Isotropic C^1-immersions in a pseudo-Riemannian manifold

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Abstract

We study the existence of isotropic C^1-immersions in a pseudo-Riemannian manifold (E, h) following the Convex Integration techniques of Gromov.

Keywords: Pseudo-Riemannian manifold; Isotropic immersion

1. Introduction

The aim of this paper is to study isometric C^1-immersions for indefinite metrics. Our study departs from the discussion by Gromov on this subject in [2, 2.4.9 (A), (B)], where the Nash–Kuiper theorem is stated in the form of h-principle and a generalisation of it to indefinite forms is outlined. In particular, Gromov derives the h-principle for isotropic immersions of a manifold V into a pseudo-Riemannian manifold (W, h) for the case when the signature of h is (r_+, r_-) for $r_+ \geqslant \dim V + 1$ and $r_- \geqslant \dim V + 1$. Recall that a C^1-map $f : V \to (W, h)$ is called *isotropic* if $f^*(h) = 0$. Here we shall be concerned with the case not covered by Gromov's discussion, namely, the case where $r_+ = \dim V$ or $r_- = \dim V$. Notice that when $r_+ = r_- = \dim V$, the situation is very rigid for the existence of the isotropic immersions. We shall prove in this article, that if W fibres over V in a certain way then it is possible to obtain h-principle for isotropic immersions when $r_+ = \dim V$ or $r_- = \dim V$.

Let us recall the Nash–Kuiper theorem [3,4].

Theorem 1.1. *Let* (V, g) *and* (W, h) *be two Riemannian manifolds such that* $\dim V < \dim W$. *If there exists a strictly short* C^∞*-immersion* $f_0 : V \to W$ *then* f_0 *can be homotoped to an isometric* C^1*-immersion* $f : (V, g) \to (W, h)$.

Consider the product manifold $E = V \times W$ with the pseudo-Riemannian metric $\bar{h} = g \ominus h$ (that is, $g \oplus -h$). Notice that the signature (r_+, r_-) of $\bar{h}$ satisfies the relations $r_+ = \dim V$ and $r_- > \dim V$. The above result can be stated in the following equivalent form.

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Pre-main Theorem. *Let V and $(E, \bar{h})$ be as in the above. If there exists a C^∞-section $f_0 : V \to V \times W = E$ such that $f_0^*(\bar{h})$ is positive definite, and $p_2 \circ f_0$ is an immersion, then f_0 can be homotoped to a C^1-isotropic section $f : V \to (E, \bar{h})$.*

Our purpose here is to extend this statement to the sections of a bundle $E \to V$, where the total space E is endowed with a pseudo-Riemannian metric h of signature (r_+, r_-), for $r_+ \geqslant \dim V$ and $r_- \geqslant \dim V + 1$. To do this, we shall start with a section $f_0 : V \to E$ which induces a Riemannian metric $g = f_0^* h$ on V, and investigate under what conditions it is possible to homotope f_0 to an isotropic immersion.

The organisation of the paper is as follows. In Section 2 we formulate the problem and state the main result of this paper as Theorem 2.4. In Section 3, we review the convex integration theory of Gromov [2], since the proof of Theorem 2.4 relies extensively on this theory. In Section 4, we prove the main result.

2. Initial set up of the problem and the statement of the Main Theorem

Let us now focus our attention on the Pre-main Theorem above. There we have:

(1) The product bundle $p : E = V \times W \to V$, where the fibre dimension is strictly greater than $\dim V$, and a pseudo-Riemannian metric $\bar{h}$ on E of signature (r_+, r_-), for $r_+ = \dim V$ and $r_- = \dim W > \dim V$.
(2) The restriction of the pseudo-Riemannian metric $\bar{h}$ to the fibres of the fibration p is negative definite.
(3) The initial map $f_0 : V \to E$ is a section satisfying the following properties:
 (i) $f_0^*(\bar{h}) > 0$, that is $f_0^*(\bar{h})$ is positive definite.
 (ii) $p_2 \circ (df_0)_x$ is an injective linear map for each $x \in V$, where $p_2 : V \times W \to W$ is the projection onto the second factor.

Definition 2.1. Let (E, h) be a pseudo-Riemannian manifold, which means that h is a non-degenerate form on E. A surjective map $p : (E, h) \to V$ is said to be a *negative submersion* if p is a submersion and $h|_{p^{-1}(v)}$ is negative definite for all $v \in V$. Note that since p is a submersion, the fibres $E_v = p^{-1}(v)$, $v \in V$, are submanifolds of E.

Definition 2.2. A map $f : V \to (E, h)$ is called *positive* if $f^*(h) > 0$ (i.e., if $f^*(h)$ is positive definite).

Observation. Given a pseudo-Riemannian manifold (E, h) with a negative submersion $p : (E, h) \to V$, one can define two distributions η and ξ on E as follows: For each $e \in E$,

$$\eta_e = T_e\big(p^{-1}(x)\big) \quad \text{and} \quad \xi_e = T_e\big(p^{-1}(x)\big)^\perp,$$

where $x = p(e)$, and $T_e(p^{-1}(x))^\perp$ denotes the orthogonal complement of $T_e(p^{-1}(x))$ relative to h. Since h is non-degenerate and $h|_{\eta_e}$ is negative definite, the subspace ξ_e is complimentary to η_e in $T_e E$. Hence the tangent bundle TE splits into the direct sum of ξ and η; $TE = \xi \oplus \eta$. Consequently, we have two orthogonal projections $p_1 : TE \to \xi$ and $p_2 : TE \to \eta$.

