ON A METHOD OF GETTING CONFOUNDED ARRANGEMENTS IN THE GENERAL SYMMETRICAL TYPE OF EXPERIMENT

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1. Introduction.

In randomised block or Latin Square designs when an experiment consists of all combinations of two or more sets of treatment factors, the number of plots per block or per row and column necessarily becomes large. In such factorial arrangements a method known as 'Confounding' has been devised which enables the experiment to be conducted with much fewer plots per block or per row and column than the number of treatment combinations.

In confounded arrangements the treatments of each replication are allotted to two or more sub-blocks in such a manner that some or all of the information is lost on unimportant comparisons, usually high-order interactions between the different factors. That is to say, if the plots of each completely replicated block are divided into a set of sub-blocks in such a manner that the comparisons between sub-block totals correspond to the comparisons representing a set of high-order interactions, then the degrees of freedom corresponding to this set of interactions will be confounded (mixed up) with sub-block differences. By confounding different sets of degrees of freedom in each replication we can recover some information on all the comparisons. This is known as *Partial Confounding*. If the same set of degrees of freedom is confounded in each replication we get *Total Confounding* in which all information is lost on this particular set of comparisons. In the former case it may well happen, if the standard error per plot is considerably reduced, that even those comparisons that are partially confounded are more accurately determined than would be the case in an un-confounded or fully orthogonal experiment.

Among the symmetrical types of factorial experiments, *i.e.*, of n factors at p levels each in all combinations, confounded designs for the cases $p=2$ and $p=3$ have been discussed by Barnard[1] and Yates[2] respectively. Yates has also indicated that the case of $p=4$, can be deduced from the case $p=2$.

In this paper I have suggested a method of getting confounded arrangements in the p^n type of experiment with the help of two systems of interchanges obtainable from a p-sided hyper-Græco-Latin Square (used in the sense of completely orthogonalised squares). It is known that a p-sided hyper-Græco-Latin Square exists when p is a prime or a power of a prime. The method of getting confounded arrangements is demonstrated in this paper for the case of $p=4$, being the smallest among powers of primes. The confounded arrangements in the case of $p=5$ (a prime) have also been worked out, and will be given in a subsequent paper. I have included the case $p=3$ in this paper partly because Yates has not explained in detail the method of getting the 3^n designs, and partly to show its connection with the general case.

2. COMPLETELY ORTHOGONALISED SQUARES.

The sum of squares

$$\sum_{i,j=1}^{p} (x_{ij}-\bar{x})^2$$

between p^2 quantities x_{ij} $(i, j=1, 2,......p)$, with mean $\bar{x}$, following the Normal Law is said to have p^2-1 degrees of freedom. This means that p^2-1 independent linear relations between these p^2 quantities, each involving a single degree of freedom, can be shown to exist.[4] Following the algebraic identity,

$$p^2 - 1 \equiv (p + 1)(p - 1)$$

it appears possible to arrange the p^2 values in p sets of p each in $p+1$ different ways orthogonal to one another, so that comparisons between the p sets of a single arrangement involve $p-1$ degrees of freedom. This completely orthogonalised arrangement into p sets of p each has been found possible so far only when p is a prime or a power of a prime. Nothing is known about other numbers except for $p=6$ in which case it has been definitely shown that this arrangement is impossible.[5]

(2·1) *Hyper-Græco-Latin Squares.*

The use of Latin square when $p=2$, of Græco-Latin square when $p=3$ and of hyper-Græco-Latin square for higher values of p very much facilitate the partition of the p^2-1 degrees of freedom into $p+1$ orthogonal sets of $p-1$ degrees of freedom each. R. A. Fisher[3] uses this method for representing the completely orthogonalised squares for $p=2, 3, 4, 5, 8$ and 9. We reproduce here the first three squares only (Table 1).

TABLE 1. COMPLETELY ORTHOGONALISED SQUARES FOR $p \times p$ $(p=2,3,4)$.

p=2			p=3			p=4			
A	B		Aα	Bβ	Cγ	A_1α	B_2β	C_3γ	D_4δ
B	A		Bγ	Cα	Aβ	B_4γ	A_3δ	D_2α	C_1β
			Cβ	Aγ	Bα	C_2δ	D_1γ	A_4β	B_3α
						D_3β	C_4α	B_1δ	A_2γ

In the 2×2 square each of the Latin letters A, B occur once and only once in each row and column. In the 3×3 square each of the Latin letters A, B, C, and each of the Greek letters α, β, γ occur once and only once in each row and column; besides, each Latin letter occurs once and only once with each Greek letter and *vice versa*. In the 4×4 square each of the Latin letters A, B, C, D, the Greek letters α, β, γ, δ and the Latin suffixes 1, 2, 3, 4, occur once and only once in each row and column. Besides, each Latin letter occurs once and only once with each Greek letter and each Latin suffix, and each Greek letter occurs once and only once with each Latin suffix. This method of denoting the orthogonalised squares of a p-sided square by means of Latin letters, Greek letters, etc., has earned for it the name of a p-sided hyper-Græco-Latin square. We see that there are 1, 2 and 3 orthogonalised squares respectively for the 2, 3 and 4-sided squares which together with the rows and columns in each case give the completely orthogonalised division of the total degrees of freedom. In general a p-sided square has $p-1$ orthogonalised squares.

(2·2) *Two Systems of Interchanges.*

The need for abandoning the hyper-Græco-Latin square method of distinguishing the $p-1$ orthogonalised squares is easily realised as we go to high values of p. I propose to adopt a system of interchanges by which given the Latin square in the standard position, the remaining $p-2$ orthogonalised squares can be obtained simply by certain interchanges of the rows or columns.

