POWER IN WEIGHTED MAJORITY GAMES*

Rana Barua, Statistics-Mathematics Division,

Satya R. Chakravarty, Economic Research Unit,

Sonali Roy, Economic Research Unit,

Indian Statistical Institute, Calcutta, India.

Abstract

This paper suggests an indicator of power in weighted majority games. An indicator of power determines the ability of a voter to influence the outcomes of the voting bodies he belongs to. In a weighted majority game each voter is assigned a certain nonnegative real number weight and there is a positive real number quota satisfying a boundedness condition such that a group of voters can pass a resolution if the sum of the weights of the group members is at least as high as the given quota. The new index is shown to satisfy all the reasonable postulates for an index of voting power. A comparison of the new index with some of the existing indices is also presented. Finally, the paper develops an axiomatic characterization of the new index.

JEL Classification Numbers: C71, D72.

Key words: Voting game, weighted majority game, voting power, indices of power, characterization.

Correspondent:
Satya R. Chakravarty
Economic Research Unit
Indian Statistical Institute
203 B.T. Road,
Calcutta – 700108
India
Fax: 91335776680
E-mail: satya@isical.ac.in

* For comments and suggestions, we are grateful to D. Felsenthal and M. Machover.

1. Introduction

A central concept of political science is power. While power is a many faceted phenomenon, here we are concerned with the notion of power as it is reflected in the formal voting system. If in a voting situation everyone has one vote and the majority rule is taken as the decisive criterion, then everyone has the same type of power. The majority rule declares a candidate as the winner if he gets the maximum number of votes among the candidates. But if some persons have more votes than others, then they can certainly manipulate the voting outcome by exercising their additional votes.

An index of voting power should reflect a voter's influence, in a numerical way, to bring about the passage or defeat of some bill. It should be based on the voter's importance in casting the deciding vote. The most well-known index of voting power is the Shapley-Shubik (1954) index. Essential to the construction of this index is the concept of swing or pivotal voter. Given an ordering of voters, the swing voter for this ordering is the person whose deletion from the coalition of voters of which he is the last member in the given order, transforms this contracting coalition from a winning to a losing one. (A coalition of voters is called winning if passage of a bill is guaranteed by 'yea' votes from exactly the voters in that coalition. Coalitions that are not winning are called losing.) The Shapley-Shubik index for voter i is the fraction of orderings for which i is the swing voter. In fact, the Shapley-Shubik index is an application of the well-known Shapley value (Shapley,1953) to a voting game, which is a formulation of a voting system in a coalitional form game.

Alternatives and variations of the Shapley-Shubik index were suggested, among others, by Banzhaf (1965), Coleman (1971), Deegan and Packel (1978) and Johnston (1978). The Banzhaf index of power of a voter is based on the number of coalitions in which the voter is swing. More precisely, it determines the number of possibilities in which a voter is in the critical position of being able to change the voting outcome by changing his vote. The two indices suggested by Coleman (1971) are proportional to the Banzhaf index, and to each other (Brams and Affuso, 1976).However, the Banzhaf - Coleman indices, which are based on the idea of a critical defection of a voter from a winning coalition, do not take into account the total number of voters whose defections from a given coalition are critical. Clearly, if a voter is the only person whose defection from a coalition is critical, then this gives a stronger indication of power than in the case where all persons' defections are critical. This is the central idea underlying the Johnston (1978) power index.

Deegan and Packel (1978) argued that only minimal winning coalitions should be considered in determining the power of a voter. (A coalition is called minimal winning if none of its proper subsets is winning.) They suggested an index under the assumptions that all minimal winning coalitions are equiprobable and any two voters belonging to the same minimal winning coalitions should enjoy the same power. However, the Banzhaf - Coleman, Johnston and Deegan-Packel indices have a common disadvantage: they all violate the transfers principle and the bloc principle (see Felsenthal and Machover, 1995). The transfers principle requires that the power of voter i, who is capable of affecting voting outcome, should decrease if he donates a part of his voting right to another voter j. According to the bloc principle, under a voluntary merger between two voters a and b, where b is capable of affecting voting outcome, the power of the merged entity will be larger than that of a. The Deegan-Packel index also violates the dominance principle, which demands that if voter j's contribution to the victory of a resolution can be equalled or bettered by another voter i, then i should not possess lower power than j. Felsenthal and Machover (1995) regarded these three postulates as the major desiderata for an index of voting power. It may be interesting to note that the Shapley Shubik index satisfies all these three postulates.

A common form of voting game is a weighted majority game, which can be described by specifying nonnegative real number weights for the voters and a positive real number quota satisfying a boundedness condition such that a coalition is winning precisely when the sum of the weights of the voters in the coalition meets or exceeds the quota. Weighted majority games arise in many contexts. Examples are: The European Economic Community and stockholder voting in corporations (see Lucas, 1982, for additional examples).

The objective of this paper is to suggest a new index of voting power in a weighted majority game. This new index is based on the number of critical defections of a voter and his weight in the game. It is in fact the second Banzhaf-Coleman index multiplied by the weight of a voter. This index is found to possess many interesting properties including satisfaction of the bloc, donation and transfer principles. When attention is restricted to weighted majority games only, this new index can, therefore, be regarded as an extended version of the Banzhaf-Coleman indices because they all make use of critical defections of a voter from coaltions, but the former does it in a more satisfactory way in the sense that it does not have the shortcomings of the latter formulae. It may be important to note that our objective is not to supplement the Shapley-Shubik index or any index of voting power which satisfies the requirements for a power index. Instead, we wish to see how the concept of critical defection and weight of a voter in a weighted majority game can be employed successfully in developing an index of power.

The paper is organized as follows. The next section discusses the properties for an index of voting power. Section 3 presents and analyses the new index in the light of the properties introduced in section 2. A comparative discussion of some of the existing indices with the new index is also presented in this section. In order to have a set of postulates that are necessary and sufficient for identifying the new index uniquely, an axiomatic characterization of the index is presented in section 4. Finally, section 5 concludes.

2. Notation, Definitions and Preliminaries

It is possible to model a voting situation as a coalitional form game, the hallmark of which is that any subgroup of players can make contractual agreements among its members independently of the remaining players. Let $N = \{A_1, A_2, ...A_n\}$ be a set of players. For any set of players *N*, $|N|$ will stand for the number of players in *N*. The power set of *N*, that is, the collection of all subsets of N is denoted by 2^N. Any member of 2^N is called a coalition. A coalitional form game with player set N is a pair $(N;V)$, where $V : 2^N \to R$ such that $V(\phi) = 0$, where R is the real line. For any coalition S, the real number $V(S)$ is the worth of the coalition, that is, this is the amount that S can guarantee to its members.

We frame a voting system as a coalitional form game by assigning the value 1 to any coalition which can pass a bill and 0 to any coalition which cannot. In this context, a player is a voter and the set $N = \{A_1, A_2, ...A_n\}$ is called the set of voters. Throughout the paper we assume that voters are not allowed to abstain from voting. A coalition S will be called winning or losing according as it can or cannot pass a resolution.

Definition1: Given a set of voters N, a voting game associated with N is a pair $(N;V)$, where $V : 2^N \to \{0,1\}$ satisfies the following conditions:

(i) $V(\phi) = 0$.

(ii) $V(N) = 1$.

(iii) If $S \subseteq T$, $S,T \in 2^N$, then $V(S) \le V(T)$.