Further, if the pseudo-Riemannian metric h is such that $r_+(h) = \dim V$, then $\dim \eta_e = r_-(h)$. Therefore, the restriction of h to the bundle ξ is positive definite.

Definition 2.3. Let (E, h) be a pseudo-Riemannian manifold and $p : (E, h) \to V$ be a negative submersion. Let η be the subbundle of TE defined by $\eta_e = T_e(p^{-1}(x))$, where $x = p(e)$, $e \in E$. A section $f : V \to E$ is said to be *co-injective* if $p_2 \circ df : TV \to TE$ is a bundle monomorphism over f, where p_2 denotes the orthogonal projection of TE onto η.

Observe that an h-isotropic section of $p : E \to V$ is necessarily co-injective. We are now ready to state the main result in this paper.

Theorem 2.4. *Let $p : E \to V$ be a fibre bundle and let h be a pseudo-Riemannian metric on E with signature (r_+, r_-) satisfying $r_+ = \dim V$ and $r_- > \dim V$. Suppose that p is a negative submersion and let $f_0 : V \to (E, h)$ be a positive* *co-injective section. Then there is a C^0-small homotopy of co-injective sections f_t, $0 \leqslant t \leqslant 1$, connecting f_0 to an h-isotropic section f_1.*

It may be noted that Theorem 1.1 is a consequence of the above result.

3. Review of convex integration techniques

In this section, we recall from [2] and [5] the basic terminology of the convex integration theory, and state the h-stability theorem for 'open relations' which is quoted below as Theorem 3.2. This result plays a central role in the theory of convex integration. Finally we discuss the case of non-open relations keeping in mind our specific problem.

Let $p : X \to V$ be a smooth fibre bundle of fibre dimension q over a manifold V of dimension m. Let $X^{(r)}$ denote the r-jet space of C^r-sections of p and let $p^r : X^{(r)} \to V$ be the natural projection map which is also a fibration.

A topological space $\mathcal{R}$ with a continuous map $\rho : \mathcal{R} \to X^{(r)}$ is said to be a *relation* over $X^{(r)}$. In particular, if $\mathcal{R}$ is a subset of $X^{(r)}$ and ρ is the inclusion map then $\mathcal{R}$ is referred as the rth order *partial differential relation* for sections of X.

The relation $\mathcal{R}$ is said to be *open* if ρ is a *microfibration*. Recall from [2, 1.4.2(B)] that a continuous map $f : A \to B$ is a microfibration (where A and B are topological spaces) if it satisfies the 'micro'-homotopy lifting property, which means that every covering homotopy problem for f relative to a pair $(P \times \{0\}, P \times I)$ admits a lift over $P \times [0, \varepsilon]$, for some $\varepsilon > 0$, where P is a compact polyhedron.

A *section* of $\mathcal{R}$ is a continuous map $\sigma : V \to \mathcal{R}$ such that $p^{(r)} \circ \rho \circ \sigma$ is the identity map of V. A section σ is called *holonomic* if $\rho \circ \sigma = j_f^r$ for a C^r-section $f : V \to X$. If $\mathcal{R}$ is a subset of $X^{(r)}$, then this section f is said to be a *solution* of $\mathcal{R}$.

The relation $\mathcal{R}$ is said to satisfy the *h-principle* if any section σ of $\mathcal{R}$ can be homotoped to a holonomic section of $\mathcal{R}$. The homotopy is a continuous map $H : V \times I \to \mathcal{R}$, such that $H(0)$ is σ and $H(1)$ is holonomic.

A relation $\tilde{\rho} : \widetilde{\mathcal{R}} \to X^{(r)}$ is said to be an *extension* of $\mathcal{R}$ if there exist an embedding $E : \mathcal{R} \to \widetilde{\mathcal{R}}$ and a retraction $\pi : \widetilde{\mathcal{R}} \to \mathcal{R}$ such that $\tilde{\rho} \circ E = \rho$ and $p^r_{r-1} \circ \rho \circ \pi = p^r_{r-1} \circ \tilde{\rho}$.

Let τ be a codimension 1 hyperplane field on V. Let f and g be germs at $x \in X$ of C^r smooth sections of X. We say that f and g are $\perp$-equivalent if

$$j_f^{r-1}(x) = j_g^{r-1}(x) \quad \text{and} \quad D\big(j_f^{r-1}\big)(x)|_\tau = D\big(j_g^{r-1}\big)(x)|_\tau,$$

where j_f^{r-1} and j_g^{r-1} are C^1 maps $V \to X^{(r-1)}$. For $r = 1$ this simply means that $f(x) = g(x)$ and $Df_x = Dg_x$ on τ_x. The $\perp$-equivalence is an equivalence relation on the r-jet space $X^{(r)}$. The equivalence class of $j_f^r(x)$ is denoted by $j_f^\perp(x)$ and is called the $\perp$-jet of f. If τ is integrable then we can choose local coordinate systems $(U; x_1, \ldots, x_n)$ so that $\{(x_1, \ldots, x_n)\colon x_n = \text{const}\}$ are integral submanifolds of τ. Moreover, we can express $j_f^r(x)$ as $(j_f^\perp(x), \partial_n^r f(x))$, where $\partial_n f$ denotes the partial derivative of f in the direction of x_n.