For $p=2$ there is only one orthogonalised square and so the question about interchanges does not arise.

For $p=3$ the Latin letters of the 3-sided Græco-Latin square of Table 1 are in the following standard position

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

and the Greek letters can be brought to the same standard position by an interchange of the columns (rows) such that the positions 1, 2, 3 of the columns (rows) are occupied by the columns (rows) 1, 3, 2 respectively. That is to say, interchange the last two columns (rows).

For $p=4$, the Latin letters of the 4-sided hyper-Græco-Latin square of Table 1 are in the following standard position

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

and the Latin suffixes and the Greek letters can be brought to the same standard position by two interchanges of the columns (rows) such that the positions 1, 2, 3, 4 of the columns (rows) are occupied by the columns (rows): 1, 4, 2, 3 and 1, 3, 4, 2 respectively or in other words by a cyclical interchange of the last three columns (rows).

The above system of interchanges for the 3×3 and 4×4 squares will be referred to on the following pages as the *first system* of interchanges for 3×3 and 4×4 respectively.

There is another system of interchanges which will be referred to as the *second system* of interchanges which consists of interchanges of the numbers 1, 2, 3,...... p of the first column (row) of a $p \times p$ square in the standard position which will give the remaining $p-1$ columns (rows). It will be seen that for the 3×3 square the second system of interchanges consists of the cyclical interchanges of the numbers 1, 2, 3. For the 4×4 square the interchanges are non-cyclic being 1, 2, 3, 4 ; 2, 1, 4, 3, ; 3, 4, 1, 2 and 4, 3, 2, 1.

Given the $p \times p$ hyper-Græco-Latin square these two systems of interchanges are uniquely determined. That these systems of interchanges are very powerful methods in getting confounded arrangements will be clear when we discuss confounding for 3^n and 4^n types of experiment.

3. Use of Orthogonalised Squares in Factorial Experiments.

(3·1) *Two factors at p levels each.*

In a factorial experiment with two factors A and B at p levels each there are p^2 treatment combinations. Suppose the experiment to be conducted in q randomised blocks of p^2 plots each. The totals (t_{ij}) of the experimental values of each treatment combination over its q replications can be arranged in a $p \times p$ square form as in Table 2.

Table 2. Experimental Values of a Two-factor Experiment.

.	a_1	a_2	.	.	a_i	.	.	a_p
b_1	t_{11}	t_{21}	.	.	t_{i1}	.	.	t_{p1}
b_2	t_{12}	t_{22}	.	.	t_{i2}	.	.	t_{p2}
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
b_j	t_{1j}	t_{2j}	.	.	t_{ij}	.	.	t_{pj}
..	.	.	.	.	.	.	.	.
..	.	.	.	.	.	.	.	.
b_p	t_{1p}	t_{2p}	.	.	t_{ip}	.	.	t_{pp}

The partition of the p^2-1 degrees of freedom between treatments into main effects of A and B and their interaction AB is given in Table 3.

Table 3. Partition of the Degrees of Freedom for Treatments.

Source of variation			d. f.
Main effect	(A)	...	$p-1$
..	(B)	...	$p-1$
Interaction	(A B)	...	$(p-1)^2$
Between treatments		...	p^2-1

The main effects of A and B correspond to comparisons among the columns and rows respectively of the $p \times p$ square in Table 2. The $p-1$ orthogonalised squares of this $p \times p$ square will enable the partition of the $(p-1)^2$ degrees of freedom of the interaction AB into $p-1$ orthogonal sets of $p-1$ degrees of freedom each. It is this possibility of splitting the interaction degrees of freedom into orthogonal sets that finds far-reaching application in the confounding of high-order interactions.

3·2. *Confounding in the General Symmetrical Type of Experiment.*

In 3·1 we considered the case of two factors at p levels each. Let us now take up the general case of n factors at p levels each in all combinations. The treatment sum of squares will now have p^n-1 degrees of freedom. It must be possible to divide the p^n treatment combinations in p sets of p^{n-1} each in $(p^{n-1}+p^{n-2}+\ldots\ldots+p+1)$ ways so that the comparisons among the p sets of any one division involve $p-1$ degrees of freedom orthogonal to the rest. The highest $[(n-1)^{th}]$ order interaction between all the factors has $(p-1)^n$ degrees of freedom. It is necessary for confounding of this interaction that the $(p-1)^{n-1}$ orthogonal sets, of $p-1$ degrees of freedom each, of this interaction should be identified out of the total $(p^{n-1}+p^{n-2}+\ldots\ldots+p+1)$ orthogonal sets. This identification is not difficult when p is a prime or a power of a prime for in these cases $p \times p$ hyper-Græco-Latin squares are known to exist.

In any single replication p sub-blocks with p^{n-1} plots in each can be made to confound any set of $p-1$ degrees of freedom out of the $(p-1)^{n-1}$ orthogonal sets of the $(n-1)^{th}$ order interaction. It will require $(p-1)^{n-1}$ replications to confound partially all the comparisons involved in the highest order interaction and thus recover information on each set of $p-1$ degrees of freedom with *equal* precision. It must however be observed that if n or p or both are large a sub-block of p^{n-1} plots may still be too big to materially increase the precision of the experiment, so that further divisions will be desirable. Such divisions will affect not only the highest order interactions but lower orders as well. Generally the main effects of all factors and the interactions between pairs of factors (first order interactions) are of great importance so that it is always welcome to keep these two treatment effects free from confounding. This can be effected if the p^2 combinations of each pair of factors occur the same number of times in each sub-block. A sub-block cannot therefore have less than p^2 plots. It will be seen presently that in the p^n type of experiment $(n>2)$ the minimum number of plots per sub-block should be p^{n-p+1} or p^2, whichever is greater. Thus p^2 will be the minimum for n not exceeding $p+1$. Considering that we are mostly interested in the main effects and first order interactions, we should choose the minimum possible size for sub-blocks, namely p^{n-p+1} or p^2, whichever is greater, so that the accuracy of the whole experiment may be the maximum attainable.

