A GEOMETRICAL NOTE ON THE USE OF RECTANGULAR CO-ORDINATES IN THE THEORY OF SAMPLING DISTRIBUTIONS CONNECTED WITH A MULTIVARIATE NORMAL POPULATION

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Introduction.

In a paper published in *Sankhyā*[1], a system of rectangular co-ordinates was defined in connection with samples drawn from a multivariate normal population and extensive use was made of this system for the study of various types of sampling distributions connected with the population. Many of the results in that paper followed from hyper-space geometry, while some of the important derivations involved the use of rather laborious algebra, including the processes of algebraic induction, which in some places were not fully carried through. It is the object of the present paper to replace by hyper-space geometry the processes referred to above. This will provide a purely geometrical structure for the whole of the previous paper and incidentally throw some light on many of the complicated algebraic expressions contained in that paper.

Section I.

In this section we propose to establish certain Lemmas, useful for our present investigations.

In the fundamental polyhedron $OZ_1 Z_2......Z_p$ of Section II, p. 8 of the *Sankhyā* paper, denote the vectors $OZ_1, OZ_2,......OZ_p$ by $l_1, l_2,......l_p$ The hypervolume of the parallelopiped formed by these may be denoted by $V(l_1, l_2......l_p)$ and the hypervolume of the parallelopiped formed by a partial set, by a corresponding symbol. If we consider only the first $r(r < p)$ of the vectors, then the angle between the flat spaces $(l_1,......l_{i-1}, l_{i+1},......l_r)$ and $(l_1,......l_{j-1}, l_{j+1},......l_r)$ will be denoted by $\theta_{ij}^{(r)}$. The perpendicular from the extremity of the vector l_i to the space[2] $(l_1,......l_{i-1}, l_{i+1},......l_r)$ considered vectorially will be denoted by $l_i^{(r)}$, and the length of the perpendicular, i. e $|l_i^{(r)}|$ by $k_i^{(r)}$ for our present purposes.

If instead of the polyhedron $OZ_1, Z_2......Z_p$ we consider the corresponding population polyhedron $OZ'_1\ Z'_2......Z'_p$ (Cf. *Sankhyā*, paper, p. 14) then the vectors OZ'_1, ... OZ'_p may be denoted by $\tau_1, \tau_2, ...\tau_p$ with a corresponding notations for volumes. In the same way the angles corresponding to $\theta_{ij}^{(r)}$ will be denoted by $\phi_{ij}^{(r)}$, and $l_i^{(r)}$ will be replaced by $\tau_i^{(r)}$, and $k_i^{(r)}$ by $\kappa_i^{(r)}$.

P. C. Mahalanobis, Raj Chandra Bose and Samarendra Nath Roy : "Normalisation of Statistical Variates and the use of Rectangular Co-ordinates in the Theory of Sampling Distributions" *Sankhyā*, Vol. III, part I, March 1937 ; pp. 1–40.

Is is easy to see that

$$a_{ij} = l_i \cdot l_j \qquad \ldots \quad (1\cdot1)$$

where the dot stands for scalar product, and a_{ij}'s are the sample co-variances as in the *Sankhyā* paper. Then clearly

$$\begin{vmatrix} a_{11} & a_{12} \ldots\ldots & a_{1p} \\ a_{21} & a_{22} \ldots\ldots & a_{2p} \\ \ldots\ldots & \ldots & \ldots\ldots \\ a_{p1} & a_{p2} \ldots\ldots & a_{pp} \end{vmatrix} = v^2(l_1, l_2, \ldots\ldots \ldots l_p). \qquad \ldots \quad (1\cdot2)$$

Lemma I.

$$A_{ij}^{(p)} = V(l_1, l_2, \ldots\ldots l_{i-1}, l_{i+1}, \ldots\ldots l_p) \cdot V(l_1, \ldots l_{j-1}, l_{j+1} \ldots l_p) \cos\theta_{ij}^{(p)}$$

where $A_{ij}^{(p)}$ is the minor of a_{ij} in the determinant on the left hand side of (1·2)

Proof: Let L be any vector in the space $(l_1, \ldots l_{i-1}, l_{i+1}, \ldots\ldots l_p)$ and M be any vector in the space $(l_1, \ldots\ldots l_{j-1}, l_{j+1}, \ldots\ldots l_p)$ Let θ be the angle be the angle between the two vectors. Then $\theta_{ij}^{(p)}$ will be the minimum value of θ. We have

$$L = \lambda_1 l_1 + \ldots\ldots \lambda_{i-1} l_{i-1} + \lambda_{i+1} l_{i+1} + \ldots\ldots\ldots \lambda_p l_p$$

$$M = \mu_1 l_1 + \ldots\ldots \mu_{j-1} l_{j-1} + \mu_{j+1} l_{j+1} + \ldots\ldots\ldots \mu_p l_p$$

where λ's and μ's are adjustable constants.