The set of all $\perp$-jets has a manifold structure [5, 6.1.1]. We denote this set by $X^\perp$. The natural projection map $p_\perp^r : X^{(r)} \to X^\perp$, taking an r-jet to its $\perp$-equivalence class, is an affine bundle where the fibres are affine space of dimension q. The fibres of this affine bundle are called *principal subspaces* relative to τ. We shall denote the principal subspace through a in $X^{(r)}$ by R_a or $R(a)$. Note that there is a unique principal subspace through each point of $X^{(r)}$. In fact, the fibre of $X^{(r)} \to X^{(r-1)}$ over any $b \in X^{(r-1)}$ is foliated by these principal subspaces and the translation map takes principal subspaces onto principal subspaces.

τ-Convex hull extension: Let τ be a hyperplane field defined on an open subset U of V. For a relation $\rho : \mathcal{R} \to X^{(r)}$ over $X^{(r)}$, let $\mathrm{Pr}_\tau(\mathcal{R})$ consist of all pairs $(a, x) \in \mathcal{R} \times X^{(r)}$ such that either $\rho(a) = x$ or x belongs to the unique principal subspace $R_{\rho(a)}$, when $\rho(a)$ lies over $v \in U$.

Let R be a principal subspace and $a \in \rho^{-1}(R)$. We denote by $\mathrm{Conv}_R(\rho^{-1}(R), a)$, the convex hull in R of the ρ-image of the path-component of a in $\rho^{-1}(R)$. If $\mathcal{R}$ is a subset in $X^{(r)}$ then $\mathrm{Conv}_R(\rho^{-1}(R); a)$ is the convex hull in R of the path-component of a in $\mathcal{R} \cap R$, and we denote it by $\mathrm{Conv}_R(\mathcal{R} \cap R; a)$.

Let $\mathrm{Conv}_\tau(\mathcal{R})$ denote the subset of $\mathrm{Pr}_\tau(\mathcal{R})$ consisting of all pairs (a, x) such that $x = \rho(a)$ if $\rho(a)$ lies over $V \setminus U$, or $x \in \mathrm{Conv}_R(\rho^{-1}(R); a)$ if $\rho(a)$ lies over U, where $R = R_{\rho(a)}$ is the unique principal subspace through $\rho(a)$.

The map $\tilde{\rho} : \mathrm{Conv}_\tau(\mathcal{R}) \to X^{(r)}$, $\tilde{\rho}(a, x) = x$, is a relation over $X^{(r)}$. If we define $E : \mathcal{R} \to \mathrm{Conv}_\tau(\mathcal{R})$ by $E(a) = (a, \rho(a))$ and $\pi : \mathrm{Conv}_\tau(\mathcal{R}) \to \mathcal{R}$ by $\pi(a, x) = a$ then it can be easily verified that $\mathrm{Conv}_\tau(\mathcal{R})$ is an extension of $\mathcal{R}$.

Proposition 3.1 [2], [5, Lemma 7.1]. *If $\mathcal{R}$ is open then* $\mathrm{Conv}_\tau(\mathcal{R})$ *is also open.*

The following result, known as the h-stability theorem, ([2, 2.4.2(B)] and [5, Theorems 7.2, 7.17]) identifies a key feature of the convex hull extension.

Theorem 3.2. *Let $\rho : \mathcal{R} \to X^{(r)}$ be an open relation and τ be a hyperplane field of codimension one on an open set $U \subset V$ which is smooth and integrable. Let $\sigma = (\alpha, j_f^r)$ be a holonomic section of* $\mathrm{Conv}_\tau(\mathcal{R})$ *(which implies that α is a holonomic section of $\mathcal{R}$ outside U). Let $\mathcal{N}$ be a neighbourhood of $j^\perp f(U)$ in $X_U^\perp$. Then there exists a homotopy of holonomic sections $\sigma_t = (\alpha_t, j_{f_t}^r)$ of* $\mathrm{Conv}_\tau(\mathcal{R})$ *such that*

(1) *$\sigma_0 = \sigma$ and $\rho \circ \alpha_1 = j_{f_1}^r$, so that α_1 is a holonomic section of $\mathcal{R}$.*
(2) *For all t, the image $j_{f_t}^\perp(U)$ is contained in $\mathcal{N}$. In particular, the homotopy f_t can be made C^{r-1}-small.*
(3) *For all $(t, x) \in [0, 1] \times (V \setminus U)$, $\sigma_t(x) = \sigma(x)$.*

Notation. Let $\mathcal{R}$ carry a metric d and let δ be a positive number. Suppose τ be a fixed hyperplane field on V as in the above and let $X^\perp$ be the corresponding set of $\perp$-jets of sections of X. For each $b \in X^\perp$, let $\mathcal{R}_b$ denote the intersection of $\mathcal{R}$ with the principal subspace over b. If $a \in \mathcal{R}_b$, $b \in X^\perp$, then we denote by $\mathrm{Conv}^\delta(\mathcal{R}_b; a)$ the convex hull (in $X_b^{(r)}$) of the ρ-image of the path-component of a in $B(a, \delta) \cap \mathcal{R}_b$.