Let us first write down as in Table 4 the p^2 combinations of the first two factors A and B in the form of a $p \times p$ square such that the columns and rows represent the levels of A and B respectively. This square has $p-1$ orthogonalised squares. By arranging the p levels $c_1, c_2, \ldots c_p$ of a third factor C in the form of one of these

TABLE 4. COMBINATIONS OF A AND B LEVELS ARRANGED AS A SQUARE.

	a_1	a_2	.	.	a_i	.	.	a_p
b_1	a_1b_1	a_2b_1	.	.	a_ib_1	.	.	a_pb_1
b_2	a_1b_2	a_2b_2	.	.	a_ib_2	.	.	a_pb_2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
b_j	a_1b_j	a_2b_j	.	.	a_ib_j	.	.	a_pb_j
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
b_p	a_1b_p	a_2b_p	.	.	a_ib_p		.	a_pb_p

squares and by superimposing that on the square in Table 4 we get p^2 combinations of the three factors A, B, C. such that all the p^2 combinations of every pair of factors occur once in the square. A fourth factor D may now be introduced by arranging its levels $d_1, d_2, \ldots d_p$ in the form of one of the remaining $p-2$ orthogonalised Latin squares and superimposing it on the (a, b, c)-square and so on till we exhaust all the $p-1$ orthogonalised squares by including $p+1$ factors in all. We thus get p^2 combinations of all the $p+1$ factors so that the p^2 combinations of every pair of factors occur once in the square. One of our sub-blocks will now consist of these p^2 combinations.

If the number of factors exceed $p+1$, first superimpose on the (a, b)-square of Table 4, $p-2$ orthogonalised squares formed with the levels of $p-2$ out of the $n-2$ factors excluding A and B. The remaining $n-p$ factors will make p^{n-p} combinations. These p^{n-p} combinations can now be split into p sets of p^{n-p-1} each such that within each set p^2 combinations of any pair of these $n-p$ factors will occur the same number of times, namely, p^{n-p-3} if n exceeds $p+2$. (If $n=p+2$ only two factors remain, the p^2 combinations of which should be divided into p sets of p each such that within each set both factors appear at all their levels once). The p sets thus obtained may now be marked $s_1, s_2, \ldots s_p$ and then arranged in the form of the $(p-1)^{th}$ orthogonalised square and superimposed on the square of Table 4, which is already occupied by combinations of p factors. Each cell of our square has now piled up p^{n-p-1} combinations of all the n factors. All the cells together contain p^{n-p+1} combinations. These combinations should be allotted to one sub-block. Each replication will consist of p^{p-1} or p^{n-2} sub-blocks according as n is $> p+1$ or $\leqslant p+1$. The method of getting the remaining sub-blocks depends on the second system of interchanges for a $p \times p$ square. Both systems of interchanges come into play when we seek to get a set of replications giving *balanced* partial confounding of the confounded interactions.

The cases considered in detail in this paper are the confounding of 3^3 and 3^4 designs in sub-blocks of 9 plots and of 4^3 and 4^4 designs in sub-blocks of 16 plots. The method that naturally suggested itself in getting the balanced replications of the 3^4 and 4^4 designs in sub-blocks of 9 and 16 plots respectively is found to be dependent on the properties of the 2×2 and 3×3 orthogonalised squares. We should therefore expect this method to be general for getting the balanced replications of a p^n design in sub-blocks of p^2 plots only when p is such that a p-sided and a $(p-1)$-sided hyper-Græco-Latin Square exist. Thus for $p=7$ the method fails.

4. THE 3^3 TYPE OF EXPERIMENT.

4.1 *Three factors at three levels each.*

Let A, B, C be the three factors at levels denoted by (a_1, a_2, a_3) ; (b_1, b_2, b_3) and (a_1, c_2, c_3) respectively. The second order interaction ABC has 8 degrees of freedom. With 3 sub-blocks of 9 plots each, 2 d.f. belonging to ABC can be confounded in one replication. For partially confounding all the 8 d.f. of ABC four balanced arrangements each consisting of 3 sub-blocks of 9 plots should be written down.

The 3×3 square in the standard position can generate three more squares by performing the first system of interchanges (3×3) on the rows and columns. These are given in Table 5.

TABLE 5. THE FOUR STANDARD 3×3 SQUARES.

1			2		
1	2	3	1	3	2
2	3	1	2	1	3
3	1	2	3	2	1
3			**4**		
1	2	3	1	3	2
3	1	2	3	2	1
2	3	1	2	1	3

Keep the levels of C arranged according to these four squares making the columns and rows represent the A levels and B levels respectively as in Table 6.

TABLE 6. THE KEY SUB-BLOCKS OF 4 REPLICATIONS IN SUB-BLOCKS OF 9 PLOTS.