$L^2 = \Sigma_s \Sigma_r \lambda_r \lambda_s a_{rs}$ where the summation extends over the values $1, 2, \ldots\ldots i-1, i+1, \ldots\ldots p$ of r and s.

Also $M^2 = \Sigma_n \Sigma_m \mu_m \mu_n a_{mn}$ where the summation extends over the values $1, 2, \ldots\ldots j-1, j+1, \ldots\ldots p$ of m and n.

Therefore, $|L| \cdot |M| \cdot \cos\theta = (L.M) = \Sigma_n \Sigma_r \lambda_r \mu_n a_{rn}$

where the summation extends over the possible values of r and n just indicated.

To minimise θ, we have by differentiation with regard to the disposable parameters.

$$\cos\theta \cdot \frac{|M|}{|L|} \cdot \Sigma \lambda_s a_{rs} = \Sigma \mu_n a_{rn} \qquad \ldots \quad (1\cdot3)$$

the summation on the left being for all possible values of s, and on the right for all possible values of n the formula being valid for $r = 1, 2, \ldots\ldots i-1, i+1, \ldots\ldots p$.

Likewise we have

$$\cos\theta \cdot \frac{|L|}{|M|} \cdot \Sigma \mu_m a_{mn} = \Sigma \lambda_r a_{rn} \qquad \ldots \quad (1\cdot4)$$

where the summation on the left is for possible values of m, and the right for possible values of r, the formula being valid for $n = 1, 2, \ldots\ldots j-1, j+1, \ldots\ldots p$, We have from (1·3) and (1·4) by elimination

$$\left|\begin{array}{cccccccc}
Ca_{11} \ldots & Ca_{1,i-1} & Ca_{1,i+1} \ldots & Ca_{1,p} & a_{11} \ldots & a_{1,j-1} & a_{1,j+1} \ldots & a_{1p} \\
\ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots \\
Ca_{i-1,1} \ldots & Ca_{i-1,i-1} & Ca_{i-1,i} \ldots & C_{i-1,p} & a_{i-1,1} \ldots & a_{i-1,j-1} & a_{i-1,j+1} \ldots & a_{i-1,p} \\
Ca_{i+1,1} \ldots & Ca_{i+1,i-1} & Ca_{i+1,i+1} \ldots & Ca_{i+1,p} & a_{i+1,1} \ldots & a_{i+1,j-1} & a_{i+1,j+1} \ldots & a_{i+1,p} \\
\ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots \\
Ca_{p1} \ldots & Ca_{p,i-1} & Ca_{p,i+1} \ldots & Ca_{pp} & a_{p1} \ldots & a_{p,j-1} & a_{p,j+1} \ldots & a_{.p} \\
Ca_{11} \ldots & a_{1,i-1} & a_{1,i+1} \ldots & a_{1,p} & Ca_{11} \ldots & Ca_{1,j-1} & Ca_{1,j+1} \ldots & Ca_{p1p} \\
\ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots \\
a_{j-1,1} \ldots & a_{j-1,i-1} & a_{j-1,i+1} \ldots & a_{j-1,p} & Ca_{j-1,1} \ldots & Ca_{j-1,j-1} & Ca_{j-1,j+1} \ldots & Ca_{j-1,p} \\
a_{j+1,1} \ldots & a_{j+1,i-1} & a_{j+1,i+1} \ldots & a_{j+1,p} & Ca_{j+1,1} \ldots & Ca_{j+1,j-1} & Ca_{j+1,j+1} \ldots & Ca_{j+1,p} \\
\ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots\ldots & \ldots\ldots \ldots & \ldots \\
a_{p1} \ldots & a_{p,i-1} & a_{p,i+1} \ldots & a_{pp} & Ca_{p1} \ldots & Ca_{p,j-1} & Ca_{p,j+1} \ldots & Ca_{pp}
\end{array}\right| = 0$$

where for shortness we have written C for $\cos\theta$.