Complement of Theorem 3.2. *Suppose that in addition to the hypothesis of Theorem 3.2, $(\mathcal{R}, d)$ is a metric space and δ is a positive function on $\mathcal{R}$, such that for each $x \in V$,*

$$j_f^r(x) \in \mathrm{Conv}^{\delta(\alpha(x))}\big(\mathcal{R}_{b(x)}; \alpha(x)\big); \qquad b(x) = j_f^\perp(x) \in X^\perp.$$

Then the homotopy σ_t in the conclusion of the above theorem can be chosen so that it also satisfies the condition $d(\alpha(x), \alpha_t(x)) < (n+1)\delta(\alpha(x))$ for all $x \in U$.

The above discussion leaves out the most interesting partial differential relations, namely the partial differential *equations*. However, if $\mathcal{R}$ is a closed subset of $X^{(r)}$ such that $\mathcal{R}_b$ is nowhere flat in the principal subspace $X_b^{(r)}$ for each $b \in X^\perp$, then it is possible to prove an h-stability theorem for $\mathcal{R}$. What helps in this particular situation is that the convex hull extension of $\mathcal{R}$ contains an open relation $\mathcal{R}^*$ over $X^{(r)}$ whose closure is $\mathcal{R}^* \cup \mathcal{R}$. We refer to [5, Chapter 9] for a detailed discussion on it.

The isotropy relation is clearly a closed one, but it does not enjoy the no-where flatness property as it is contained in a (proper) affine subbundle of $p_\perp^r$ (see Section 4). Consequently, the convex hull extension is not sufficiently large to contain any open relation. But, by taking successive convex hull extensions we can get away with this difficulty. The problem can be accommodated in the convex integration theory for *non-open* relations as developed in [2, 2.4.6], where Gromov has used more general convex hull extensions. The construction of $\mathcal{R}^*$ in this connection is very technical. However, it is possible to avoid this technicality altogether as long as we are only concerned with Theorem 2.4, which is a variant of Nash–Kuiper C^1-isometric immersion theorem. We shall proceed in the line of Eliashberg–Mishachev [1, Chapter 21].

Finally, one has to remember that the h-stability theorem does not solve the problem completely. We must ensure that $\mathrm{Conv}_\tau(\mathcal{R})$ has a holonomic section σ for Theorem 3.2 to be applicable.

A special case: Let the affine bundle $p_\perp^r : X^{(r)} \to X^\perp$ admit a complete metric d such that the restriction of d to each fibre is Euclidean (translation invariant). Moreover, let $\mathcal{R}$ contain a sphere bundle C. This means that, for each $b \in X^\perp$, C_b is a sphere in some affine subspace Y_b of $X_b^{(r)}$ relative to this metric d. It may be recalled that in the Nash–Kuiper problem the relevant relation $\mathcal{R}$ itself is a sphere bundle.

Suppose, there exists a section $f : V \to X$ such that $j_f^r(x)$ lies in the convex hull of $C_{b(x)}$ for each $x \in V$, where $b(x) = j_f^\perp(x)$; also suppose that $j_f^r(x)$ is neither the centre of $C_{b(x)}$, nor does it lie on $C_{b(x)}$. Then we can uniquely choose an $\alpha(x) \in C_{b(x)}$, so that (α, j_f^r) is a holonomic section of $\mathrm{Conv}_\tau(\mathcal{R})$. Indeed, we can choose $\alpha(x)$ on the radius vector through $j_f^r(x)$ for each $x \in V$.

Let $\widetilde{C}_b$ denote the complement of C_b in its convex hull and let $\widetilde{C}_b^\delta$ denote the intersection of the δ-neighbourhood of C_b with $\widetilde{C}_b$. If $\bar{r} = \bar{r}(b)$ is the radius of C_b, then $\widetilde{C}_b$ is an open ball of radius $\bar{r}$ (possibly in a lower dimensional affine subspace of $X_b^{(r)}$). On the other hand if $\delta < \bar{r}$, then $\widetilde{C}_b^\delta$ is an open spherical shell with outer radius $\bar{r}$ and inner radius $\bar{r} - \delta$. It can be seen easily that $\bigcup_{a \in C_b} \mathrm{Conv}^{\sqrt{2\bar{r}\delta}}(C_b; a)$ contains $\widetilde{C}_b^\delta$. For each $\varepsilon > 0$, we now consider the relation $\widetilde{\mathcal{R}}^\varepsilon = \bigcup_{b \in X^\perp} \widetilde{C}_b^\varepsilon$. A solution of $\widetilde{\mathcal{R}}^\varepsilon$ will be referred as an ε-approximate solution of $\mathcal{R}$.

Let δ be the positive function on V defined as follows:

$$\delta(x) = d\big(\alpha(x), j_f^r(x)\big) \quad \text{for each } x \in V.$$

This function can be pulled back by p^r to define $\bar{\delta}$ on $\mathcal{R}$. Note that f is a solution of $\widetilde{\mathcal{R}}^{\delta_1}$ for any $\delta_1 > \bar{\delta}$. On the other hand, observe that the set $C_{b(x)} \cap B(\alpha(x), \sqrt{2\bar{r}\delta(x)}\,)$ is connected (where $\bar{r} = \bar{r}(b(x))$ is the radius of $C_{b(x)}$), and $j_f^r(x)$ lies in the convex hull of this set so that $j_f^r(x) \notin C_{b(x)}$. Consider a circular arc γ on $C_{b(x)} \cap B(\alpha(x), \sqrt{2\bar{r}\delta(x)}\,)$ whose convex hull contains $j_f^r(x)$. If we push this curve γ normally inward to C_b within $\widetilde{C}_b$, then after a small deformation it will still contain $j_f^r(x)$ within its convex hull (since $j_f^r(x)$ does not lie on $C_{b(x)}$). Then, it follows from the above discussion that j_f^r can be lifted to a holonomic section (β, j_f^r) of $\mathrm{Conv}_\tau(\widetilde{\mathcal{R}}^\varepsilon)$ for sufficiently small $\varepsilon > 0$, such that $j_f^r(x) \in \mathrm{Conv}^{\sqrt{2\bar{r}\delta(x)}}(\widetilde{\mathcal{R}}^\varepsilon_{b(x)}; \beta(x))$.