	1			2			3			4		
	a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3
b_1	c_1	c_2	c_3	c_1	c_3	c_2	c_1	c_2	c_3	c_1	c_3	c_2
b_2	c_2	c_3	c_1	c_2	c_1	c_3	c_3	c_1	c_2	c_3	c_2	c_1
b_3	c_3	c_1	c_2	c_3	c_2	c_1	c_2	c_3	c_1	c_2	c_1	c_3

These four squares represent the 9 combinations for one sub-block from each of the four balanced replications. The remaining two sub-blocks of each replication are obtained by performing the second system of interchanges (3×3) on the rows (columns) of each key sub-block. The four sets of 2 d.f. of ABC obtained by this method are denoted by s_1, s_2, s_3 and s_4 and Table 7 (following Yates) gives the levels of the three factors occurring in each sub-block. A and B are taken as the first two factors and C as the third factor.

TABLE 7. 3^3 DESIGNS IN SUB-BLOCKS OF 9 PLOTS.

Levels of first and second factors		s_1			s_2			s_3			s_4		
		Level of third factor											
11	1	1	2	3	1	3	2	1	2	3	1	3	2
12	2	2	3	1	2	1	3	3	1	2	3	2	1
13	3	3	1	2	3	2	1	2	3	1	2	1	3
21	4	2	3	1	3	2	1	2	3	1	3	2	1
22	5	3	1	2	1	3	2	1	2	3	2	1	3
23	6	1	2	3	2	1	3	3	1	2	1	3	2
31	7	3	1	2	2	1	3	3	1	2	2	1	3
32	8	1	2	3	3	2	1	2	3	1	1	3	2
33	9	2	3	1	1	3	2	1	2	3	3	2	1

4.2. *Four factors at three levels each.*

Let A, B, C, D be the four factors at levels denoted by (a_1, a_2, a_3) ; (b_1, b_2, b_3) (c_1, c_2, c_3) and (d_1, d_2, d_3) respectively. In sub-blocks of 9 plots, 8 d.f. get confounded in one replication. These we should expect to be distributed among the second order interactions ABC, ABD, ACD, BCD and the third order interaction ABCD. The 8 d.f. confounded can be split into 4 sets of 2 d.f. each. It will be found that these four sets belong only to the second order interactions ABC, ABD, ACD and BCD. Thus all main effects, first order and third order interactions remain undisturbed.

Take a 3×3 square with the columns and rows representing the A and B levels. The C and D levels should appear in the cells of this square in the form of two orthogonalised Latin squares. The four squares of Table 5 fall into two groups G_1 and G_2 so that squares in one group are orthogonal only to the squares in the other group. G_1 consists of squares 1 and 4 and G_2 of squares 2 and 3. Taking for the C levels square 1 and for the D levels square 2 we get a key sub-block of one replication (Table 8).

Table 8. Key Sub-blocks of a 3^4 Design in Sub-blocks of 9 Plots.

	a_1	a_2	a_3
b_1	c_1d_1	c_2d_2	c_3d_3
b_2	c_2d_3	c_3d_1	c_1d_2
b_3	c_3d_2	c_1d_3	c_2d_1

Arranging the C levels according to the two squares of G_1 and the D levels according to the two squares of G_2 we get four squares of the type of Table 8. Next arrange the C levels according to the squares in G_2 and the D levels according to the squares in G_1, We get in all eight squares of the type of Table 8. The method of their generation is clear from Table 9.

Table 9. Squares combined for C and D Levels to get the 8 Balanced Replications.

	I		II	
	G_1 (c)	G_2 (d)	G_2 (c)	G_1 (d)
1	1	2	2	1
2	1	3	2	4
3	4	2	3	1
4	4	3	3	4

The eight squares give the key sub-blocks of eight balanced replications. By performing the second system of interchanges on the *c* rows (columns) and the *d* rows (columns) of a single square we get nine squares which are the nine sub-blocks of the replication of which that square is the key sub-block.

The treatment combinations of the nine sub-blocks of the replication of which the square in Table 8 is the key sub-block are shown in Table 10.

TABLE 10. A 3^4 DESIGN IN SUB-BLOCKS OF 9 PLOTS CONFOUNDING SECOND ORDER INTERACTIONS.

Levels of first and second factors		Levels of third and fourth factors								
11	1	1	5	9	6	7	2	8	3	4
12	2	5	9	1	7	2	6	3	4	8
13	3	9	1	5	2	6	7	4	8	3
21	4	6	7	2	8	3	4	1	5	9
22	5	7	2	6	3	4	8	5	9	1
23	6	2	6	7	4	8	3	9	1	5
31	7	8	3	4	1	5	9	6	7	2
32	8	3	4	8	5	9	1	7	2	6
33	9	4	8	3	9	1	5	2	6	7

The last nine columns represent the nine sub-blocks. The key sub-block is given in the first column. Table 10 is in the form of a 9 × 9 Latin square with an internal pattern of a 3 × 3 Latin square. It is easy to see that given the first column the remaining columns can be formed with little effort. The key sub-blocks of the eight replications are given in Table 11 so that by sticking to the pattern of Table 10 the reader may himself generate the remaining 8 sub-blocks of each replication. The 8 d.f. between the nine sub-blocks can be split into 4 sets of 2 d.f. each by arranging the sub-block numbers 1 to 9 in the form of a 3 × 3 square. It will be easily recognised that the 4 sets of 2 d.f. of a replication belong to the four second order interactions ABC, ABD, ACD and BCD. By referring to Table 7 we can easily write down which of the four sets: s_1, s_2, s_3, s_4, of the interactions ABC, ABD, ACD, BCD are confounded in each replication. These are given in Table 12.

TABLE 11. KEY SUB-BLOCKS OF 9 PLOTS OF 8 REPLICATIONS OF A 3^4 DESIGN.

