A NEW CHARACTERIZATION OF THE BANZHAF INDEX OF POWER *

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Abstract

This paper develops a new axiomatic characterization of the Banzhaf index of power using four axioms from four different contributions to the area.A nice feature of the characterization is independence of the axioms showing importance of each of them in the exercise.

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1. Introduction

A central concept of political science is power. While power is a many faceted phenomenon, here we are concerned with the notion of power as it is reflected in the formal voting system. If in a voting situation everyone has one vote and the majority rule is taken as the decisive criterion, then everyone has the same type of power. The majority rule declares a candidate as the winner if he gets the maximum number of votes among the candidates. But if some persons have more votes than others, then they can certainly manipulate the voting outcome by exercising their additional votes.

An index of voting power should reflect a voter's influence, in a numerical way, to bring about the passage or defeat of some bill. It should be based on the voter's importance in casting the deciding vote. The most well-known indices of voting power are the Shapley-Shubik (1954) and the Banzhaf(1965) indices. The latter index is based on the number of coalitions in which the concerned voter is swing. A swing voter in a coalition is the person whose deletion from the coalition transforms it from a winning to a losing one. (A coalition of voters is called winning if passage of a bill is guaranteed by 'yea' votes from exactly the voters in that coalition. Coalitions that are not winning are called losing.) The Shapley-Shubik index for voter i is the fraction of orderings for which i is the swing voter. In fact, the Shapley-Shubik index is an application of the well-known Shapley value (Shapley,1953) to a voting game, which is a formulation of a voting system in a coalitional form game.

The voting power indices can give quite different results. One index may give considerably more power to some voters than another. In view of this, it is necessary to characterize alternative indices axiomatically for understanding which indices become more appropriate in which situation. An axiomatic characterization gives us an insight of the underlying index in a more elaborate way through the axioms employed in the characterization exercise. Interesting characterizations of the Shapley-Shubik and the Banzhaf indices were developed and discussed, among others, by Dubey(1975), Straffin(1977,1994), Owen(1978,1978a), Dubey and Shapley(1979), Lehrer(1988), Roth(1988), Haller(1994), Brink and Laan(1998), Felsenthal and Machover(1998), Nowak and Radzik(2000), Burgin and Shapley(2001) and Laruelle and Valenciano(2001) .

This paper develops a new characterization of the Banzhaf index using four axioms. These axioms or their themes are taken directly from four different contributions to the area.Thus our characterization shows the importance of this set of existing axioms from a new perspective.A very attractive feature of this characterization is independence of the axioms. By independence we mean that if one of these axioms is dropped, then there will be a power index other than the Banzhaf index that will satisfy the remaining three axioms, but not the dropped one. That is, independence says that none of the axioms implies or implied by another.

The next section discusses the background material. Section 3 presents the characterization theorem and demonstrates independence of the axioms. Finally, section 4 concludes.

2. Notation, Definitions and Preliminaries

It is possible to model a voting situation as a coalitional form game, the hallmark of which is that any subgroup of players can make contractual agreements among its members independently of the remaining players. Let $N = \{A_1, A_2, ...A_n\}$ be a set of players. For any set of players *N*, $|N|$ will stand for the number of players in *N*. The power set of *N*, that is, the collection of all subsets of N is denoted by 2^N. Any member of 2^N is called a coalition. A coalitional form game with player set N is a pair $(N;V)$, where $V : 2^N \to R$ such that $V(\phi) = 0$, where R is the real line. For any coalition S, the real number $V(S)$ is the worth of the coalition, that is, this is the amount that S can guarantee to its members.

We frame a voting system as a coalitional form game by assigning the value 1 to any coalition which can pass a bill and 0 to any coalition which cannot. In this context, a player is a voter and the set $N = \{A_1, A_2, ...A_n\}$ is called the set of voters. Throughout the paper we assume that voters are not allowed to abstain from voting. A coalition S will be called winning or losing according as it can or cannot pass a resolution.

Definition 1: Given a set of voters N, a voting game associated with N is a pair $(N;V)$, where $V : 2^N \to \{0,1\}$ satisfies the following conditions:

(i) $V(\phi)=0$.

(ii) $V(N)=1$.