We first multiply each of the first $p-1$ rows by C and divide each of the last $p-1$ columns by C. This leaves the determinantal equation unchanged. In the resulting determinant C^2 is common to and occurs only in the elements of the leading minor of $(p-1)$-th order. It is also readily seen that except for C^2, the first $(p-1)$ rows and the last $(p-1)$ rows, have $(p-2)$ rows common, each to each. Subtracting the latter rows, from the corresponding rows of the former, we have (C^2-1) i.e. $-\sin^2\theta$ as factors in the first $(p-1)$ columns of these common upper $(p-2)$ rows, and zeros in the last $(p-1)$ columns of the same rows. Hence $(-\sin^2\theta)^{p-2}$ comes out as a factor. Dividing by this (since $\theta \neq 0$), it follows from Laplace's development of a determinant that

$$\cos^2\theta = (A_{ij}^{(p)})^2/v^2(l_1, l_2, \ldots\ldots l_{i-1}, l_{i+1}, \ldots\ldots l_p) \cdot v^2(l_1, \ldots\ldots l_{j-1}\ l_{j+1}, l_p).$$

This minimised θ is $\theta_{ij}^{(p)}$,

One thing more should be noticed in this connection. It is well known and may also be proved from the above results that corresponding to this minimised θ there is a unique pair of lines l and m both of which are perpendicular to $(l_1, l_2 \ldots l_{i-1}, l_{i+1}, \ldots l_{j-1}, l_{j+1}, \ldots\ldots l_p)$, which let us call S for our present purpose.

Then $l_i^{(p)}$ being perpendicular to (S, l_j) is perpendicular to l which lies in that space. Similarly $l_j^{(p)}$ is perpendicular to M. Also $l_i^{(p)}$, $l_j^{(p)}$, l and m are co-planar, all lying in (S, l_i, l_j) and all being perpendicular to S.

Therefore, $\theta_{ij}^{(p)}$ = angle between l and m = angle between $l_i^{(p)}$ and $l_j^{(p)}$.

Therefore, $A_{ij}^{(p)} = v(l_1, \ldots\ldots l_{i-1}, l_{i+1}, \ldots\ l_p) \cdot v(l_1 \ldots\ldots l_{j-1}, l_{j+1}, \ldots\ldots l_p) \cdot \cos\theta_{ij}^{(p)}$ (1·5)

Corollary: If $A_{ij}^{(r)}$ denotes the minor of a_{ij} ($i \leqslant r, j \leqslant r$) in the leading minor of order r, then

$$A_{ij}^{(r)} = v(l_1, \ldots\ldots l_{i-1}, l_{i+1}, \ldots\ldots l_r) \cdot v(l_1, \ldots\ldots l_{j-1}, l_{j+1}, \ldots\ldots l_r) \cdot \cos\theta_{ij}^{(r)} \quad \ldots \quad (1·6)$$

Lema II.

$$a_{ij} = \frac{\cos\theta_{ij}^{(r)}}{k_i^{(p)}\, k_j^{(p)}} \quad \cdots \quad (1·7)$$

where a^{ij} is the co-factor of a_{ij} divided by $|a_{ij}|$, as defined in the *Sankhyā* paper.

Proof:

$$a_{ij} = \Lambda_{ij}^{(p)} / |a_{ij}| = \frac{\cos \theta_{ij}^{(p)} \cdot v(l_1, \ldots l_{i-1}, l_{i+1}, \ldots l_p) \cdot v(l_1, \ldots l_{j-1}, l_{j+1}, \ldots l_p)}{v(l_1, l_2, \ldots\ldots l_p) \cdot v(l_1, l_2, \ldots, l_p)} = \frac{\cos \theta_{ij}^{(p)}}{h_i^{(p)} h_j^{(p)}}$$

Corollary: If $a_{(r)}^{ij}$ denotes $\Lambda_{ij}^{(r)}$ divided by the leading minor of r-th order in $|a_{ij}|$, then

$$a_{(r)}^{ij} = \frac{\cos \theta_{ij}^{(r)}}{k_i^{(r)} k_j^{(r)}} \qquad (1 \cdot 75)$$

Lemma III.

$$\frac{|T_{(r)}^{i, p-r}|}{\sqrt{|T_{(r)}^{p-r, p-r}|}} = \frac{\cos \phi_{i, p-r}^{(p-r)}}{\kappa_i^{(p-r)}} \qquad (1 \cdot 8)$$

the quantities on the left hand side of the equation being defined by equations (11.21 to 11·26) of p. 18 of the *Sankhyā* paper. It was noted on p. 16 of the *Sankhyā* paper that T_{ij}'s are the same as a_{ij}'s, where $||a_{ij}||$ is the dispersion matrix for the population and accordingly $T^{ij} = a^{ij}$.