(iv) For $S,T \in 2^N$, if $V(S) = V(T) = 1$, then $S \cap T \ne \phi$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. It follows that the empty coalition ϕ is losing (condition (i)) and the grand coalition N is winning (condition (ii)). All other coalitions are either winning or losing. Condition (iii) ensures that if a coalition S can pass a bill, then any superset T of S can pass it as well. According to condition (iv) two winning coalitions cannot be disjoint. Disjointness of two winning coalitions implies that two s can be passed simultaneously.

Definition 2: The unanimity game $(N;U_N)$ associated with a given set of voters N is the game whose only winning coali N.

Given a set of voters N, let $(N;V)$ be a voting game.

(i) For any coalition $S \in 2^N$, we say that $i \in N$ is swing in S if $V(S) = 1$ but $V(S - \{i\}) = 0$.

(ii) For any coalition $S \in 2^N, i \in N$ is said to be swing outside S if $V(S) = 0$ but $V(S \cup \{i\}) = 1$.

(iii) A coalition $S \in 2^N$ is said to be minimal winning if $V(S) = 1$ but there does not exist $T \subset S$ such that $V(T) = 1$.

Thus, voter i is swing, also called pivotal or key, in the winning coalition S if his deletion from S makes the resulting coalition $S - \{i\}$ losing. Similarly, voter i is swing outside the losing coalition S if his addition to S makes the resulting coalition $S \cup \{i\}$

winning. For any voter i, the number of winning coalitions in which he is swing is same as the number of losing coalitions outside which he is swing (Burgin and Shapley, 2001, Corollary 4.1).

Definition 4: For a set of voters N, let $(N;V)$ be a voting game. A voter $i \in N$ is called a dummy in $(N;V)$ if he is never swing in the game. A voter $i \in N$ is called a nondummy in $(N;V)$ if he is not dummy in $(N;V)$.

Following Burgin and Shapley (2001) we have

Definition 5: For a voting game $(N;V)$ with the set of voters $N,$ a voter $i \in N$ is called a dictator if $\{i\}$ is a winning coalition.

A dictator in a game is unique. If a game has a dictator, then he is the only swing voter in the game.

As stated earlier, an extremely important voting game is a weighted majority game.

Definition 6: For a set of voters $N = \{A_1, A_2, ...A_n\}$, a weighted majority game is a quadruplet $G = (N;V;\ \mathbf{w};q)$, where $V : 2^N \to \{0,1\}$, $\mathbf{w} = (w_1, w_2, ..., w_n)$ is the vector of nonnegative weights of the $|N|$ voters in N, q is a positive real number quota such that $\sum_{i=1}^{|N|} \frac{w_i}{2} < q \le \sum_{i=1}^{|N|} w_i$ and for any $S \in 2^N$,

$$V(S) = 1 \text{ if } \sum_{i \in S} w_i \ge q,$$

$$= 0 \text{ otherwise.} \qquad (1)$$

That is, here the i^{th} voter has the weight w_i and q, fulfilling the boundedness condition $\sum_{i=1}^{|N|} \frac{w_i}{2} < q \le \sum_{i=1}^{|N|} w_i$, is the quota of weights needed to pass a resolution. The games $G = (N;V;\ \mathbf{w};q)$ and $G_a = (N;V;\ a\mathbf{w}\ ;aq)$, where $a > 0$, are equivalent in the sense that for any $S \in 2^N$, $\sum_{i \in S} aw_i \ge aq$ if and only if $\sum_{i \in S} w_i \ge q$. Thus, a coalition is winning in G if and only if it is winning in $G_a, a > 0$. Clearly, every weighted majority game satisfies conditions (i)-(iv) of definition 1. (See Felsenthal and Machover, 1995, for additional discussions on definitions 1 and 6.)

It should be noted that not all voting games are weighted majority games. An important example is the United States legislative scheme in which a winning coalition has to contain the President and a majority of both the Senate and the House of Representatives or two-thirds of both the Senate and the House.

The collection of all weighted majority games is denoted by **F**. An index of voting power of voter i in a weighted majority game is a nonnegative real valued function P_i defined on **F**, that is, P_i : $\mathbf{F} \rightarrow R_+$, the nonnegative part of the real line. Such an index should fulfil certain desirable properties. Since here we are dealing with power in weighted majority games, we will state these properties in terms of such games. General formulation (more precisely, formulation in the context of general voting games) of the properties **MIN**, **ANY**, **DEP**, **MON**, **BOP** and **TRP** presented below are available in Felsenthal and Machover (1995).

The first property we consider is:

Minimality (MIN): For all $G = (N;V;\,\mathbf{w};q) \in \mathbf{F}$ for any $i \in N$, $P_i\,(G)$ achieves its minimum value, zero, if and only if i is a dummy.

By definition, a power index of a voter i is nonnegative. **MIN** says that the necessary and sufficient condition that the index attains its lower bound, zero, is that the concerned voter i is a dummy. To understand this more explicitly, let us look at the game $G_0 = (N;V;1,2,6;5)$, where $N = \{A_1, A_2, A_3\}$. In G_0, none of the voters 1 and 2 is in a critical position of making a winning (losing) coalition losing (winning). If we view power simply in terms of the weight enjoyed by a voter, then in G_0, voter 2 has a higher power than voter 1. But in terms of their ability of switching a coalition from winning (losing) to losing (winning), they are identical because they are both dummy. As argued in the literature (Taylor, 1995), since the essence of power of a voter lies in his capability of being a key voter, we appeal that a voter's power should be minimal (zero) if he is dummy (see also Dubey, 1975; Dubey and Shapley, 1979; and Burgin and Shapley, 2001). A similar argument can be given from the reverse side. Since the power index has been assumed to be nonnegative, and a voter is either dummy or nondummy, **MIN** implies that for a nondummy voter, the value of the power index is positive.

Now, if in a game no voter other than i is capable of affecting voting outcome, then i does not have to share power with anybody. In other words, a voter i possesses maximum power if he is the only nondummy voter in the game, more precisely, if he is a dictator. Thus, we have,

Maximality (MAX): For all $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$, for any $i \in N, P_i\ (G)$ achieves its maximum value whenever i is a dictator.

Since a dummy can never affect the outcome of voting, it is natural to expect that if a dummy is excluded from a voting game, the power of the remaining voters remain unaltered. In view of this we can state the following postulate:

Dummy Exclusion Principle (DEP): For all $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$, and for any dummy $d \in N$, $P_i\ (G) = P_i\ (G_{-d})$, where G_{-d} is the $(|N|-1)$ weighted majority game obtained from G by excluding the dummy $d \in N$ and $i \in N - \{d\}$ is arbitrary.

Likewise, we can have a **dummy inclusion principle**, which requires that if d is not a voter of $G \in \mathbf{F}$, then the power of any voter $i \in N$, in the expanded game G_{+d} obtained from G by including d as a dummy, is the same as the power possessed by i in the game G.

The fourth property is anonymity.

Anonymity (ANY): For any $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$, where $|N| = n$, let $(\Pi_1, \Pi_2, ...\Pi_n)$ be any reordering of voters and let the corresponding reordering of the weights $(w_1, w_2, ...w_n)$ be $(w_{\Pi_1}, w_{\Pi_2}, ...w_{\Pi_n}) = \Pi\mathbf{w}$. Then $P_i\ (G) = P_{\Pi_i}(\Pi G)$, where ΠG is the game $(N;V;\Pi\ \mathbf{w};q)$.

Anonymity means that the power of a voter remains invariant under the same permutation of the voters and their weights. Thus, any characteristic other than the weights of the voters (e.g., their living conditions) is irrelevant to the measurement of voting power. For instance, in calculating the voting power of a member of the European Economic Community (say, France), the only consideration is its weight.