Now, if C is a sphere bundle of maximum possible dimension, then $\widetilde{\mathcal{R}}^\varepsilon$ is an open relation. Consequently, the h-stability theorem ensures that there exists a solution f_1 of $\widetilde{\mathcal{R}}^\varepsilon$ so that $j_{f_1}^r(x)$ lies in the convex hull of $C_{b(x)}$ for each $x \in V$. If ε is less than $\bar{\delta}$ then the 'approximate solution' f_1 of $\mathcal{R}$ is better than the initial approximation f. Moreover, the C^0-distance between $j_{f_1}^r$ and α can be made less than $(n+1)\sqrt{2r\delta}$ (see Complement of Theorem 3.2). This implies that the C^r-distance between f and f_1 is less than $(n+1)\sqrt{2r\delta} + \delta$. Thus, a successive application of the h-stability theorem gives rise to a C^r-Cauchy sequence $\{f_n\}$ which converges to a solution of $\mathcal{R}$, since $\mathcal{R}$ (being closed) is complete relative to the chosen metric d.

Since $\widetilde{\mathcal{R}}^\varepsilon$ is not open, in general, we cannot directly apply the h-stability theorem to it. Moreover, for practical reasons we need to include the possibility that $j_f^r(x)$ may lie on $C_{b(x)}$ for some $x \in V$. In such cases, we modify $\widetilde{\mathcal{R}}^\varepsilon$ by redefining $\widetilde{C}_b^\varepsilon$ as the ε-neighbourhood of C_b in $X_b^{(r)}$. Then, clearly $\widetilde{\mathcal{R}}^\varepsilon$ is an open relation and $\sigma = (\alpha, j_f^r)$ is a holonomic section of $\mathrm{Conv}_\tau(\widetilde{\mathcal{R}}^\varepsilon)$ for every $\varepsilon > 0$. Let δ be defined as in the above. If we apply the h-stability theorem to the section σ and $\widetilde{\mathcal{R}}^\varepsilon$ for some positive function $\varepsilon < \bar{\delta}$, then we obtain, as before, an ε-approximate solution f_1 of $\mathcal{R}$; but, in this case $j_{f_1}^r(x)$ may not lie in the convex hull of $C_{b(x)}$ as it does not contain $\widetilde{C}_{b(x)}^\varepsilon$ in the new set up. Consequently, we can not pass on to the next iterative step. On the other hand, the C^r-distance between f and f_1 is now less than $(n+1)\sqrt{2r\delta} + \delta + \rho$, where the error term ρ depends on ε and can be made as small as we want by taking ε small enough. We refer the reader to [1, §21.5] for an illustrative explanation for the appearance of ρ. □

4. From a positive co-injective section to an isotropic section

Let E be a smooth manifold with a pseudo-Riemannian metric h such that $r_+(h) = \dim V$, and let $p : E \to V$ be a negative submersion onto a smooth manifold V.

Suppose there exists a strictly positive section of p, namely $f : V \to (E, h)$, which is also co-injective (see Definition 2.3). We want to prove that f can be homotoped to an isotropic C^1-section of p.

Our main task is to adjust to the present framework Nash's approach for the construction of isometric C^1 immersions between Riemannian manifolds [4].

Lemma 4.1 (Approximation lemma). *Let f be as stated above, and let $g = f^*h$. Then for every ε, $0 < \varepsilon < 1$, there exists a C^0-small homotopy of co-injective sections connecting f_0 to f_ε, where f_ε^*h is positive definite and is arbitrarily close to εg in the C^0-topology. Moreover, the C^1-distance between f and f_ε can be measured in terms of the C^0-norm of g.*

We shall prove the above lemma by an application of the Convex Integration techniques discussed in the previous section. Before proceeding to prove Lemma 4.1, we observe that the main result of this paper follows almost directly by successive application of the above approximation lemma. Indeed, starting with the given section f, we can construct a sequence of positive, co-injective sections $\{f_i\}$ of the fibration p, where $f_0 = f$ and f_i is obtained from f_{i-1} by applying the above lemma with $\varepsilon = 1/2$. Thus, $f_i^*h \approx f_{i-1}^*h/2 \approx \frac{g}{2^i}$ for $i > 1$, and the C^1-distance between

f_{i-1} and f_i can be expressed in terms of the C^0-norm of $\frac{g}{2^{i-1}}$. Indeed, it will follow from our estimates that $\{f_i\}$ is Cauchy in the C^1-topology, and hence it must converge to a C^1-section f. This f is the required isotropic section of $p: E \to V$, as $f^*h = \lim_{i\to\infty} f_i^*h = \lim_{i\to\infty} g/2^i = 0$. Therefore, it is enough for our purpose to prove the above approximation lemma.