Levels of *a* and *b*			I				II			
			1	2	3	4	1	2	3	4
a	*b*		Levels of *c* and *d*							
1	1	1	1	1	1	1	1	1	1	1
1	2	2	5	6	8	9	5	6	8	9
1	3	3	9	8	6	5	9	8	6	5
2	1	4	6	5	9	8	8	9	5	6
2	2	5	7	7	4	4	3	2	3	2
2	3	6	2	3	2	3	4	4	7	7
3	1	7	8	9	5	6	6	5	9	8
3	2	8	3	2	3	2	7	7	4	4
3	3	9	4	4	7	7	2	3	2	3

Table 12. Sets of Second Order Interactions Confounded.

Interaction	I				II			
	1	2	3	4	1	2	3	4
A B C	s_0	s_1	s_1	s_4	s_1	s_2	s_3	s_0
A B D	s_3	s_2	s_3	s_3	s_2	s_0	s_3	s_1
A C D	s_3	s_4	s_3	s_3	s_2	s_1	s_4	s_2
B C D	s_4	s_4	s_1	s_1	s_4	s_3	s_2	s_3

It will be seen from Table 12 that the same set of each second order interaction is confounded in two replications. Since there are only 4 sets of 2 d.f for a second order interaction and since one set is confounded in every replication complete balancing can be achieved with four replications. So the four replications under I and II of Table 9 have to be reshuffled as shown in Table 13, yielding two independent balanced sets I′ and II′ of four replications. It is instructive to notice in Table 13 the formation of a 2×2 Latin square by the numbers I and II. Also the replications 1, 2, 3, 4 under I and II if arranged in the form of a 2×2 square thus:

Table 13. Two Balanced Sets of 4 Replications.

	I′					II′				
		ABC	ABD	ACD	BCD		ABC	ABD	ACD	BCD
1	I	s_4	s_1	s_2	s_4	II	s_3	s_4	s_2	s_4
4		s_4	s_1	s_3	s_1		s_0	s_0	s_2	s_0
2	II	s_3	s_1	s_1	s_2	I	s_1	s_2	s_1	s_3
3		s_3	s_1	s_4	s_2		s_1	s_2	s_3	s_3

1 2
3 4

will show that two replications occurring in the same row or column confound the same set of ABC or ABD interactions. It is necessary therefore to segregate the four replications of I and II in two groups taken diagonally *viz.* 1 and 4; 2 and 3. Thus in arriving at Table 13 we are obliged to make use of the property of the 2×2 Latin square in two places.

5. The 4^n Type of Experiment.

5.1. *Three factors at four levels each.*

The factors A, B, C are now at four levels each (a_1, a_2, a_3, a_4) etc. There are 27 d.f. belonging to the second order interaction ABC. In sub-blocks of 16 plots one replication will have 4 sub-blocks thus confounding 3 d.f. belonging to ABC. Nine replications are needed for balanced partial confounding.

The 4 × 4 square in the standard position can generate 8 more squares by performing the first system of interchanges (4 × 4) on the rows and columns. These are given in Table 14. Keep the levels of C arranged according to these 9 squares making the columns and rows represent the A levels and B levels respectively. These 9 (a, b, c)-squares represent the key sub-blocks of the 9 balanced replications. The remaining three sub-blocks of each replication are obtained by performing the second system of interchanges on the rows (columns) of each key sub-block. The 9 sets of 3 d. f. of ABC obtained thus are denoted by s_1, s_2,.........s_9 and Table 15 gives the levels of the three factors occurring in each sub-block of the nine replications. A and B are taken as the first two factors and C as the third factor.

TABLE 14. THE NINE STANDARD 4 × 4 SQUARES.

1				2				3			
1	2	3	4	1	4	2	3	1	3	4	2
2	1	4	3	2	3	1	4	2	4	3	1
3	4	1	2	3	2	4	1	3	1	2	4
4	3	2	1	4	1	3	2	4	2	1	3
4				**5**				**6**			
1	2	3	4	1	4	2	3	1	3	4	2
4	3	2	1	4	1	3	2	4	2	1	3
2	1	4	3	2	3	1	4	2	4	3	1
3	4	1	2	3	2	4	1	3	1	2	4
7				**8**				**9**			
1	2	3	4	1	4	2	3	1	3	4	2
3	4	1	2	3	2	4	1	3	1	2	4
4	3	2	1	4	1	3	2	4	2	1	3
2	1	4	3	2	3	1	4	2	4	3	1

5.2. *Four factors at four levels each.*

Let A, B, C, D be the four factors at levels represented by (a_1, a_2, a_3, a_4) etc. There are 27 d.f. belonging to each of the second order interactions A B C, A B D, A C D and B C D and 81 d.f. belonging to the third order interaction A B C D. In sub-blocks of 16 plots, 15 d.f. get confounded among the 16 sub-blocks of a single replication. These 15 d. f. can be separated into 5 orthogonal sets of 3 d. f. each. Four of these sets belong one each to A B C, A B D, A C D and B C D and the fifth set to A B C D. For balanced partial confounding of A B C D, 27 replications are necessary. In this section a method of getting these 27 replications with 16 sub-blocks of 16 plots will be discussed and it will also be shown that the 27 replications fall into three sets of 9 replications, each set confounding with *balance* all the second order interactions.

TABLE 15. 4^3 DESIGN IN SUB-BLOCKS OF 16 PLOTS CONFOUNDING SECOND ORDER INTERACTIONS.