Proof: We have from equation (11·25) p. 18 of the *Sankhyā* paper.

$$|T_{(r)}| \cdot |T_{(r)}^{ij}| = \begin{vmatrix} T^{ij} & T^{i, p-r+1} & T^{i, p-r+2} & \ldots\ldots & T^{i, p} \\ T^{p-r+1, j} & T^{p-r+1, p-r+1} & T^{p-r+1, p-r+2} & \ldots & T^{p-r+1, p} \\ T^{p-r+2, j} & T^{p-r+2, p-r+1} & T^{p-r+2, p-r+2} & \ldots & T^{p-r+2, p} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ T^{p, j} & T^{p, p-r+1} & T^{p, p-r+2} & \ldots\ldots & T^{pp} \end{vmatrix}$$

The right hand side of this equation is easily seen to be equal to

$$\frac{\text{Minor of } T_{ij} \text{ in the leading diagonal of the } (p-r)\text{-th order in } |T_{ij}|}{|T_{ij}^{(p)}|}$$

$$= \frac{\text{Minor of } T_{ij} \text{ in } |T_{ij}^{(p-r)}|}{T_{ij}^{(p)}} \qquad \ldots \quad (1 \cdot 81)$$

in the notation as we have laid down.

Also from equations (11·21 to 11·24) p. 18 of the *Sankhyā* paper

$$|T_{(r)}| = \begin{vmatrix} T^{p-r+1, p-r+1} & \ldots\ldots & T^{p-r+1, p} \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots \\ T^{p, p-r+1} & \ldots\ldots & T^{pp} \end{vmatrix} = \frac{|T_{ij}^{(p-r)}|}{|T_{ij}^{(p)}|} \qquad \ldots \quad (1 \cdot 82)$$

Also

$$|T_{(r)}^{p-r, p-r}| = \frac{|T_{(r+1)}|}{|T_{(r)}|} \qquad \ldots \quad (1 \cdot 83)$$

Therefore,

$$\frac{|T_{(r)}^{i, p-r}|}{\sqrt{|T_{(r)}^{p-r, p-r}|}} = \frac{\text{Minor of } T_{ij} \text{ in } |T_{ij}^{(p-r)}|}{|T_{ij}^{(p)}| \cdot |T_{(r)}|} \cdots \frac{\sqrt{|T_{(r)}|}}{\sqrt{|T_{(r+1)}|}} \text{ from } 1 \cdot 81, \text{ and } (1 \cdot 83).$$

$$= \frac{v(r_1, \ldots r_{i-1}, r_{i+1} \ldots r_{p-r}) \cdot v(r_1, \ldots r_{p-r-1}) \cdot \cos \phi_{i, p-r}^{(p-r)}}{v(r_1, \ldots r_{p-r}) \cdot v(r_1, \ldots r_{p-r-1})} \quad \text{from } (1 \cdot 5)$$

$$= \frac{\cos \phi_{i, p-r}^{(p-r)}}{\kappa_i^{(p-r)}} \quad \text{which proves } (1.8)$$

Section II

In the *Sankhyā* paper the matrix of rectangular co-ordinates $||l_{ij}||$ was connected by means of algebraic induction with the dispersion matrix $||a_{ij}||$, by means of the following fundamental relation [p. 12, equation (6·5)]

$$l_{ij} = \frac{\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1,i-1} & \vdots & a_{1j} \\ a_{21} & a_{22} & \ldots\ldots & a_{2,i-1} & \vdots & a_{2j} \\ \ldots & \ldots & \ldots\ldots & \ldots & & \ldots \\ a_{i-1,1} & a_{i-1,2} & \ldots\ldots & a_{i-1,i-1} & \vdots & a_{i-1,j} \\ a_{i1} & a_{i2} & \ldots\ldots & a_{i,i-1} & \vdots & a_{ij} \end{vmatrix}}{\begin{vmatrix} a_{11} & \ldots\ldots & a_{1,i-1} \\ \ldots & \ldots\ldots & \ldots \\ a_{i-1,1} & \ldots\ldots & a_{i-1,i-1} \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} a_{11} & \ldots\ldots & a_{1i} \\ \ldots & \ldots\ldots & \ldots \\ a_{i1} & \ldots\ldots & a_{ii} \end{vmatrix}^{\frac{1}{2}}} \qquad (2{\cdot}1)$$

