_G|}{(|\mathbf{L}_G| + \Delta|\mathbf{L}_G|)|\mathbf{L}_G|}$$

$$= \frac{\Delta m_i|\mathbf{L}_G| - m_i(G).\Delta|\mathbf{L}_G|}{(|\mathbf{L}_G| + \Delta|\mathbf{L}_G|)|\mathbf{L}_G|}$$

≥ 0 (Using (9)).

Thus power to initiate action of player i does not decrease after i receives some voting right from another player.

(ii) For the donor j the proof is straightforward because $\Delta m_j \leq 0$ and $\Delta|\mathbf{L}_G| \geq 0$.

Proof of satisfaction of **DOM** and **BOP** by I_i is similar to the proof in theorem 3.

(b) If the game has a dictator then it becomes a proper game and $|\mathbf{W}_G| = |\mathbf{L}_G| = 2^{|N|-1}$. Also by the definition of a dictator, he is critical player in all the winning coalitions. Therefore $I_i = 1$. Conversely, $I_i = 1$ implies that $m_i = |\mathbf{L}_G|$. Since m_i is also the number of losing coalitions outside which i is critical, $m_i = |\mathbf{L}_G|$ implies that i is critical outside every losing coalition in the game. Since the game is proper, $m_i \leq |\mathbf{W}_G| \leq |\mathbf{L}_G| = m_i$. Therefore, $m_i = |\mathbf{W}_G| = |\mathbf{L}_G|$. So i is a critical member of each winning coalition of the game. This means i's 'yes' vote is necessary to pass the bill. Following definition 1 we know that ϕ is a losing coalition. Since i is critical outside every losing coalition in the game, therefore $\{i\}$ is a winning coalition. This means i's 'yes' vote is sufficient to pass the bill. So i is a dictator.

(c) The proof of this part of the theorem is similar to the proof of part **(c)** of theorem 3. □

It is important to note that while P_i reaches its maximum in the case of an ordinary blocker, I_i is maximum if the blocker is a dictator.

The previous two theorems show that both the Coleman index of the power to prevent action and the power to initiate action can be used to get an idea of the extent of influence that an individual voter enjoys over the outcome of the decision making body. However, if we modify Coleman's Preventive Power Index by replacing $|\mathbf{W}_G|$ by $|\mathbf{W}_G^i|$ in the denominator, the resulting index also qualifies as a measure of influence of an individual voter.

Let the transformed preventive index be defined as

$$T_i = \frac{m_i(G)}{|\mathbf{W}_G^i|}. \qquad (10)$$

Clearly, T_i is an attainable upper bound on P_i. More precisely, for all $G \in \mathbf{F}$, $P_i(G) \leq T_i(G)$.

Theorem 5:

(a) The Transformed Coleman index of the power to prevent act, T_i, satisfies VJD, INV, IGD, DOM, TRP, BOP and ABP for all voting games.

(b) T_i achieves its upper bound 1 if and only if i is a blocker.

Proof:

(a) The proof that T_i satisfies **VJD, INV, IGD** is similar to the proof in theorem 3 and 4. We now prove that T_i satisfies **TRP**. Suppose the game G' is derived from the game G

by transferring some voting right of voter j to voter i. Following the discussion in the proof of theorem 3, we have

$\Delta m_i = \alpha_1 + \alpha_3$; $\Delta\left|\mathbf{W}_G^i\right| = \alpha_1$.

$\Delta m_j = -(\alpha_2 + \alpha_4)$; $\Delta\left|\mathbf{W}_G^j\right| = -\alpha_2$.

Adopting a similar approach as in the proof of theorem 3 and 4, it is easy to see that after the transfer the donor can never gain power (i.e. T_j can never rise) and the recipient can never lose power (i.e. T_j can never fall). Satisfaction of **DOM** and **BOP** by T_i is easy to check now.

The proof that T_i satisfies **ABP** is similar to the proof of part **(c)** of theorem 3.

(b) If i is a blocker, then by definition he is a critical member of all the winning coalitions in the game. So $m_i = \left|\mathbf{W}_G\right| = \left|\mathbf{W}_G^i\right|$. This means T_i=1. Conversely, suppose T_i=1. This implies $m_i = \left|\mathbf{W}_G^i\right|$. That is, i is a critical member of all winning coalitions that he belongs to. Further, it is easy to note that there can be no winning coalitions in which i is not a member. For, if there exists such a coalition S, such that $S \in \mathbf{W}_G$ and $i \notin S$, then by monotonicity $S \cup \{i\} \in \mathbf{W}_G$, and i is not a critical member of this coalition. This contradicts $m_i = \left|\mathbf{W}_G^i\right|$. So we have $m_i = \left|\mathbf{W}_G\right|$ = $\left|\mathbf{W}_G^i\right|$. This implies that i is a blocker.

Hence the proof of the theorem. □

5. Conclusions

An index of voting power is a measure of the extent to which a voter is able to influence the passage or defeat of a resolution. Coleman (1971) suggested two such indices, power to prevent an action and power to initiate an action. The former gives an indication of the chance a voter has to block a bill and the latter is concerned with the voter's probability to initiate action. However, there has not been much discussion of these indices in the literature on voting power. This paper rigorously examines these indices in the light of different properties for an index of voting power suggested by Felsenthal and Machover (1995, 1998) and demonstrates their suitability in this context. A relationship between these two indices is also established in the paper. It is further shown that a transformation of the power to prevent an action index can also be considered as an approximate voting power index.

A great deal remains to be done. For instance, an attempt to characterize these indices using two sets of independent axioms will be a worthwhile exercise.

Note

1. Although Felsenthal and Machover (1998, p. 245) argued that 'any reasonable measure of a priori power …must respect dominance', some authors (e.g. Laruelle and Valenciano, 2002) questioned desirability of this postulate. See also Braham and Steffen (2002).

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