Levels of 1st & 2nd factors		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
		Level of third factor								
1 1	1	1 2 3 4	1 2 3 4	1 2 3 4	1 4 2 3	1 4 2 3	1 4 2 3	1 3 4 2	1 3 4 2	1 3 4 2
1 2	2	2 1 4 3	2 1 4 3	2 1 4 3	4 1 3 2	4 1 3 2	4 1 3 2	3 1 2 4	3 1 2 4	3 1 2 4
1 3	3	3 4 1 2	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	4 2 1 3	4 2 1 3	4 2 1 3
1 4	4	4 3 2 1	4 3 2 1	4 3 2 1	3 2 4 1	3 2 4 1	3 2 4 1	2 4 3 1	2 4 3 1	2 4 3 1
2 1	5	2 1 4 3	4 3 2 1	3 4 1 2	2 3 1 4	4 1 3 2	3 2 4 1	2 4 3 1	4 2 1 3	3 1 2 4
2 2	6	1 2 3 4	3 4 1 2	4 3 2 1	3 2 4 1	1 4 2 3	2 3 1 4	4 2 1 3	2 4 3 1	1 3 4 2
2 3	7	4 3 2 1	2 1 4 3	1 2 3 4	1 4 2 3	3 2 4 1	4 1 3 2	3 1 2 4	1 3 4 2	2 4 3 1
2 4	8	3 4 1 2	1 2 3 4	2 1 4 3	4 1 3 2	2 3 1 4	1 4 2 3	1 3 4 2	3 1 2 4	4 2 1 3
3 1	9	3 4 1 2	2 1 4 3	4 3 2 1	3 2 4 1	2 3 1 4	4 1 3 2	3 1 2 4	2 4 3 1	4 2 1 3
3 2	10	4 3 2 1	1 2 3 4	3 4 1 2	2 3 1 4	3 2 4 1	1 4 2 3	1 3 4 2	4 2 1 3	2 4 3 1
3 3	11	1 2 3 4	4 3 2 1	2 1 4 3	4 1 3 2	1 4 2 3	3 2 4 1	2 4 3 1	3 1 2 4	1 3 4 2
3 4	12	2 1 4 3	3 4 1 2	1 2 3 4	1 4 2 3	4 1 3 2	2 3 1 4	4 2 1 3	1 3 4 2	3 1 2 4
4 1	13	4 3 2 1	3 4 1 2	2 1 4 3	4 1 3 2	3 2 4 1	2 3 1 4	4 2 1 3	3 1 2 4	2 4 3 1
4 2	14	3 4 1 2	4 3 2 1	1 2 3 4	1 4 2 3	2 3 1 4	3 2 4 1	2 4 3 1	1 3 4 2	4 2 1 3
4 3	15	2 1 4 3	1 2 3 4	4 3 2 1	3 2 4 1	4 1 3 2	1 4 2 3	1 3 4 2	2 4 3 1	3 1 2 4
4 4	16	1 2 3 4	2 1 4 3	3 4 1 2	2 3 1 4	1 4 2 3	4 1 3 2	3 1 2 4	4 2 1 3	1 3 4 2

Take a 4 × 4 square with the columns and rows representing the A and B levels. The C and D levels should appear in the cells of this square in the form of two orthogonalised Latin squares. The 9 squares of Table 14 fall into three groups G_1, G_2, G_3 so that the squares in any group are orthogonal only to the squares of the other two groups. G_1 consists of squares 1, 5, 9; G_2 of squares 2, 6, 7 and G_3 of squares 3, 4, 8. Arrange the letters G_1, G_2, G_3 in the form of a 3 × 3 Latin square in the standard position:—

$$\begin{matrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_1 \\ G_3 & G_1 & G_2 \end{matrix}$$

Allow the C levels to be arranged according to the squares of the groups G_1, G_2, G_3 of the first column. For D levels choose the squares in the group of the second column belonging to the same row as the group to which the square selected for the C levels belong. This process will yield 27 (*abcd*)-squares representing one sub-block from each of the 27 replications that we need for balancing the third order interaction A B C D. The method of generation is clear from Table 16. (By choosing for D levels the squares belonging to the groups in the third column we have another set of 27 balanced replications.)

TABLE 16. SQUARES COMBINED FOR C AND D LEVELS TO GET THE 27 BALANCED REPLICATIONS.

	I		II		III	
	$G_1(c)$	$G_1(d)$	$G_2(c)$	$G_2(d)$	$G_3(c)$	$G_3(d)$
1	1	2	2	3	3	1
2	1	6	2	4	3	5
3	1	7	2	8	3	9
4	5	2	6	3	4	1
5	5	6	6	4	4	5
6	5	7	6	8	4	9
7	9	2	7	3	8	1
8	9	6	7	4	8	5
9	9	7	7	8	8	9

From each of the 27 squares, 15 squares more can be generated by performing the second system of interchanges (4 × 4) on the *c* rows (columns) and the *d* rows (columns). Thus we get all the 16 sub-blocks of the 27 replications.

The treatment combinations of the 16 sub-blocks of one of these replications are given in Table 17 for purpose of illustration. The last 16 columns represent the 16

TABLE 17. A 4^4 DESIGN IN SUB-BLOCKS OF 16 PLOTS CONFOUNDING SECOND AND THIRD ORDER INTERACTIONS.