The next property is concerning the power of a voting bloc. Suppose that a set of two voters in a game, say $L=\{i,j\}$ forms a bloc and operates as a single voter. Evidently, this generates a new voting game, which is obtained by replacing the two voters by the new voter representing the bloc, whom we denote by *b*. The bloc's weight in the new game, which is denoted by $\hat{G}$, is $w_b=\sum_{k\in L} w_k$. The distinction between a bloc and a coalition should be clear. A coalition is a subset of voters in the same game and members of the coalition, whose separate identities exist as voters, may vote together in a play of the game. On the other hand, a bloc is a new single voter in a new game and separate identities of the components of the bloc do not exist as voters. The bloc principle is then stated as

Block Principle (BOP): For any $G\in\mathbf{F}$, for any block b of two voters i and j and for any voter $k\in L=\{i,j\}$, $P_b\left(\hat{G}\right)>P_k\left(G\right)$, given that $l\in L-\{k\}$ is nondummy.

BOP can be interpreted as follows. When a voter k acquires the voting power of a nondummy voter l, then voting power of the bloc consisting of these two voters should be higher than that of k. In other words, a voter (k) is gaining power by swallowing the power of a nondummy voter (l). This is quite reasonable intuitively. A person will not join a bloc if the voting right of the bloc is not larger than his own voting right. If P_i satisfies **BOP**, then it takes on a positive value whenever i is a nondummy (Felsenthal and Machover, 1995, theorem 5.10).

However, **BOP** does not say anything explicitly about the number of critical defections of the bloc or merged voter. To determine this number, we first have the following:

Definition 7: Let $(N;V)$ be a voting game associated with the voter set N. Suppose that the voters $i,j\in N$ are amalgamated into one voter ij. Then the post-merger voting game is the pair $(N';V')$, where $N'=N-\{i,j\}\cup\{ij\}$ and

$$
\begin{aligned}
V'(S) &= V(S) \text{ if } S\subseteq N'-\{ij\}\\
&= V\left((S-\{ij\})\cup\{i,j\}\right) \text{ if } ij\in S.
\end{aligned}
$$

Given any voting game $(N;V)$, let m_i be the number of winning coalitions in which $i \in N$ is pivotal or swing. That is, m_i is the number of winning coalitions from which i's defection is critical. Equivalently, m_i is the number of losing coalitions outside which i is swing. These two equivalent statements will be represented by the statement that m_i is the number of swings of voter i.

The following proposition gives the number of swings of the bloc voter ij in a general voting game $(N;V)$.

<u>Proposition 1</u>: Suppose that $(N;V)$ is a voting game with the voter set N. Assume that the voters $i, j \in N$ are merged into one voter ij. Then the number of swings of the bloc voter ij in the post-merger game $(N';V')$ is $(m_i + m_j)/2$, where $m_i(m_j)$ is the number of swings of voter $i(j)$ in the original game $(N;V)$.

<u>Proof</u>: $m_i + m_j = \sum_{S \subseteq N-\{i\}} [V(S \cup \{i\}) - V(S)] + \sum_{S \subseteq N-\{j\}} [V(S \cup \{j\}) - V(S)]$

$$= \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i\}) - V(S)] + \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S \cup \{j\})] + \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{j\}) - V(S)]$$

$$+ \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S \cup \{i\})]$$

$$= 2 \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S)]$$

$$= 2 \sum_{S \subseteq N-\{ij\}} [V'(S \cup \{ij\}) - V'(S)]$$

$$= 2\ m_{ij},$$

where m_{ij} is the number of swings of the voter ij in the merged game $(N';V')$. Hence $m_{ij} = (m_i + m_j)/2$. This completes the proof of the proposition[1]. □

Since proposition 1 holds in a general voting game, it holds in a weighted majority game as well. An interesting implication of this proposition is that the sum $(m_i + m_j)$ is either zero or an even positive integer.

The sixth postulate we consider is concerning the dominance of the contribution of a voter over that of another to the victory of a coalition. If the contribution of voter i is more than or equal to that of voter j, to any coalition, then i' s power should not be less than j's. In terms of weighted majority games, this dominance principle becomes a monotonicity condition, which states that voting power should be a nondecreasing function of weight. Thus, in the European Economic Community since the weights of Germany and Luxembourg are 10 and 2 respectively, the voting power of Germany should not be less than that of Luxembourg. We thus have

Monotonicity (MON): For any $G=(N;V;\,\mathbf{w};q)\in\mathbf{F}$, if $w_j \le w_i$, then $P_j\ (G) \le P_i\ (G)$, where i, $j\ \in N$.

For any $G=(N;V;\,\mathbf{w};q)\in\mathbf{F}$, we say that $G'=(N;V;\,\mathbf{u};q)\in\mathbf{F}$, is obtained from G through transfer of weights from voter i to voter j if

$$\begin{aligned} u_i &= w_i - \delta, \\ u_j &= w_j + \delta, \\ u_k &= w_k, \forall k \ne i, j, \end{aligned} \tag{2}$$

where, $0<\delta \le w_i$ and $i, j \in N$. That is, given a voting body G, a new voting body G' is obtained through a donation of weights from i to j. Such a situation may arise if a share holder in a company sells a part of his shares to another share holder. Clearly, the power of i, who is the donor of weights in the transfer, should decrease if i is nondummy. Formally,

Transfers Principle (TRP): For any $G\in\mathbf{F}$, suppose $G'\in\mathbf{F}$ is obtained from G through a transfer of weights from i to j, where $i, j \in\ N$ and i is not dummy. Then $P_i\ (G') < P_i\ (G)$.

Likewise, the power of voter j, who is the recipient of weights in (2), should not reduce under the transfer. Certainly, if j is dummy and remains dummy with the additional weight, then j's power should not decrease. But if the additional weight transforms j from a dummy to a nondummy or if he was already nondummy before receiving the additional weight, then j's power should go up under the transfer. Thus, several possibilities regarding change of statuses of i and j may arise in going from G to G'. Clearly, i may lose some swing roles and j may gain some new swing roles. It is also likely that the transfer does not change their swing positions at all. For further discussions on these properties and related issues, see Kilgour (1974), Brams (1975), Brams and Affuso ((1976), Fischer and Schotter (1978), Dreyer and Schotter (1980), Straffin (1982) and Felsenthal and Machover (1995,1998).

3. The New Index of Voting Power and Its Properties

Since the new index is closely related to the Banzhaf-Coleman indices, we begin this section with a discussion of the latter indices. Although these indices are well-defined for any arbitrary voting game, for the purpose of comparison with the new index we define them on the set of weighted majority games. Following Owen (1978) and Burgin and Shapley (2001) we define the first Banzhaf-Coleman power index of voter i, $B_{1,i}$, as m_i, the number of swings of voter i. Formally, $B_{1i} : \mathbf{F} \to R_+$ is defined by

$$B_{1,i}(G) = m_i, \qquad (3)$$

where $G = (N; V; \mathbf{w}; q) \in \mathbf{F}$ is arbitrary. The second Banzhaf-Coleman power index $B_{2,i}$ is obtained by dividing $B_{1,i}$ by $2^{|N|-1}$, the maximal value that m_i takes, that is, the value of m_i when i is a dictator. More precisely,

$$B_{2,i}(G) = \frac{m_i}{2^{|N|-1}}. \qquad (4)$$

If in a voting model, each voter i's probability p_i of voting 'yes' or ‘no’ on a bill is chosen independently from the uniform distribution [0,1], then the power of the voter i is estimated by $B_{2,i}$ (Straffin, 1977). Since $B_{2,i}$ does not involve numbers of coalitions in which voters other than i are swing, Dubey and Shapley (1979) regarded it as an absolute index of voter i's power. The third Banzhaf-Coleman index $B_{3,i}$ of voter i is the index $B_{1,i}$ (or $B_{2,i}$) normalized to make the indices of all voters add upto unity. That is,

$$B_{3,i}(G) = \frac{m_i}{\sum_{j=1}^{|N|} m_j}. \qquad (5)$$

Since the third index involves swings of all voters in the game, it is regarded as a relative index. As stated by Felsenthal and Machover (1995), in contrast to the first two indices, it is rather difficult to handle the third index mathematically[2].