where each of i and j varies from 1 to p, p being the number of variates. The object of the present section is to prove this relation by pure geometry. Consider the two spaces $(l_1, l_2, \ldots\ldots l_{i-1}, l_i)$ and $(l_1, l_2, \ldots\ldots l_{i-1}, l_j)$ respectively, and the volumes $v(l_1, l_2, \ldots\ldots l_{i-1}, l_i)$ and $(l_1, l_2, \ldots\ldots l_{i-1}, l_j)$. Let θ be the angle between the two spaces i.e. the angle between the normals to these spaces in the space $(l_1, l_2, \ldots l_{i-1}, l_i, l_j)$ Let ϕ be the angle between the vector l_j and the perpendicular to the space $(l_1, l_2, \ldots l_{i-1})$ in the space $(l_1, l_2, \ldots l_{i-1}, l_j)$.

Then it is easily seen from equations (1·2) and (1·5) that the right hand side of (2·1).

$$= \frac{v\,(l_1, l_2, \ldots\ldots l_{i-1}, l_i) \,.\, v\,(l_1, l_2, \ldots\ldots l_{i-1}, l_j) \,.\, \cos\theta}{v\,(l_1, l_2, \ldots\ldots l_{i-1}) \,.\, v\,(l_1, l_2, \ldots\ldots l_{i-1}, l_i)}$$

$$= \frac{v\,(l_1, l_2, \ldots\ldots l_{i-1}) \,.\, |l_j| \,.\, \cos\phi \,.\, \cos\theta}{v\,(l_1, l_2, \ldots\ldots l_{i-1})}$$

Since $v(l_1, l_2, \ldots\ldots l_{i-1}, l_j) = v(l_1, l_2, \ldots\ldots l_{i-1}) \,.\, |l_j| \,.\, \cos\phi \quad = |l_j| \,.\, \cos\phi \,.\, \cos\theta.$

Equation (2·1) comes out to be equivalent to $\quad l_{ij} = |l_j| \,.\, \cos\phi \,.\, \cos\theta. \quad \ldots \quad (2{\cdot}2)$

To establish (2·1) geometrically we have therefore only to deduce (2·2) by geometry. To do this we proceed as follows:

The space $(l_1, l_2, \ldots l_{i-1})$ will be referred to as S_{i-1} for the present. In fig. 1 let OA be the perpendicular to S_{i-1} lying in the space (S_{i-1}, l_i); and OB be perpendicular to S_{i-1} in the space (S_{i-1}, l_j). The line OB is common to the plane OAB and the plane containing OB and l_j. Consider the line of intersection of the plane (OB, l_j) with the space S_{i-1}. This line is perpendicular to the plane OAB because it lies in S_{i-1} which is absolutely perpendicular to both OA and OB. Therefore, the plane defined by this line and OB, that is, the plane containing l_j and OB, is perpendicular to the plane OAB.

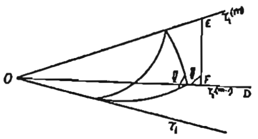

Fig. 1

It will be remembered that l_{ij} was originally defined in the *Sankhyā* paper as the projection of l_j in a direction perpendicular to S_{i-1} in S_i.

Therefore, $l_{ij} = l_j \, . \cos\psi$, where ψ is the angle between l_j and OB. Since, $\cos\psi = \cos\theta \, . \cos\varphi$, we have $l_{ij} = l_j \, . \cos\varphi \, . \cos\theta$ which proves (2.2) and hence (2·1).

SECTION III.

In *Sankhyā* paper the fundamental quadric (p. 16, 9·9) was reduced to a sum of squares by means of a transformation which introduced new variables defined by equation (11.6), p. 20 of the *Sankhyā* paper.

This transformation and the consequent reduction of the fundamental quadric form involved algebraic induction. In this section we shall establish by pure geometry that the proposed transformation leads to the reduction of the fundamental quadric form (p. 16, equation (9·9)) to a sum of squares.