Proof of Lemma 4.1. Since the metric g is positive definite, $(1-\varepsilon)g$ $(0<\varepsilon<1)$ admits a decomposition, which is typical of Nash's construction [4]:

$$(1-\varepsilon)g = \sum_i \phi_i^2\, d\psi_i^2, \tag{1}$$

where ϕ_i and ψ_i are C^∞ functions on V such that the support of ϕ_i is contained in some contractible open set U_i for some locally finite open covering $\{U_i\}_{i\in\mathbb{N}}$ of V.

For the sake of simplicity, assume that V is a closed manifold, in which case the above decomposition can be taken to be finite; suppose there are N monomials in the decomposition. Let us define a sequence of Riemannian metrics $\{g_k\}_{0\leqslant k\leqslant N}$ on V as follows: $g_0 = g$, and $g_k = g_{k-1} - \phi_k^2\, d\psi_k^2$ for $1 \leqslant k \leqslant N$. We shall construct a sequence of positive C^1-sections $\{\bar f_k\}_{k=0,\dots,N}$, such that $\bar f_k^* h$ is sufficiently close to g_k (in notation, $\bar f_k^* h \approx g_k$), where $\bar f_0 = f$. Note that $\bar f_N$ is our desired f_ε.

Thus a single step consists of the following: We have a section f of $p: E \to V$ such that

(1) f is co-injective, and
(2) $f^*h - \phi_1^2\, d\psi_1^2 = g_1 > 0$.

We shall prove the existence of a positive, co-injective C^1-section $\bar f_1$ such that $\bar f_1^* h \approx g_1$, and give an estimate for the C^1-distance between f and $\bar f_1$ in terms of the C^0-norm of g. This is where we shall apply the method of convex integration.

Let $\mathcal{R}$ denote the first order differential relation consisting of 1-jets of germs of g_1-isometric sections of p which are also co-injective. We shall show that $\mathcal{R}$ has an 'approximate' solution; explicitly we prove the existence of a positive, co-injective section $\bar f_1 : V \to E$ such that the induced metric $\bar f_1^* h$ is sufficiently close to g_1.

Recall that $g_1 = f^*h - \phi_1^2\, d\psi_1^2$, where ϕ_1 and ψ_1 are smooth functions on V and the support of ϕ_1 lies in a contractible open subset U_1 of V. The function ψ_1 defines an integrable hyperplane distribution τ on U_1 whose integral submanifolds are given by the level sets of ψ_1. Now, consider the fibration $p_\perp^1 : E^{(1)} \to E^\perp$ relative to this distribution τ. We shall first study the intersections of $\mathcal{R}$ with the fibres of the fibration $p_\perp^1 : E^{(1)} \to E^\perp$. At a point $x \in U_1$, let $f^*h = g_1 + l^2$, where l is a non-zero linear functional on the tangent space T_xV. Then $\tau_x = \ker l$. Choose a tangent vector v at x which is transversal to τ and satisfies $g_1(v,\tau) = 0$. Then $h(f_*v, f_*v) = g_1(v,v) + \|v\|^2_{g-g_1}$, where $g = f^*h$.

Let $b \in E^\perp$. Then b is of the form (x, y, β), where $\beta : \tau_x \to E_y$ is a linear map. If α is a 1-jet over $b \in E^\perp$, then α is completely determined by its value at v. Thus one can identify the principal subspace over b (corresponding to the hyperplane τ) with the tangent space T_yE of E at y. Consequently, $E^{(1)}$ can be endowed with a metric d such that, under the above identification, its restriction to the fibre $E_b^{(1)}$ is induced from $\bar h = h_1 - h_2$, where $h_1 = h|_\xi$ and $h_2 = h|\eta$. Recall from Section 2 that TE admits an orthogonal decomposition $TE = \xi \oplus \eta$, such that the restrictions of h to ξ and η are respectively positive definite and negative definite. Therefore, $\bar h = h_1 - h_2$ defines a Riemannian metric on TE.

In accordance with this identification $\mathcal{R} \cap E_b^{(1)}$ corresponds to the set of all vectors $w \in T_yE$ satisfying the following properties:

(1) $w \in dp_x^{-1}(v)$;
(2) p_2 restricted to $\beta(\tau_x) \oplus \langle w\rangle$ is injective;
(3) $h(w, \beta(\tau_x)) = 0$;
(4) $h(w,w) = g_1(v,v)$.

Indeed, condition (1) implies that the corresponding 1-jet comes from a *section* of the fibre-bundle p, while (2) translates into the co-injectivity condition. Conditions (3) and (4) are equivalent to the isotropy criterion.

A few comments are now in order. Recall that $dp|_\xi : \xi \to TV$ is a bundle isomorphism. Given a vector $v \in TV$, let $\bar{v}$ denote the unique vector in ξ which is mapped onto v by dp. Thus, if a vector $w \in dp_x^{-1}(v)$, then there is a unique vector $w' \in \eta_x$ such that $w = \bar{v} + w'$.