Now, consider the game $\hat{G}_0 = (N;V;2,2,5;5)$ derived from G_0 by a transfer of weight from voter 3 to voter 1. We note that the power of voter 3, as measured by the first Banzhaf-Coleman index $B_{1,i}$, is the same in both G_0 and $\hat{G}_0$. This shows that $B_{1,i}$ may not satisfy **TRP**. The same remark applies to $B_{2,i}$ as well. Felsenthal and Machover (1995) showed that $B_{3,i}$ may increase under a transfer from a nondummy voter i to another voter j. They observed a similar counterintuitive behavior of $B_{3,i}$ with respect to **BOP**.

In view of these observations, it may be worthwhile to suggest an extension of the Banzhaf-Coleman indices that will respond correctly to the postulates considered in section 2. We have already argued that w_i, the weight possessed by voter i may not be an appropriate indicator of power of i. However w_i along with scaled down m_i, that is, $m_i / 2^{|N|-1}$, may give us an adequate information on the level of power, since the two together can tell us with what weight the voter is capable of making m_i winning (losing) coalitions losing (winning). More precisely, as an index of power of voter i, we suggest the use of $I_i : \mathbf{F} \to R_+$, where for any $G = (N;V;\mathbf{w};q) \in \mathbf{F}$,

$$I_i(G) = \frac{m_i w_i}{2^{|N|-1}}. \qquad (6)$$

Thus, I_i is simply the second Banzhaf-Coleman power index $B_{2,i}$ multiplied by the weight of the voter i. In view of Corollary 4.1 of Burgin and Shapley (2001), like the three Banzhaf-Coleman indices, the new index I_i is symmetric- they remain the same whether we define them by the number of swing positions in or outside coalitions. Suppose in a weighted majority voting situation, each voter's probability of voting for or against a bill is selected independently from the uniform distribution [0,1]. Then following Straffin (1977), we can show that I_i becomes the weight of voter i multiplied by the probability that other voters will vote such that the bill will pass or fail according as i votes in favour of or against it.

The following theorem summarizes the behaviour of I_i with respect to the properties discussed in section 2.

<u>Theorem1</u>: The index I_i satisfies **MIN, MAX, DEP, ANY, BOP, MON** and **TRP**.

<u>Proof:</u> We first show that if $w_j \le w_i$, then $m_j \le m_i$. Let T_1 be the set of all winning coalitions containing i and j. Since sum of the weights is the only criterion to judge whether a coalition is winning or not, $w_j \le w_i$ means that the contribution j makes to the victory of coalition $S \in T_1$ can be equalled or bettered by i. Therefore, if j is a swing voter in *S*, i must be swing in it. Clearly, there may exist coalitions in T_1 in which i is swing but j is not. Thus, we have $m_{j1} \le m_{i1}$, where m_{i1} (m_{j1}) is the number of coalitions in T_1 in which i (j) is swing.

Next, let T_{2i} (T_{2j}) be the set of all winning coalitions that contain i (j) but not j (i). If j is swing in the coalition $C_j \in T_{2j}$, then since $w_j \le w_i$, i becomes swing in the coalition $C_i = C_j - \{j\} \cup \{i\} \in T_{2i}$ generated from C_j. That is, if j is swing in a coalition $C_j \in T_{2j}$, then i must be swing in the coalition $C_i \in T_{2i}$ that results from C_j when j is replaced by i. Since $C_j \in T_{2j}$ is arbitrary, it follows that $m_{j2} \le m_{i2}$, where m_{i2} (m_{j2}) is the number of coalitions in T_{2i} (T_{2j}) in which i (j) is swing. Observing that $m_i = m_{i1} + m_{i2}$ and $m_j = m_{j1} + m_{j2}$, we have $m_j \le m_i$.

Given that $w_i \ge w_j$ implies $m_i \ge m_j$, we have $\frac{w_i m_i}{2^{|N|-1}} \ge \frac{w_j m_j}{2^{|N|-1}}$. This in turn implies that I_i satisfies **MON.** I_i is obviously anonymous.

We shall now show that I_i satisfies **DEP**. Note that exclusion of a dummy *d* from a game *G* does not change w_i. Let G_{-d} be the weighted majority game obtained from *G* by excluding the dummy $d \in N$ and let $i \in N - \{d\}$ be arbitrary.

Let $\psi_i' = \{S \subseteq N - \{i,d\} : i$ is a swing in S in $G_{-d}\}$

and $\psi_i = \{S \subseteq N - \{i\} : i$ is a swing in S in $G\}$.

Clearly, if i is a swing in S in G_{-d}, then i is a swing in S in G. Also, if i is a swing in S in G_{-d}, then i is a swing in $S \cup \{d\}$ in the game G. Hence, $\psi_i = \psi_i' \cup \{S \cup \{d\} : S \in \psi_i'\}$. Thus, $|\psi_i| = 2|\psi_i'|$, that is, $m_i = 2\overline{m}_i$, where $\overline{m}_i$ is the number of swings of i in G_{-d}. Hence, $I_i(G) = \frac{m_i w_i}{2^{|N|-1}} = \frac{2\overline{m}_i w_i}{2^{|N|-1}} = \frac{\overline{m}_i w_i}{2^{|N|-2}} = I_i(G_{-d})$. So I_i verifies **DEP**. (Note that this can also be proved using proposition 1.)

To check verification of **TRP**, suppose that the game $G' = (N;V;\mathbf{u};q)$ is generated from the game $G = (N;V;\mathbf{w};q)$ by a transfer from a nondummy voter i to another voter j (see equation (2)). Now,

$$I_i(G') = \frac{m_i'(w_i - \delta)}{2^{|N|-1}}, \tag{7}$$

where m_i' is the number of swings of voter i in G'. Since $w_i - \delta < w_i$, it is not hard to see that if i is a swing in S in the game G', then i is also a swing in S in the game G irrespective of whether S contains j or not. Hence, $m_i' \le m_i$. A direct comparison shows that $I_i(G') < I_i(G)$. Hence I_i satisfies **TRP**. If voter i is a dummy in the game G then $m_i = 0$, which in turn shows $I_i(G) = 0$. Conversely, let $I_i(G) = 0$. In this case, given $w_i > 0$, we must have $m_i = 0$, that is, i must be a dummy. (Note that if $w_i = 0$, then m_i is certainly zero.) Thus, I_i verifies **MIN**. Next, given w, $I_i(G)$ takes on the maximal value if m_i is maximized, that is, when i is a dictator. Thus, I_i fulfils **MAX**. (Note that if m_i is maximized, w_i is also maximized.) Felsenthal and Machover (1995, theorem 7.10) demonstrates that **TRP**, in the presence of **ANY** and **DEP**, implies **BOP**. Since I_i meets **TRP**, **ANY** and **DEP**, it meets **BOP** also. (Satisfaction of **BOP** by I_i can also be verified using proposition 1.) This completes the proof of the theorem. □