(iii) If $S \subseteq T$, $S,T \in 2^N$, then $V(S) \le V(T)$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. It follows that the empty coalition ϕ is losing (condition (i)) and the grand coalition N is winning (condition (ii)). All other coalitions are either winning or losing. Condition (iii) ensures that if a coalition S can pass a bill, then any superset T of S can pass it as well. A voting game is **proper** if for $S,T \in 2^N$, $V(S)=V(T)=1$ implies that $S \cap T \neq \phi$. According to this condition two winning coalitions cannot be disjoint. The collection of all voting games is denoted by $\boldsymbol{F}$.

Definition 2: The unanimity game $(N;U_N)$ associated with a given set of voters *N* is the game whose only winning coalition is the grand coalition *N*.

Definition 3: Let $(N;V) \in F$ be arbitrary.

(i) For any coalition $S \in 2^N$, we say that $i \in N$ is swing in S if $V(S)=1$ but $V(S-\{i\})=0$.

(ii) For any coalition $S \in 2^N, i \in N$ is said to be swing outside S if $V(S)=0$ but $V(S \cup \{i\})=1$.

(iii) A coalition $S \in 2^N$ is said to be minimal winning if $V(S)=1$ but there does not exist $T \subset S$ such that $V(T)=1$.

Thus, voter i is swing, also called pivotal or key, in the winning coalition S if his deletion from S makes the resulting coalition $S-\{i\}$ losing. Similarly, voter i is swing outside the losing coalition S if his addition to S makes the resulting coalition $S \cup \{i\}$ winning. For any voter i, the number of winning coalitions in which he is swing is same as the number of losing coalitions outside which he is swing (Burgin and Shapley, 2001, Corollary 4.1). For any game $G=(N;V)$, we denote this common number by $m_i(G)$,or, simply by m_i.

Definition 4: For any $(N;V)\in F$, voter $i\in N$ is called a dummy in $(N;V)$ if he is never swing in the game. A voter $i\in N$ is called a non-dummy in $(N;V)$ if he is not dummy in $(N;V)$.

Following Felsenthal and Machover (1998) we have

Definition 5: For any $(N;V)\in F$, voter $i\in N$ is called a dictator if $\{i\}$ is the sole minimal winning coalition in the game.

By definition, a dictator in a game is unique. If a game has a dictator i, then i is the only swing voter in the game. That is, if i is a dictator, m_i is maximized .

A common form of voting game is a weighted majority game $(N;V;q;w_1,w_2,.....,w_n)$. Here voter i casts w_i votes and q ,where $\sum_{i\in N} w_i \geq q$, is the quota of votes needed to pass a bill.That is, $V(S)=1$ if and only if $\sum_{i\in S} w_i \geq q$ .A weighted majority game will be proper if $2q > \sum_{i\in N} w_i$.

3. The Characterization Theorem

The **Banzhaf power index** of voter i, I_i, in a voting game $G=(N;V)$ is defined as m_i, the number of swings of voter i ,divided by $2^{|N|-1}$. Formally, $I_i : F \to R_+$ is defined by

$$I_i(G) = m_i / 2^{|N|-1}, \qquad (1)$$

where R_+ is the nonnegative part of the real line.

For any $G=(N;V)\in F$, for any $i\in N$, I_i achieves its minimum value, zero, if and only if i is a dummy. It remains invariant under any permutation of the voters. If a dummy is excluded from the game, I_i does not change. Similarly, it remains unaltered if a dummy is included in the game (see Felsenthal And Machover,1998). If in a voting game, each voter $i's$ probability p_i of voting 'yes' or ‘no’ on a bill is chosen independently from the uniform distribution [0,1], then the power of the voter i is estimated by I_i (Straffin, 1977). Since I_i does not involve numbers of coalitions in which voters other than i are swing, Felsenthal and Machover (1998) regarded it as an absolute index of voter i's power.

The following definitions will be necessary for presenting one of the axioms.

Definition 6: Given $G_1 = (N_1;V_1)$, $G_2 = (N_2;V_2) \in F$, we define, $G_1 \vee G_2$ as the game with the set of voters $N_1 \cup N_2$ and in which a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if either $V_1(S \cap N_1) = 1$ or $V_2(S \cap N_2) = 1$.

Definition 7: Given $G_1 = (N_1;V_1)$, $G_2 = (N_2;V_2) \in F$, we define, $G_1 \wedge G_2$ as the game with the set of voters $N_1 \cup N_2$ and in which a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ and $V_2(S \cap N_2) = 1$.