Let $T = p_2(\beta(\tau_x))$. Then T is an $(n-1)$-dimensional subspace of η_y, and condition (2) on w is equivalent to saying that $w' \notin T$. On the other hand the third condition is equivalent to $h_2(w', \beta'(t)) = -h_1(\bar{v}, \bar{t})$ for all $t \in \tau$, where $\beta'(t)$ is the unique element in η_y such that $\beta(t) = \bar{t} + \beta'(t)$. This represents a system of equations for the vector $w' \in \eta_y$, which defines an affine subspace of η_y of codimension $(n-1)$ (because h_2 is negative definite and h_1 is positive definite). Further note that this affine subspace is a translate of the orthogonal complement of T in η_y relative to $-h_2$. Thus, $\mathcal{R}_b$ lies in an affine subspace X_b of $E_b^{(1)}$ such that the restriction of d to X_b is induced from $-h_2$.

Finally, condition (4) on w may be expressed as $h_2(w', w') = g_1(v, v) - h_1(\bar{v}, \bar{v})$. This equation for w' represents a non-empty set in η_y, provided $g_1(v, v) - h_1(\bar{v}, \bar{v}) < 0$, in which case it corresponds to a sphere in a codimension $(n-1)$ affine subspace of η_y, which is connected if we also have $q > n$.

Now, let $b = j_f^{\perp}(x)$ and $w_0 = df_x(v)$. Then w_0 satisfies conditions (1), (2) and (3) stated above for w. Moreover, if we write $w_0 = \bar{v} + w_0'$, then $h_2(w_0', w_0') = g_1(v, v) - h_1(\bar{v}, \bar{v}) + \|v\|^2_{g-g_1} < 0$ since h_2 is negative definite. Therefore, it follows from the preceding paragraph that $\mathcal{R} \cap E_b^{(1)}$ is a non-empty set and $df_x(v)$ lies in the interior of the convex hull of $\mathcal{R} \cap E_b^{(1)}$.

Note that the affine subspace of η_y determined by (2) intersects T in a unique point c. So c is the only point on this affine subspace which does not correspond to a co-injective jet. Clearly, c is different from w_0' ($w_0' \notin T$). The ray from c through w_0' intersects the subset of η_y determined by (3) and (4) above in a unique point w_1'. This point in turn defines a unique 1-jet α_x in $\mathcal{R}$ over $j_f^{\perp}(x)$. Thus we can define a section α of $p_\perp^{(1)}$ covering $j_f^{\perp}$. Moreover, $d(c, \alpha_x'(v)) \leqslant \sqrt{h_1(\bar{v}, \bar{v}) - g_1(v, v)}$.

The above discussion summarises to the following:

(1) The fibre $\mathcal{R}_b$ lies in an affine subspace X_b of $E_b^{(1)}$.
(2) The restriction of the metric d to X_b is induced from $-h_2$.
(3) $\mathcal{R}_b$ represents a sphere in X_b relative to d.
(4) For every $x \in V$, df_x lies in the convex hull of $\mathcal{R}_b$, where $b = j_f^{\perp}(x)$.

Therefore, we are in the special situation discussed in Section 3 (see the special case). Let $\widetilde{\mathcal{R}}$ be a fibrewise open tubular neighbourhood of $\mathcal{R}$ relative to the fibre bundle $p_\perp'$. We can choose $\widetilde{\mathcal{R}}$ so that it lies entirely within the set of 1-jets of positive, co-injective sections of $p : (E, h) \to V$, because the positivity and the co-injectivity conditions are open conditions. Note that $\sigma = (\alpha, j_f')$ is a section of $\mathrm{Conv}_\tau(\mathcal{R})$ and hence is a section of $\mathrm{Conv}_\tau(\widetilde{\mathcal{R}})$. For any $v \in T_x V$,

$$d\big(\alpha_x(v), df_x(v)\big) = \big\|\alpha_x(v) - df_x(v)\big\|_{\tilde{h}} = \big\|\alpha_x'(v) - df_x'(v)\big\|_{-h_2}.$$

Now, since c, $\alpha_x'(v)$ and $df_x'(v)$ are collinear

$$\big\|\alpha_x'(v) - df_x'(v)\big\|_{-h_2} \leqslant \sqrt{\big\|\alpha_x'(v) - c\big\|^2_{-h_2} - \big\|df_x'(v) - c\big\|^2_{-h_2}}.$$

On the other hand since $c \in T$, and $\alpha_x'(v) - c$ and $df_x'(v) - c$ belong to the orthogonal complement of T relative to the metric $-h_2$, the right-hand side term is equal to

$$\sqrt{\big\|\alpha_x'(v)\big\|^2_{-h_2} - \big\|df_x'(v)\big\|^2_{-h_2}} = \sqrt{-h\big(\alpha_x(v), \alpha_x(v)\big) + h\big(df_x(v), df_x(v)\big)}.$$

The last equality can be deduced from the following observation. If we write $\alpha_x(v) = \bar{v} + \alpha_x'(v)$, and $df_x(v) = \bar{v} + df_x'(v)$, where $\bar{v} \in \xi$ and $\alpha_x'(v), df_x'(v) \in \eta$, then $-h(\alpha_x(v), \alpha_x(v)) = -h_1(\bar{v}, \bar{v}) - h_2(\alpha_x'(v), \alpha_x'(v))$ and $h(df_x(v), df_x(v)) = h_1(\bar{v}, \bar{v}) + h_2(df_x'(v), df_x'(v))$. Thus we arrive at the following inequality:

$$d\big(\alpha_x(v), df_x(v)\big) \leqslant \|v\|_{g-g_1}.$$

Now, fix a Riemannian metric $\bar{g}$ on V. Since $g - g_1 \leqslant (1-\varepsilon)g \leqslant g$, it follows from the above that

$$d_{\bar{g}}\big(\alpha_x, j^1_f(x)\big) = \sup_{\|v\|_{\bar{g}}=1} d\big(\alpha_x(v), df_x(v)\big) \leqslant \sup_{\|v\|_{\bar{g}}=1} \|v\|_g = \delta,$$

where $\delta = \sqrt{\|g\|_0}$ is the C^0-norm of g relative to the metric $\bar{g}$.