Although I_i does not involve coalitions in which voters other than i are swing and their weights, it may change due to changes in weights of these other voters. For instance, a transfer of weight between voters j and k, where $i \neq j \neq k$, that changes swing positions of different voters, will change I_i. Similarly, an increase in w_j, where $j \neq i$, may change I_i. This shows that I_i has a relative flavour although it is not normalized in the sense that powers of all voters should add upto one. (See Felsenthal and Machover, 1995, for a discussion on the normalization principle.) This shows that power of voter i, as measured by I_i, does not involve i isolatedly, but incorporates the entire structure of a voting game. In other words, I_i involves situations of the other voters, in particular, the availability of other voters with whom i can form winning coalitions. In fact, I_i can be given a relative colour by noting that it satisfies a relative version of **TRP**. According to this version, in any game, power of a nondummy voter i relative to that of another voter j will decrease under a transfer of weight from i to j. We state this modified transfers principle as

Relative Transfers Principle (RTP): For any $G \in \mathbf{F}$, let $G' \in \mathbf{F}$ be the game obtained from G through a transfer of weight from a nondummy voter i to another voter j. Then,

$$\frac{P_i\left(G'\right)}{P_j\left(G'\right)} < \frac{P_i\left(G\right)}{P_j\left(G\right)} , \qquad (8)$$

where P_j 's are assumed to be positive.

Since **TRP** implies **RTP**, a voting power index that meets **TRP**, e.g., the Shapley-Shubik index, will meet **RTP** as well. But **RTP** does not imply **TRP**, because a transfer from i to j that does not change i's power but increases j's power shows that the index fulfils **RTP** but not **TRP**. For instance, if $I_{3,i}$ is the normalized power index I_i analogous to the third Banzhaf-Coleman index, then $I_{3,i}$ satisfies **RTP** but not **TRP**.

Relativity of I_i can also be brought about through its satisfaction of the following modification of **DEP**.

Relative Dummy Exclusion Principle (RDP): For any $G = (N;V;\ \mathbf{w};q) \in$ **F**, let G_{-d} be the ($|N|$-1) weighted majority game obtained from G be excluding the dummy $d \in N$. Then for any $i, j \in N - \{d\}$,

$$\frac{P_i\,(G)}{P_j\,(G)} = \frac{P_i\,(G_{-d})}{P_j\,(G_{-d})} \quad , \qquad (9)$$

where P_j 's are assumed to be positive.

RDP says that the power of a voter i relative to another voter j remains unaltered if a dummy is excluded from the game. Obviously, we can analogously formulate a relative dummy inclusion principle. Clearly, all indices that satisfy **DEP**, for example, $B_{2,i}$ and the Shapley-Shubik index, will satisfy **RDP**. But **RDP** is weaker than **DEP** in the sense that there may exist indices that may fulfil the former but not the latter. One such index is $B_{1,i}$. A second example that satisfies **RDP** but not **DEP** is the power index given by $m_i w_i$.

We conclude this section with a discussion on the Shapley-Shubik, the Deegan-Packel and Johnston indices. Although like the Banzhaf-Coleman indices, these three power indices have been defined in the context of general voting games, we restrict our attention to weighted majority games. For any $G = (N;V;\,\mathbf{w};q) \in \mathbf{F}$, the Shapley-Shubik power index of voter *i* is

$$H_i(G) = \frac{\text{The number of orders in which } i \text{ is swing}}{|N|!}.$$

We can rewrite $H_i(G)$ in the combinatorial form as,

$$H_i(G) = \sum_{\substack{i \\ swings \\ for \\ S \subseteq N}} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}. \qquad (10)$$

If in a voting model, each voter i's probability p_i of voting 'yes' or 'no' on a resolution is a random variable and $p_i = p$ for all i, where p is chosen from the uniform distribution [0,1], then the power of voter i is estimated by H_i (Straffin, 1977). It meets **MIN, MAX, ANY, MON, DEP, BOP and TRP**.

There are some important differences between the new index and the Shapley-Shubik index. The latter makes use of permutations of the voters and relies on the orders in which the winning coalitions are formed. It attaches importance to a voter whose deletion from a coalition of which he is the last member in a given ordering of voters, converts the coalition from a winning to a losing one. On the other hand, the new index is concerned with alternative combinations of voters and considers the number of coalitions in which a voter is swing or key. It does not deal with the chronological orders in which the winning coalitions are formed.

For any $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$, the Deegan-Packel index for voter i is defined as follows. First, we take the reciprocal of the number of voters in each minimal winning coalition to which i belongs and sum up these reciprocals. The resulting sum, TDP_i, is called the total Deegan-Packel power of voter i in the game G. Then the Deegan- Packel index for i is

$$D_i(G) = \frac{TDP_i(G)}{\sum_{j=1}^{|N|} TDP_j(G)} \ . \tag{11}$$

The Deegan-Packel index violates **MON**, an observation made by Deegan and Packel (1982) themselves. It also violates **TRP** and **BOP**. But it fulfils **ANY**.

Finally, for any $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$, the definition of the Johnston index of power for voter i proceeds as follows. For any coalition S in which i is swing, find the reciprocal of the number of voters who are swing in S. Add up these reciprocals over all coalitions in which i is swing and call it the total Johnston power of voter i (TJP_i, for short). TJP_i for $G = (N;V;\ \mathbf{w};q) \in \mathbf{F}$ is given by

$$TJP_i(G) = \sum_{S \subseteq N} \{ p(S)^{-1} : S \text{ is winning but } S\text{- } \{i\} \text{ is losing}\}, \tag{12}$$

where $p(S)$ = number of voters who are swing in S. Then the Johnston power index of voter i for the game G is

$$J_i(G) = \frac{TJP_i(G)}{\sum_{j=1}^{|N|} TJP_j(G)} \ . \tag{13}$$

The Johnston index meets **ANY** and **MON**. However, it violates **TRP** and **BOP**.

4. The Characterization Theorem

The voting power indices can give quite different results. One index can give considerably more power to some voters than another. In view of this, it is necessary to characterize alternative indices axiomatically for understanding which indices become most appropriate in which situation. An axiomatic characterization gives us an insight of the underlying index in a more elaborate way through the axioms employed in the characterization exercise. Interesting characterizations of the Shapley-Shubik and Banzhaf-Coleman indices have been developed and discussed by several researchers, including, Dubey (1975), Straffin (1977, 1994), Owen (1978, 1978a), Dubey and Shapley (1979), Lehrer (1988), Roth (1988), Haller (1994), Brink and Laan (1998) and Burgin and Shapley (2001). For the Deegan-Packel index, a characterization was developed by the authors themselves.