Thus, in order to win in $G_1 \vee G_2$, a coalition must win in either G_1 or in G_2, whereas to win in $G_1 \wedge G_2$, it has to win in both G_1 and G_2.

As Lehrer(1988), Nowak and Radzik(2000) and Felsenthal and Machover(1998) pointed out, formation of blocs by voters is an important issue in voting game. We now formally define the reduced game when two voters i and j form a bloc ij.

Definition 8: Let $(N;V) \in F$ be arbitrary. Suppose that the voters $i, j \in N$ form a bloc ij. Then the resulting voting game is the pair $(N';V')$, where $N' = N - \{i,j\} \cup \{ij\}$ and

$V'(S) = V(S)$ if $S \subseteq N' - \{ij\}$,

$\quad = V((S - \{ij\}) \cup \{i,j\})$ if $ij \in S$.

We are now in a position to present four axioms on a power index $P_i : F \to R_+$ that uniquely determines the new index I_i in (1). The first axiom is taken from Dubey (1975) (see also Dubey and Shapley ,1979). It shows that the sum of powers of voter i in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ is equal to the sum of his powers in G_1 and G_2.

Axiom A1 (Sum Principle): For $G_1 = (N_1;V_1), G_2 = (N_2;V_2) \in F$,

$$P_i(G_1 \vee G_2) + P_i(G_1 \wedge G_2) = P_i(G_1) + P_i(G_2). \qquad (2)$$

The idea of the next axiom, which makes a specification about a dictator's power, is taken from Felsenthal and Machover(1998).A dictator, if there is one, should posses maximum power in the game since he can be characterized as the only non-dummy voter.

Axiom A2 (Maximal Power Specification): For any game $G = (N;V) \in F$, if i is a dictator in the game, then

$$P_i(G) = 1 . \qquad (3)$$

The third axiom, which is formulated in terms of substitutability between two voters, is taken from Lehrer(1988). Two voters in a game are said to be substitutes if the worth of an arbitrary coalition in the game becomes the same when they join the coalition separately (Shapley, 1953). Therefore, it is reasonable to expect that their powers are the same. More precisely, we have the following axiom.

Axiom A3 (Equal Treatment): Let voters i and j be substitutes in the game $G = (N;V) \in F$, that is, $V(S \cup \{i\}) = V(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$. Then,

$$P_i(G) = P_j(G) \qquad (4)$$

The next axiom shows the relationship between the power of a bloc and its constituents in an unanimity game. It is similar to axiom A5 of Nowak and Radzik (2000) (see also Lehrer, 1988).

Axiom A4 (Two-Voter Bloc Principle): Let $G' = (N', V') \in F$ be the $(|N| - 1)$-person game obtained from the game $G = (N;V)$ when the voters $i, j \in N$ form a bloc ij, where $V = U_N$, N' and V' are same as in the definition 8.Then

$$P_{ij}(G') = P_i(G) + P_j(G) \quad . \qquad (5)$$

<u>Theorem 1</u>: A power index P_i satisfies axioms A1-A4 if and only if P_i is the index I_i in (1).

<u>Proof</u>: We will first show that I_i satisfies all the axioms **A1** through **A4**.

To show that **A1** is satisfied by I_i, first let $i \in N_1 - N_2$. Now, any subset S' of $N_2 - N_1$ can be appended to a swing coalition $S \subseteq N_1$ for i in G_1 to obtain a swing coalition $S \cup S'$ for i in $G_1 \vee G_2$ unless $(S \cup S') \cap N_2$ is winning in G_2. Hence the number of swings for voter i in $G_1 \vee G_2$ is

$m_i^{G_1 \vee G_2} = m_i^{G_1} 2^{|N_2 - N_1|} - m_i^{G_1 \wedge G_2}$, where $m_i^{G_1 \wedge G_2}$ is the number of swings of i in $G_1 \wedge G_2$. Since for $i \in N_1 - N_2$, $m_i^{G_2}$=0, we rewrite $m_i^{G_1 \vee G_2}$ as $m_i^{G_1 \vee G_2} = m_i^{G_1} 2^{|N_2 - N_1|} + m_i^{G_2} 2^{|N_1 - N_2|} - m_i^{G_1 \wedge G_2}$.