Therefore, if we apply the h-stability theorem to σ and $\widetilde{\mathcal{R}}$, we obtain a C^0-small homotopy of sections joining f to $\bar{f}_1 : V \to (E, h)$ such that $\bar{f}_1$ is co-injective and $\bar{f}_1^* h \approx g_1$. Moreover, the C^1-distance between f and $\bar{f}_1$ relative to $\bar{g}$ is $d_{\bar{g}}(f, \bar{f}_1) \leqslant (n+1)\sqrt{2r\delta} + \delta + \rho$, where $r = \sup_{\|v\|_{\bar{g}}=1} \sqrt{h_1(\bar{v}, \bar{v})}$. Note that $r > \sqrt{h_1(\bar{v}, \bar{v}) - g_1(v, v)} > d(c, \alpha'_x(v))$ whenever $\|v\|_{\bar{g}} = 1$.

It may be noted that the homotopy connecting f and $\bar{f}_1$ constructed above need not consist of co-injective maps, because the image of $p_2 : \mathrm{Conv}_\tau \widetilde{\mathcal{R}} \to E^{(1)}$ may not be contained within the set $\mathcal{C}$ of co-injective jets.

However, in the present situation we can do away with this difficulty by restricting the relation $\widetilde{\mathcal{R}}$ as follows: Recall that the fibres $\mathcal{R}_b$, $b \in E^{\perp}$, are spherical, and that the centres of these spheres are the only points in the convex hulls of these spherical fibres $\mathcal{R}_b$ which do not correspond to co-injective jets. Since f is a co-injective map, we can choose a connected, compact subset K_b in $\mathcal{R}_b$, $b = j^{\perp}_f(x)$, whose convex hull contains $j^1_f(x)$ and is contained in the set $\mathcal{C}$; (see the illustration in [1, p. 186]). If we take a sufficiently small neighbourhood of this K_b then its convex hull will also be contained in the set of co-injective jets. Therefore we can obtain an open relation $\widetilde{\mathcal{R}}$ for which the image of $p_2 : \mathrm{Conv}_\tau \widetilde{\mathcal{R}} \to E^{(1)}$ is contained within $\mathcal{C}$, and moreover (α, j^1_f) is a section of $\mathrm{Conv}_\tau(\widetilde{\mathcal{R}})$. If we apply the h-stability theorem to this $\widetilde{\mathcal{R}}$, then the homotopy f_t will consist of co-injective sections of $p : E \to V$.

Now, if we repeat the above steps for the map $\bar{f}_1$ and the metric $g'_2 = \bar{f}_1^* h - \phi_2^2 d\psi_2^2$ (which is sufficiently C^0-close to g_2), we obtain a map $\bar{f}_2$ such that $\bar{f}_2^* h \approx g_2$. Moreover, $d_{\bar{g}}(\bar{f}_1, \bar{f}_2) \leqslant (n+1)\sqrt{2r\delta} + \delta + \rho$. Proceeding in this way, we obtain $f_\varepsilon = \bar{f}_N$ at the end of the Nth iterative step. Observe that

$$d_{\bar{g}}(f, f_\varepsilon) \leqslant N\big((n+1)\sqrt{2r\delta} + \delta + \rho\big),$$

where $\delta = \sqrt{\|f^* h\|_0}$, and ρ can be made as small as we want. On the other hand, f_ε can be obtained so that $f_\varepsilon^* h$ is arbitrarily close to εg in the C^0-topology.

If the manifold V is open, then choose a locally finite open covering $\{U_i\}_{i \in \mathbb{N}}$ of V such that each point $x \in V$ has a neighbourhood U_x which intersects $(n+1)$ members of the covering. Proceeding as in the above, we obtain a sequence $\{\bar{f}_i\}_{i \in \mathbb{N}}$ such that on any of the open set U_x, $x \in V$, the sequence is eventually constant. Therefore we can define f_ε by $f_\varepsilon = \lim_{i \to \infty} f_i$. Moreover, on each U_x, $d_{\bar{g}}(f, f_\varepsilon) \leqslant (n+1)((n+1)\sqrt{2r\delta} + \delta + \rho)$, since only $(n+1)$ many U_i's intersect each U_x.

This completes the proof of Lemma 4.1. □

As we have remarked earlier, by successive application of the above lemma with $\varepsilon = 1/2$, we can obtain a sequence $\{f_i\}$ such that

$$f_i^* h \approx \frac{g}{2^i} \quad \text{and} \quad d_{\bar{g}}(f_i, f_{i-1}) \leqslant N\big((n+1)\sqrt{2r\delta_i} + \delta_i + \rho_i\big),$$

where $\delta_i = \delta/(\sqrt{2})^{i-1}$, and ρ_i is arbitrarily small. Then clearly $\{f_i\}$ is a Cauchy sequence in the C^1 topology, and therefore, converges to an isotropic C^1-section of the bundle $p : E \to V$ proving Theorem 2.4.

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