The objective of this section is to characterize the new index using a set of intuitively appealing axioms. For this purpose, we need to extend the domain of the power index I_i. We shall assume that the power index I_i given by (6) is defined on all "weighted voting games" $G = (N;V;\mathbf{w})$, where $(N;V)$ is a voting game considered in definition 1 and $\mathbf{w} = (w_1, w_2, ..., w_n)$ is the vector of nonnegative weights of the *n* voters in *N*. The set of all weighted voting games will be denoted by $\tilde{\mathbf{F}}$. Since a weighted majority game can also be regarded as a weighted voting game, $\mathbf{F} \subseteq \tilde{\mathbf{F}}$. In the literature often 'weighted voting games' and 'weighted majority games' are used synonymously. However, in this paper the former represents a more general type of games than the latter. For instance, suppose that in an organization, where each member has a nonnegative weight, a coalition is winning if and only if it is winning in the sense of definition 6 and it contains the chairperson or two vice-chairpersons of the organization. This can be regarded as a weighted voting game although it is not representable as a weighted majority game of the type given by definition 6. Additional similar examples can be constructed. Although the index I_i is well-defined on a weighted voting game, as discussed earlier, it has interesting and desirable properties when restricted to the weighted majority games. Our characterization of the power index is with this extended domain. Throughout the section we will assume that, if $G_1 = (N_1;V_1;\ \mathbf{w}_1)$ and

$G_2 = (N_2; V_2;\ \mathbf{w}_2)$ are two weighted voting games, then for a voter $i \in N_1 \cap N_2$, the weights in both the games are the same. The weight of voter $i \in N_j$ will be denoted by $w_i^{G_j}$, where j =1,2. Thus if $i \in N_1 \cap N_2$, then $w_i^{G_1} = w_i^{G_2}$. We will write $m_i^{G_j}$ for the number of swings of voter i in $G_j, i \in N_j, j = 1,2.$

Definition 8: Given $G_1 = (N_1; V_1;\ \mathbf{w}_1)$, $G_2 = (N_2; V_2;\ \mathbf{w}_2) \in \tilde{\mathbf{F}}$, we define, $G_1 \vee G_2$ as the game with the set of voters $N_1 \cup N_2$, weight vector $\mathbf{w} = \{w_i : i \in N_1 \cup N_2\}$, where,

$$
\begin{aligned}
w_i &= w_i^{G_1} && \text{if } i \in N_1 \\
&= w_i^{G_2} && \text{if } i \in N_2,
\end{aligned}
$$

and in which a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if either $V_1(S \cap N_1) = 1$ or $V_2(S \cap N_2) = 1$.

Definition 9: Given $G_1 = (N_1; V_1;\ \mathbf{w}_1), G_2 = (N_2; V_2;\ \mathbf{w}_2) \in \tilde{\mathbf{F}}$, we define, $G_1 \wedge G_2$ as the game with the set of voters $N_1 \cup N_2$, weight vector $\mathbf{w} = \{w_i : i \in N_1 \cup N_2\}$, where,

$$
\begin{aligned}
w_i &= w_i^{G_1} && \text{if } i \in N_1 \\
&= w_i^{G_2} && \text{if } i \in N_2,
\end{aligned}
$$

and in which a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ and $V_2(S \cap N_2) = 1$.

Thus, in order to win in $G_1 \vee G_2$, a coalition must win in either G_1 or in G_2, whereas to win in $G_1 \wedge G_2$, it has to win in both G_1 and G_2. It may be noted that if G_1 and G_2 are two unanimity weighted majority games, then $G_1 \wedge G_2$ is also a weighted majority game.

We are now in a position to present four axioms on a power index P_i that uniquely determines the new index I_i in (6). The first axiom we consider is the axiom A4 considered in Dubey and Shapley (1979) (see also Dubey, 1975). It shows that the sum of powers of voter i in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ is equal to the sum of his powers in G_1 and G_2.

Axiom A1 (Sum Principle): For $G_1 = (N_1; V_1;\ \mathbf{w}_1), G_2 = (N_2; V_2;\ \mathbf{w}_2) \in \tilde{\mathbf{F}}$,

$$P_i(G_1 \vee G_2) + P_i(G_1 \wedge G_2) = P_i(G_1) + P_i(G_2). \qquad (14)$$

In order to state the next axiom, let us consider the marginal contribution $V(S \cup \{i\}) - V(S)$ of voter i when he joins an arbitrary coalition S in $G = (N; V;\ \mathbf{w})$. Now voter i's worth in the game is $V(i)$. Equating this worth with the marginal contribution, we note that voter i is either a dummy $(V(i) = 0)$ or a dictator $(V(i) = 1)$. As stated earlier, the power index of a voter should be minimum (maximum) if he is a dummy (dictator). We consider these two extreme cases of power for voter i in the following axiom.

Axiom A2 (Extreme Powers): For every $G = (N; V;\ \mathbf{w}) \in \tilde{\mathbf{F}}$, if $V(S \cup \{i\}) = V(S) + V(i)$ for all $S \subseteq N - \{i\}$, then

$$P_i(G) \big/ w_i = V(i), \qquad (15)$$

where $w_i > 0$.

The third axiom is formulated in terms of substitutability between two voters. Two voters in a game are said to be substitutes if the worth of an arbitrary coalition in the game becomes the same when they join the coalition separately(Shapley,1953). Therefore, it is reasonable to expect that their powers, as fractions of individual weights, are the same. More precisely, we have the following axiom.

Axiom A3 (Weight Proportionality): Let voters i and j be substitutes in the game $G = (N; V;\ \mathbf{w}) \in \tilde{\mathbf{F}}$, that is, $V(S \cup \{i\}) = V(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$. Then,

$$P_i(G) \big/ w_i = P_j(G) \big/ w_j, \qquad (16)$$

where w_i and w_j are positive[3].

The next axiom is concerning the merger of two voters into one. It shows the relationship between the power of a bloc or a merged voter and his constituents. It is similar to axiom A5 of Nowak and Radzik (2000) (see also Lehrer, 1988).

Axiom A4 (Two-Voter Merging Principle): Let $G' = (N'; V'; \mathbf{w}') \in \tilde{\mathbf{F}}$ be the $(|N|-1)$-person merged game associated with a pair (i, j) of voters in the game $G = (N; V; \mathbf{w}) \in \mathbf{F}$, where $V = U_N$, N' and V' are same as in the definition 7, and

$$w'_k = w_k \qquad \text{if } k \neq i, j$$

$$w'_{ij} = w_i + w_j \ .$$

Then,

$$P_{ij}(G') \big/ w'_{ij} = \left[P_i(G) \big/ w_i + P_j(G) \big/ w_j \right], \qquad (17)$$

where w_i and w_j are positive.

<u>Theorem 2</u>: A power index P_i satisfies A1-A4 if and only if P_i is the index I_i in (6).

<u>Proof</u>: We will first show that I_i satisfies all the axioms **A1** through **A4**.

To show that **A1** is satisfied by I_i, first let $i \in N_1 - N_2$. Now, any subset S' of $N_2 - N_1$ can be appended to a swing coalition $S \subseteq N_1$ for i in G_1 to obtain a swing coalition $S \cup S'$ for i in $G_1 \vee G_2$ unless $(S \cup S') \cap N_2$ is winning in G_2. Hence the number of swings for voter i in $G_1 \vee G_2$ is

$$m_i^{G_1 \vee G_2} = m_i^{G_1} 2^{|N_2 - N_1|} - m_i^{G_1 \wedge G_2} \quad ,$$

where $m_i^{G_1 \wedge G_2}$ is the number of swings of i in $G_1 \wedge G_2$. Since for $i \in N_1 - N_2$, $m_i^{G_2}$ =0, we rewrite $m_i^{G_1 \vee G_2}$ as

$$m_i^{G_1 \vee G_2} = m_i^{G_1} 2^{|N_2 - N_1|} + m_i^{G_2} 2^{|N_1 - N_2|} - m_i^{G_1 \wedge G_2} .$$

The same expression for $m_i^{G_1 \vee G_2}$ will be obtained if $i \in N_1 \cap N_2$ or $i \in N_2 - N_1$.

Therefore,

$$I_i(G_1 \vee G_2) = \frac{m_i^{G_1} w_i}{2^{|N_1|-1}} + \frac{m_i^{G_2} w_i}{2^{|N_2|-1}} - \frac{m_i^{G_1 \wedge G_2} w_i}{2^{|N_1 \cup N_2|-1}}, \qquad (18)$$

which in turn gives

$$I_i(G_1 \vee G_2) + I_i(G_1 \wedge G_2) = I_i(G_1) + I_i(G_2).$$

This shows that I_i verifies **A1**.