Levels of first and second factors			Levels of third and fourth factors															
1	1	1	1	6	11	16	8	3	14	9	10	13	4	7	15	12	5	2
1	2	2	6	1	16	11	3	8	9	14	13	10	7	4	12	15	2	5
1	3	3	11	16	1	6	14	9	8	3	4	7	10	13	5	2	15	12
1	4	4	16	11	6	1	9	14	3	8	7	4	13	10	2	5	12	15
2	1	5	8	3	14	9	1	6	11	16	15	12	5	2	10	13	4	7
2	2	6	3	8	9	14	6	1	16	11	12	15	2	5	13	10	7	4
2	3	7	14	9	8	3	11	16	1	6	5	2	15	12	4	7	10	13
2	4	8	9	14	3	8	16	11	6	1	2	5	12	15	7	4	13	10
3	1	9	10	13	4	7	15	12	5	2	1	6	11	16	8	3	14	9
3	2	10	13	10	7	4	12	15	2	5	6	1	16	11	3	8	9	14
3	3	11	4	7	10	13	5	2	15	12	11	16	1	6	11	9	8	3
3	4	12	7	4	13	10	2	5	12	15	16	11	6	1	0	11	3	8
4	1	13	15	12	5	2	10	13	4	7	8	3	14	9	1	0	11	16
4	2	14	12	15	2	5	13	10	7	4	3	8	9	11	6	1	16	11
4	3	15	5	2	15	12	4	7	10	13	14	9	8	3	11	16	1	6
4	4	16	2	5	12	15	7	4	13	10	0	14	3	8	16	11	6	1

TABLE 18. KEY SUB-BLOCKS OF 27 BALANCED REPLICATIONS OF A 4^4 DESIGN IN SUB-BLOCKS OF 16 PLOTS.

Levels of a and b			I									II									III								
			1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
a	b		Levels of c and d																										
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	6	8	7	14	16	15	10	12	11	6	8	7	14	16	15	10	12	11	6	8	7	14	16	15	10	12	11
1	3	3	11	10	12	7	6	8	15	14	16	11	10	12	7	6	8	15	14	16	11	10	12	7	6	8	15	14	16
1	4	4	16	15	14	12	11	10	8	7	6	16	15	14	12	11	10	8	7	6	16	15	14	12	11	10	8	7	6
2	1	5	8	7	6	16	15	14	12	11	10	15	14	16	11	10	12	7	6	8	10	12	11	6	8	7	14	16	15
2	2	6	3	2	4	3	2	4	3	2	4	12	11	10	8	7	6	16	15	14	13	13	13	9	9	9	5	5	5
2	3	7	14	16	15	10	12	11	6	8	7	5	5	5	13	13	13	9	9	9	4	3	2	4	3	2	4	3	2
2	4	8	9	9	9	5	5	5	13	13	13	2	4	3	2	4	3	2	4	3	7	6	8	15	14	16	11	10	12
3	1	9	10	12	11	6	8	7	14	16	15	8	7	6	16	15	14	12	11	10	15	14	16	11	10	12	7	6	8
3	2	10	13	13	13	9	9	9	5	5	5	3	2	4	3	2	4	3	2	4	12	11	10	8	7	6	16	15	14
3	3	11	4	3	2	4	3	2	4	3	2	14	16	15	10	12	11	6	8	7	5	5	5	13	13	13	9	9	9
3	4	12	7	6	8	15	14	16	11	10	12	9	9	9	5	5	5	13	13	13	2	4	3	2	4	3	2	4	3
4	1	13	15	14	16	11	10	12	7	6	8	10	12	11	6	8	7	14	16	15	8	7	6	16	15	14	12	11	10
4	2	14	12	11	10	8	7	6	16	15	14	13	13	13	9	9	9	5	5	5	3	2	4	3	2	4	3	2	4
4	3	15	5	5	5	13	13	13	9	9	9	4	3	2	4	3	2	4	3	2	14	16	15	10	12	11	6	8	7
4	4	16	2	4	3	2	4	3	2	4	3	7	6	8	15	14	16	11	10	12	9	9	9	5	5	5	13	13	13

sub-blocks. Table 17 is in the form of a 16 × 16 Latin square with an internal pattern of a 4 × 4 Latin square. It is easy to see that given the first column (*i.e.*, the key sub-block) the remaining columns can be formed with little effort. The key sub-blocks of the 27 replications are given in Table 18.

Table 19 gives details about the 5 sets of 3 d. f. confounded in each replication. The 27 sets of 3 d.f. of A B C D are classified under three heads: X, Y, Z, there having 9 sets in each. In fact we are introducing X, Y, Z as alternative third factors to A and B and identifying the 9 sets of 3 d.f. of A B X, A B Y and A B Z with the help of Table 15. The levels of X, Y, Z are defined in Table 20 where the numbers 1, 2, 3,............16 stand for the CD combinations as marked in Table 21.

TABLE 19. SETS OF SECOND AND THIRD ORDER INTERACTIONS CONFOUNDED IN THE 27 BALANCED REPLICATIONS.

		I									II									III								
		1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
ABC		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
ABD		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
ACD		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
BCD		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
ABCD	X		[illegible]				[illegible]	[illegible]				[illegible]				[illegible]	[illegible]				[illegible]				[illegible]	[illegible]		
	Y	[illegible]				[illegible]				[illegible]	[illegible]				[illegible]				[illegible]	[illegible]				[illegible]				[illegible]
	Z			[illegible]	[illegible]				[illegible]				[illegible]	[illegible]				[illegible]				[illegible]	[illegible]				[illegible]	

TABLE 20. LEVELS OF X, Y, Z.