The same expression for $m_i^{G_1 \vee G_2}$ will be obtained if $i \in N_1 \cap N_2$ or $i \in N_2 - N_1$.

Therefore,

$$I_i(G_1 \vee G_2) = \frac{m_i^{G_1}}{2^{|N_1|-1}} + \frac{m_i^{G_2}}{2^{|N_2|-1}} - \frac{m_i^{G_1 \wedge G_2}}{2^{|N_1 \cup N_2|-1}}, \tag{6}$$

which in turn gives

. $$I_i(G_1 \vee G_2) + I_i(G_1 \wedge G_2) = I_i(G_1) + I_i(G_2).$$

This shows that I_i verifies **A1**.

To check satisfaction of **A2** by I_i, note that if i is a dictator in the game $G = (N;V)$, then i is the only non-dummy voter in the game, that is, m_i is maximized, which means that $m_i = 2^{|N|-1}$ and $m_j = 0$ for all $j \neq i$. Substituting this value of m_i in (1), we get $I_i(G) = 1$, which shows that I_i meets **A2**.

Next we verify fulfillment of **A3** by I_i.

Now, let $\zeta = \{S \subseteq N - \{i\}\}$. Clearly, we can write ζ as $\zeta_1 \cup \zeta_2$, where $\zeta_1 = \{S \subseteq N - \{i, j\}\}$ and $\zeta_2 = \{S \subseteq N - \{i\}$ and $j \in S\}$. We rewrite $S \in \zeta_2$ as $S' \cup \{j\}$, where $S' \subseteq N - \{i, j\}$. Then,

$$\begin{aligned} m_i &= \sum_{S \subseteq N - \{i\}} [V(S \cup \{i\}) - V(S)] \\ &= \sum_{S \in \zeta} [V(S \cup \{i\}) - V(S)] \\ &= \sum_{S \in \zeta_1} [V(S \cup \{i\}) - V(S)] + \sum_{S \in \zeta_2} [V(S \cup \{i\}) - V(S)] \\ &= \sum_{S \subseteq N - \{i,j\}} [V(S \cup \{i\}) - V(S)] + \sum_{S' \subseteq N - \{i,j\}} [V(S' \cup \{i, j\}) - V(S' \cup \{j\})]. \end{aligned} \tag{7}$$

We can rewrite m_i in (7) as

$$m_i = \sum_{S \subseteq N - \{i,j\}} [V(S \cup \{i\}) - V(S)] + \sum_{S \subseteq N - \{i,j\}} [V(S \cup \{i, j\}) - V(S \cup \{j\})], \tag{8}$$

which on simplification becomes $m_i = \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S)]$, since by hypothesis $V(S \cup \{i\}) = V(S \cup \{j\}), \forall S \subseteq N - \{i,j\}$.

By a similar calculation we get $m_j = \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S)]$.

Hence $m_i = m_j$. Therefore, $I_I(G) = m_i / 2^{|N|-1} = m_j / 2^{|N|-1} = I_J(G)$, which shows that I_i meets **A3**.

Finally, let $G = (N;V)$ be such that $V = U_N$. Let $G' = (N';V')$ be the $(|N|-1)$-person game when the voters $i, j \in N$ form a bloc ij. Then since $V' = U_{N'}$, $I_{ij}(G') = m'_{ij} / 2^{|N|-2} = 1 / 2^{|N|-2}$,where $m'_{ij} = m_{ij}(G')$.

Also, $I_i(G) + I_j(G) = m_i / 2^{|N|-1} + m_j / 2^{|N|-1} = 1 / 2^{|N|-1} + 1 / 2^{|N|-1} = 1 / 2^{|N|-2}$, since $V = U_N$. Thus, I_i satisfies **A4.**

We now show that if a power index P_i satisfies **A1-A4**, then it must be I_i. First observe that any P_i is uniquely determined by its values on unanimity games. This is because, for any game $G = (N;V)$, $G = G_{S_1} \vee G_{S_2} \vee .... \vee G_{S_k}$, where $S_1, S_2, ..., S_k$ are minimal winning coalitions of G and G_{S_i} is the unanimity game corresponding to $S_i, i = 1,2,...,k$. Thus, by **A1**, $P_i(G)$ is determined if $P_i(G_{S_1})$, $P_i(G_{S_2} \vee G_{S_3} \vee .... G_{S_k})$ and $P_i(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are known. But, $G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}) = G_{S_1 \cup S_2} \vee \; ... \vee G_{S_1 \cup S_k}$ and hence, by induction hypothesis both $P_i(G_{S_2} \vee G_{S_3} \vee .... G_{S_k})$ and $P_i(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are determined. So $P_i(G)$ is determined.