To check satisfaction of **A2** by I_i, note that $V(S \cup \{i\}) - V(S) = V(i)$ gives

$$
\begin{aligned}
m_i &= \sum_{S \subseteq N-\{i\}} \left[V(S \cup \{i\}) - V(S) \right] \\
&= \sum_{S \subseteq N-\{i\}} V(i) \\
&= 2^{|N|-1} V(i).
\end{aligned}
$$

Hence,

$$
I_i(G) \Big/ w_i = m_i w_i \Big/ w_i 2^{|N|-1} = m_i \Big/ 2^{|N|-1} = V(i), \tag{19}
$$

which shows that I_i meets **A2**.

Next we verify fulfillment of **A3** by I_i.

Now, let $\zeta = \{S \subseteq N - \{i\}\}$. Clearly, we can write ζ as $\zeta_1 \cup \zeta_2$, where $\zeta_1 = \{S \subseteq N - \{i, j\}\}$ and $\zeta_2 = \{S \subseteq N - \{i\}$ and $j \in S\}$. We rewrite $S \in \zeta_2$ as $S' \cup \{j\}$, where $S' \subseteq N - \{i, j\}$. Then,

$$
\begin{aligned}
m_i &= \sum_{S \subseteq N-\{i\}} \left[V(S \cup \{i\}) - V(S) \right] \\
&= \sum_{S \in \zeta} \left[V(S \cup \{i\}) - V(S) \right] \\
&= \sum_{S \in \zeta_1} \left[V(S \cup \{i\}) - V(S) \right] + \sum_{S \in \zeta_2} \left[V(S \cup \{i\}) - V(S) \right] \\
&= \sum_{S \subseteq N-\{i,j\}} \left[V(S \cup \{i\}) - V(S) \right] + \sum_{S' \subseteq N-\{i,j\}} \left[V(S' \cup \{i, j\}) - V(S' \cup \{j\}) \right] \qquad (20)
\end{aligned}
$$

We can rewrite m_i in (20) as

$$
m_i = \sum_{S \subseteq N-\{i,j\}} \left[V(S \cup \{i\}) - V(S) \right] + \sum_{S \subseteq N-\{i,j\}} \left[V(S \cup \{i, j\}) - V(S \cup \{j\}) \right], \tag{21}
$$

which on simplification becomes $m_i = \sum_{S \subseteq N-\{i,j\}} \left[V(S \cup \{i, j\}) - V(S) \right]$, since by hypothesis $V(S \cup \{i\}) = V(S \cup \{j\}), \forall S \subseteq N - \{i, j\}$.

By a similar calculation we get $m_j = \sum_{S \subseteq N-\{i,j\}} \left[V(S \cup \{i, j\}) - V(S) \right]$.

Hence $m_i = m_j$. Therefore, $I_i(G) \Big/ w_i = m_i \Big/ 2^{|N|-1} = m_j \Big/ 2^{|N|-1} = I_j(G) \Big/ w_j$, which shows that I_i meets **A3**.

Finally, let $G=(N;V;\,\mathbf{w})$ be a weighted voting game where $V=U_N$. Let $G'=(N';V';\mathbf{w}')$ be the $(|N|-1)$ merged game associated with the pair of voters (i,j) in the game G. Then since $V'=U_{N'}$,

$$I_{ij}(G')\big/w'_{ij} = m'_{ij}w'_{ij}\big/2^{|N|-2}\,w'_{ij} = w'_{ij}\big/2^{|N|-2}\,w'_{ij} = 1\big/2^{|N|-2}.$$

Also,

$$I_i(G)\big/w_i + I_j(G)\big/w_j = m_i w_i\big/2^{|N|-1}\,w_i + m_j w_j\big/2^{|N|-1}\,w_j = w_i\big/2^{|N|-1}\,w_i + w_j\big/2^{|N|-1}\,w_j = 1\big/2^{|N|-2},$$

since $V=U_N$. Thus, I_i satisfies **A4.**

We now show that if a power index P_i satisfies **A1-A4**, then it must be I_i. First observe that any P_i is uniquely determined by its values on unanimity games. This is because, for any game $G=(N;V;\,\mathbf{w})\in\tilde{\mathbf{F}}$, $G=G_{S_1}\vee G_{S_2}\vee\ldots\vee G_{S_k}$, where $S_1,S_2,\ldots,S_k$ are minimal winning coalitions of G and G_{S_i} is the unanimity game corresponding to $S_i, i=1,2,\ldots,k$. Thus, by **A1**, $P_i(G)$ is determined if $P_i(G_{S_1})$, $P_i(G_{S_2}\vee G_{S_3}\vee\ldots G_{S_k})$ and $P_i(G_{S_1}\wedge(G_{S_2}\vee\ldots\vee G_{S_k}))$ are known. But, $G_{S_1}\wedge(G_{S_2}\vee\ldots\vee G_{S_k})=G_{S_1\cup S_2}\vee\ \ldots\vee G_{S_1\cup S_k}$ and hence, by induction hypothesis both $P_i(G_{S_2}\vee G_{S_3}\vee\ldots G_{S_k})$ and $P_i(G_{S_1}\wedge(G_{S_2}\vee\ldots\vee G_{S_k}))$ are determined. So $P_i(G)$ is determined.

In view of the above discussion, we can say that it is enough to determine $P_i(N;U_N;\,\mathbf{w})$ for any unanimity game. We shall prove by induction on $|N|$ that,

$$P_i(N;U_N;\,\mathbf{w}) = w_i\big/2^{|N|-1}.$$

If $|N|=1$, then by **A2**, $P_i(N;U_N;\mathbf{w})\big/w_i=1$. So assume $|N|>1$. Let $i\neq j$ be two voters in N and for the merged game $(N',U_{N'},\mathbf{w}')$ associated with the pair (i,j), we have, by **A4**,

$$\frac{P_i(N;U_N;\mathbf{w})}{w_i} + \frac{P_j(N;U_N;\mathbf{w})}{w_j} = \frac{P_{ij}(N';U_{N'};\mathbf{w}')}{w'_{ij}}. \qquad (22)$$

By induction hypothesis,

$$\frac{P_{ij}\left(N';U_{N'};\mathbf{w}'\right)}{w'_{ij}}=\frac{1}{2^{|N|-2}}\,. \qquad (23)$$

Also by **A3**,

$$\frac{P_i\left(N;U_N;\mathbf{w}\right)}{w_i} = \frac{P_j\left(N;U_N;\mathbf{w}\right)}{w_j}\,. \qquad (24)$$

Hence by (22)-(24), we have

$$2\,\frac{P_i\left(N;U_N;\mathbf{w}\right)}{w_i} = \frac{1}{2^{|N|-2}}\,,$$

which gives $P_i\left(N;U_N;\mathbf{w}\right) = {w_i}/{2^{|N|-1}}$.