X		Y		Z	
x_1	1 + 6 + 11 + 16	y_1	1 + 7 + 12 + 14	z_1	1 + 8 + 10 + 15
x_2	2 + 5 + 12 + 15	y_2	2 + 8 + 11 + 13	z_2	2 + 7 + 9 + 16
x_3	3 + 8 + 9 + 14	y_3	3 + 5 + 10 + 16	z_3	3 + 6 + 12 + 13
x_4	4 + 7 + 10 + 13	y_4	4 + 6 + 9 + 15	z_4	4 + 5 + 11 + 14

TABLE 21. THE COMBINATIONS OF C AND D LEVELS.

	c_1	c_2	c_3	c_4
d_1	1	5	9	13
d_2	2	6	10	14
d_3	3	7	11	15
d_4	4	8	12	16

While consulting Table 15, A and B were taken as the first and second factors for interactions ABC, ABD, ABX, ABY and ABZ; C and D were taken as the first and second factors for interactions ACD and BCD.

The 27 replications of Table 19 balance all the second and third order interactions. Since the third order interaction is less important than the second order interactions it is useful to get sets of 9 replications which will balance all the second order interactions. The nine replications of any of the groups I, II and III of Table 19 may be arranged in the form of a 3×3 square:

1	2	3
4	5	6
7	8	9

It will be found that the three replications of any row or column confound the same set of ABC or ABD interactions. It is necessary therefore to split the 9 replications under I, II and III in 3 groups of 3 each according to the two orthogonal squares for 3×3. The reshufflings of the 27 replications under I, II and III to yield three sets of 9 balanced replications were found possible in three independent ways. These are given in Tables 22 to 24. It is instructive to notice how the numbers I, II, III arrange themselves in the form of a 3×3 Latin square in all the three Tables. Similar sets can be got from the other 27 balanced replications obtainable by keeping the D levels according to the last column of the G-square.

Table 22. Sets I_a, II_a, III_a of 9 Replications balancing Second Order Interactions.

		I_a								II_a								III_a						
		ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD		
						X	Y	Z						X	Y	Z						X	Y	Z
1	I	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		III	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		II	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
5		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
9		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
2	II	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			I	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			III	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
6		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
7		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
3	III	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	II	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	I	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
4		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
8		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]

It will be noticed how throughout this section we had to bring in the properties of the 3×3 Græco-Latin square.

Table 23. Sets I_b, II_b, III_b of 9 Replications balancing Second Order Interactions.

	I_b								II_b								III_b							
		ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD		
						X	Y	Z						X	Y	Z						X	Y	Z
1	I	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		II	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		III	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
5		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
9		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
2	III	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			I	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			II	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
6		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
7		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
3	II	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	III	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	I	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
4		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
8		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]

Table 24. Sets I_c, II_c, III_c of 9 Replications balancing Second Order Interactions.

	I_c								II_c								III_c							
		ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD				ABC	ABD	ACD	BCD	ABCD		
						X	Y	Z						X	Y	Z						X	Y	Z
1	I	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		II	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		III	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
6		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
8		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
2	III	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			I	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			II	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
4		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]		[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
9		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
3	II	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	III	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]	I	[illegible]	[illegible]	[illegible]	[illegible]			[illegible]
5		[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]			[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	
7		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		

5.3. *Five factors at four levels each.*

If there are five factors A, B, C, D, E at four levels each, sub-blocks of 16 plots can be formed by arranging the levels of C, D and E according to the three orthogonalised Latin squares of the 4 × 4 square whose columns and rows are the A and B levels. One such square is given in Table 25.

The 64 sub-blocks of a single replication are obtained by performing the second system of interchanges 4 × 4) on the *c* rows (columns), *d* rows (columns) and *e* rows (columns). Here there are 10 second order interactions each with 27 d.f., 5 third order interactions each with 81 d.f. and 1 fourth order interaction with 243 d.f. I have not investigated how the 63 d.f. confounded in a replication are distributed among these interactions. There is reason to believe that only 3 d.f. of ABCDE get confounded in a single replication so that balancing of this interaction requires 81 replications. A key sub-block of each of these 81 replications can be easily obtained by adopting for the C, D and E levels the squares of columns 1, 2 and 3 respectively of the G-square such that all the three squares used in one key sub-block belong to the three groups of the same row. We will get from each row 27 key sub-blocks. The 64 sub-blocks of a single replication are obtained by performing the second system of interchanges (4×4) on the c rows (columns), the d rows (columns) and the e rows (columns) of the key sub-block of that replication. It has yet to be investigated how these 81 replications can be split into sets which balance (say) only the second order interactions.

Table 25. A Key Sub-block of 16 Plots of a 4^3 Confounded Design.

	a_1	a_2	a_3	a_4
b_1	$c_1d_1e_1$	$c_2d_2e_2$	$c_3d_3e_3$	$c_4d_4e_4$
b_2	$c_3d_4e_2$	$c_4d_3e_1$	$c_1d_2e_4$	$c_2d_1e_3$
b_3	$c_4d_2e_3$	$c_3d_1e_4$	$c_2d_4e_1$	$c_1d_3e_2$
b_4	$c_2d_3e_4$	$c_1d_4e_3$	$c_4d_1e_2$	$c_3d_2e_1$

Summary.

The paper enunciates the problem of confounding in the general symmetrical type of experiment, of n factors at p levels each and develops a method based on two systems of interchanges derivable from the p-sided hyper-Græco-Latin square. The working of this method has been demonstrated in getting confounded arrangements for the p^n type of experiment in sub-blocks of p^2 plots, for $p=3$, $n=3$ and 4; and $p=4$, $n=3$, 4 and 5. In getting *balanced* replications a method has been developed which works only when p is such that $p-1$ also is either a prime or a power of a prime.

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