In view of the above discussion, we can say that it is enough to determine $P_i(N;U_N)$ for any unanimity game. We shall prove by induction on $|N|$ that $P_i(N;U_N) = 1 / 2^{|N|-1}$. If $|N|=1$, then by **A2**, $P_i(N;U_N) = 1 / 2^{|N|-1}$. So assume $|N|>1$. Let $i \neq j$ be two voters in N and for the game $(N';U_{N'})$ when $i, j \in N$ form a bloc ij, we have, by **A4,**

$$P_i(N;U_N) + P_j(N;U_N) = P_{ij}(N';U_{N'}) . \qquad (9)$$

By induction hypothesis,

$$P_{ij}(N';U_{N'}) = \frac{1}{2^{|N|-2}} . \qquad (10)$$

Also by **A3**,

$$P_i(N;U_N) = P_j(N;U_N) . \qquad (11)$$

Hence by (9)-(11), we have

$$2P_i(N;U_N) = \frac{1}{2^{|N|-2}} ,$$

which gives $P_i(N;U_N) = \frac{1}{2^{|N|-1}}$ .Thus the values of P_i coincide with I_i on unanimity games and hence on all voting games. Δ

We now give an example to see how the power of a voter in a game can be calculated using his powers in the minimal winning coalitions of the game.Consider the weighted majority game G with the voter set $N = \{A_1, A_2, A_3, A_4\}$, where $w_1 = 4, w_2 = 3, w_3 = 2, w_4 = 1$ and $q = 7$ (see Straffin,1994).The minimal winning coalitions here are $S_1 = \{A_1, A_2\}$and $S_2 = \{A_1, A_3, A_4\}$. Denoting the unanimity game for S_j by G_{S_j} $(j = 1,2)$,we get $I_i(G) = I_i(G_{S_1}) + I_i(G_{S_2}) - I_i(G_N)$,where G_N is the unanimity game related to N .Suppose now that $i = A_1$ .Then $I_{A_1}(G) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$ .Similarly we can calculate the powers of other voters.

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Finally, we demonstrate independence of the axioms. Independence means that the given set of axioms is minimal in the sense that none of its proper subset will characterize the Banzhaf index.

<u>**Theorem 2**</u>**: Axioms A1-A4 are independent.**

<u>**Proof**</u>:(1) Since the index given by

$$B_{1i}(G) = \frac{\log_2(m_i)}{|N| - 1},$$

where $|N|$ >2,is nonlinear in m_i , it fails to satisfy A1, but it satisfies A2-A4.

(2) Because the index

$$B_{2i} = \frac{m_i}{2^{|N|}},$$

is not appropriately normalized, it is a violator of A2. Nevertheless, it verifies A1, A3 and A4.

(3) Consider the index

$$B_{3i} = \frac{m_i}{2^{|N|-1}} + \alpha_i,$$

where α_i satisfies the following conditions:(a) α_i =0 or <0 according as i is a dictator or not, (b) $\alpha_i \neq \alpha_j$ if $i \neq j$ and (c) if two voters i and j form a bloc ij ,then $\alpha_{ij} = \alpha_i + \alpha_j$. Since α_i's are different across voters, A3 is violated. However, A1, A2 and A4 are satisfied by B_{3i}.

(4) Finally, the index given by

$$B_{4i}(G) = \frac{m_i}{2^{|N|}} + \frac{1}{2},$$

does not meet A4 because of the presence of ½ on the right-hand side. However, it fulfils A1-A3. Δ

4. Concluding Remarks

Power of an individual voter depends on the chance he has of being critical to the passage or defeat of a resolution. The well-known Banzhaf index is a normalized value of the number of coalitions in which the voter is in the critical position of making winning(losing) coalitions losing(winning).Several characterizations of this index have been proposed in the literature. In this paper we provide a new characterization of the index using four axioms from four different contributions to the area. Independence of the axioms is also demonstrated.

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