Thus the values of P_i coincide with I_i on unanimity games and hence on all weighted voting games. □

We may now give an example to illustrate how the power of a voter in a game can be calculated from his powers in the minimal winning coalitions in it. Consider the weighted majority game $G=(N;V;\ \mathbf{w};q)$, where $N=\{A_1,A_2,A_3,A_4\}$, $\mathbf{w}=(4,3,2,1)$ and the quota $q=7$. (This example is due to Straffin, 1994). The minimal winning coalitions are $S_1=\{A_1,A_2\}$ and $S_2=\{A_1,A_3,A_4\}$. Hence, denoting the unanimity game for S_i by G_{S_i} (i=1,2), and letting $(N;U_N)=G_N$, we get

$$I_i(G)=I_i\left(G_{S_1}\right)+I_i\left(G_{S_2}\right)-I_i\left(G_N\right).$$

Suppose now, that $i=A_1$, that is, we want to find the power of voter A_1 in G. Then,

$$I_{A_1}(G)=\frac{4}{2}+\frac{4}{2^2}-\frac{4}{2^3}$$

$$= 2.5.$$

Likewise, we determine powers for other voters.

5. Concluding Remarks

Power of an individual voter depends on the chance he has of being critical to the passage or defeat of a resolution. This paper suggests an index of power for an individual voter in a weighted majority game, a very popular and common type of voting game. The index is found to satisfy all the intuitively compelling axioms suggested in the literature for a voting power index. We then relate the new index with some of the existing indices in order to investigate its relative performance. An axiomatic characterization of the new index has also been carried out for getting a greater insight of it.

An interesting extension of our analysis will be to check independence of the axioms A1-A4. By independence we mean that if one of these axioms is dropped, then there will be a power index other than I_i in (6) that will satisfy the remaining three axioms but not the dropped one. That is, independence says that none of the axioms A1-A4 implies or is implied by another. This is left as a future research programme.

One limitation of our index is that it is applicable to weighted games only and dependent on the weights chosen. However, weighted games are extremely easy to imagine and arise in real life quite frequently. From this perspective, our index has a very clear merit.

Notes

1. Felsenthal and Machover (1995, theorem 11.1) implicitly demonstrated proposition1 while determining the value of the second Banzhaf-Coleman index for a bloc voter. However, our proof of the proposition is different from their proof.
2. Strictly speaking, the two Coleman indices are given by $E_{1i} = {}^{m_i}\!/_{\omega}$ and $E_{2i} = {}^{m_i}\!/_{\left(2^{|N|} - \omega\right)}$, where ω is the total number of winning coalitions in the game. We, however, follow the convention adopted in the literature and refer to B_{1i} - B_{3i} as the Banzhaf-Coleman indices. It is easy to see that E_{1i} and E_{2i}, which are called indices of power to prevent action and power to initiate action and whose harmonic mean becomes B_{2i}, may behave in the same way as B_{1i} with respect to **TRP**.
3. Nowak and Radzik (2000) suggested a similar 'Weight Proportionality' axiom for mutually dependent voters, where two voters i and j in a game $(N;V)$ are called mutually dependent if $V(S \cup \{i\}) = V(S) = V(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$.

References

1. Banzhaf, J. F. (1965): Weighted Voting Doesn't Work: A Mathematical Analysis, Rutgers Law Review, 19, 317-343.

2. Brams, S.J.(1975): Game Theory and Politics, Free Press, New York.
3. Brams, S.J. and P.J. Affuso (1976): Power and Size: A New Paradox, Theory and Decision, 7, 29-56.
4. Brink, R. van den and G. van der Laan (1998): Axiomatizations of the Normalized Banzhaf Value and the Shapley Value, Social Choice and Welfare 15,567-582.
5. Burgin, M. and L.S. Shapley (2001): Enhanced Banzhaf Power Index and Its Mathematical Properties, WP-797, Department of Mathematics, UCLA.
6. Coleman, J.S. (1971): Control of Collectives and the Power of a Collectivity to Act, in B. Lieberman (ed.) Social Choice, Gordon and Breach, New York, pp. 269-298.
7. Deegan, J. and E.W. Packel (1978): A New Index of Power for Simple n- Person Games, International Journal of Game Theory, 7, 113-123.
8. Deegan J. and E.W. Packel (1982): To the (Minimal Winning) Victors Go the (Equally Divided) Spoils: A New Index of Power for Simple n-Person Games, in S.J. Brams , W.F. Lucas and P.D. Straffin (eds) Political and Related Models, Springer-Verlag, New York, pp. 239-255.
9. Dreyer, J and A. Schotter (1980): Power Relationships in the International Monetary Fund: The Consequences of Quota Changes, Review of Economics and Statistics, 62,97-106.
10. Dubey, P. (1975): On the Uniquness of the Shapley Value, International Journal of Game Theory, 4, 131-140.
11. Dubey, P. and L.S. Shapley (1979): Mathematical Properties of the Banzhaf Power Index, Mathematics of Operations Research, 4,99-131.
12. Felsenthal, D.S. and M. Machover (1995): Postulates and Paradoxes of Relative Voting Power – A Critical Reappraisal, Theory and Decision, 38, 195 – 229.
13. Felsenthal, D. and M. Machover (1998): The Measurement of Voting Power, Edward Elgar, Cheltenham.
14. Fischer, D. and A Schotter(1978): The Inevitability of the Paradox of Redistribution in the Allocation of Voting Weights, Public Choice, 33, 49-67.
15. Haller, H. (1994): Collusion Properties of Values, International Journal of Game Theory, 23, 261-281.
16. Johnston, R.J. (1978): On the Measurement of Power: Some Reactions to Laver, Environment and Planning A, 10, 907-914.
17. Kilgour, D.M. (1974): A Shapley Value for Cooperative Games with Quarrelling , in A. Rapoport (ed.) Game Theory as a Theory of Conflict Resolution, Reidel, Boston.
18. Lehrer, E. (1988): An Axiomatization of the Banzhaf Value, International Journal of Game Theory, 17, 89-99.
19. Lucas, W. F. (1982): Measuring Power in Weighted Voting Systems, in S.J. Brams, W.F. Lucas and P.D. Straffin (eds.), Political and Related Models, Springer-Verlag, New York, pp. 183-238.
20. Nowak, A.S. and T. Radzik (2000): An Alternative Characterization of the Weighted Banzhaf Value, International Journal of Game Theory, 29, 127-132.
21. Owen, G. (1978): Characterization of the Banzhaf- Coleman Index, SIAM Journal of Applied Mathematics, 35, 315-327.
22. Owen, G. (1978a): A Note on the Banzhaf – Coleman Axioms, in P. Ordeshook (ed.) Game Theory and Political Science, New York University Press, New York, pp. 451-461.
23. Roth, A.E. (1988): Introduction to the Shapley Value, in A.E. Roth (ed.) The Shapley Value, Essays in Honor of L. Shapley, Cambridge University Press, Cambridge, pp.1-27.
24. Shapley, L.S. (1953): A Value for n-Person Games, in H.W. Kuhn and A.W. Tucker (eds.)Contributions to the Theory of Games II (Annals of Mathematics Studies), Princeton University Press, Princeton, pp. 307-317.
25. Shapley, L.S. and M.J. Shubik (1954): A Method for Evaluating the Distribution of Power in a Committee System, American Political Science Review, 48, 787-792.

26. Straffin,, P.D. (1977): Homogeneity, Independence and Power Indices, Public Choice , 30, 107-118.
27. Straffin, P.D.(1982): Power Indices in Politics , in S.J. Brams, W.F. Lucas and P.D. Straffin (eds.) Political Science and Related Models, Springer-Verlag, New York, pp. 256-321.
28. Straffin, P.D. (1994): Power and Stability in Politics, in R.J. Aumann and S. Hart (eds.) Handbook of Game Theory with Economic Applications, Vol. 2, North Holland, Amsterdam, pp. 1128-1151.
29. Taylor, A.D. (1995): Mathematics and Politics: Strategy, Voting, Power and Proof, Springer-Verlag, New York.