It is easily seen from (11·9), p. 20 of the *Sankhyā* paper that l_{ip}'s are the components of a vector in the space $(t_1, t_2, \ldots\ldots t_p)$ which let us call the vector l_p. Likewise it is easily seen from (11·37, p. 19) that $l_{i,p-1}$ are the components of a vector in the space $(t_1, t_2, \ldots\ldots t_{p-1})$ which let us call the vector l_{p-1}, and so on. We have now a system of p vectors $l_1, l_2, \ldots\ldots l_p$ of the *Sankhyā* paper. It is now easily seen from equation (11·6), p. 20 of the *Sankhyā* paper and from (1·8) of the present paper that

$$\left.\begin{array}{lcl}
l_p & = & \dfrac{\cos\varphi_{1p}^{(p)}}{\kappa_1^{(p)}} \, . \, t_1 + \dfrac{\cos\varphi_{2p}^{(p)}}{\kappa_2^{(p)}} \, . \, t_2 + \ldots\ldots \dfrac{\cos\varphi_{pp}^{(p)}}{\kappa_p^{(p)}} \, . \, t_p \\
l_{p-1} & = & \dfrac{\cos\varphi_{1.p-1}^{(p-1)}}{\kappa_1^{(p-1)}} \, . \, t_1 + \dfrac{\cos\varphi_{2.p-1}^{(p-1)}}{\kappa_2^{(p-1)}} \, . \, t_2 + \ldots\ldots \dfrac{\cos\varphi_{p-1.p-1}^{(p-1)}}{\kappa_{p-1}^{(p-1)}} \, . \, t_{p-1} \\
\ldots\ldots & & \ldots\ldots\ldots \qquad \ldots\ldots\ldots \\
l_2 & = & \dfrac{\cos\varphi_{12}^{(2)}}{\kappa_1^{(2)}} \, . \, t_1 + \dfrac{\cos\varphi_{22}^{(2)}}{\kappa_2^{(2)}} \, . \, t_2 \\
l_1 & = & \dfrac{\cos\varphi_{11}^{(1)}}{\kappa_1^{(1)}} \, . \, t_1
\end{array}\right\} \quad \ldots \quad (3·1)$$

Let us verify that this transformation reduces the fundamental quadric to a sum of squares $l_1^2+l_2^2 \ldots\ldots l_p^2$. The fundamental quadric (9.9, p. 16) can be easily written in the form

$$\Sigma^p_{j=1}\Sigma^p_{i=1}(l_i . l_j) . \frac{\cos\phi_{ij}^{(p)}}{\kappa_i^{(p)} . \kappa_j^{(p)}} \quad \cdots \quad (3{\cdot}2)$$

Squaring the expression on the right hand side of (3·1) and adding we have to verify by comparing with (3 2), that the coefficients of $(l_i . l_j)$ agree on both sides. The coefficients of $(l_1 . l_p)$, $(l_2 . l_p)$,......$(l_{p-1} . l_p)$, $(l_p . l_p)$ are easily seen to agree on both sides.

Consider now from (3·2), the coefficient of $(l_i . l_j)$ which is

$$\cos\phi_{ij}^{(p)}/\kappa_i^{(p)}\kappa_j^{(p)} \quad \ldots \quad (3.3)$$

Also consider the coefficient of $(l_i . l_j)$ in $l_1^2+l_2^2+\ldots\ldots l_p^2$, which, (from 3·1) is seen to be equal to

$$\frac{\cos\phi_{ip}^{(p)}\cos\phi_{jp}^{(p)}}{\kappa_i^{(p)}\kappa_j^{(p)}} + \frac{\cos\phi_{i,p-1}^{(p-1)}\cos\theta_{j,p-1}^{(p-1)}}{\kappa_i^{(p-1)} . \kappa_j^{(p-1)}} + \ldots\ldots \frac{\cos\phi_{ij}^{(j)} . \cos\phi_{jj}^{(j)}}{\kappa_i^{(j)}\kappa_j^{(j)}} \quad (3{\cdot}35)$$

To establish the identity of $\Sigma_i{}^p{}_{=1}\Sigma_{j=1}{}^p\frac{\cos\phi_{ij}{}^p}{\kappa_i^{(p)}\kappa_j^{(p)}}(l_i . l_j)$ with $l_1^2 + l_2^2 + \ldots\ldots l_p^2$

we have to show that the coefficient of $(l_i . l_j)$ agree on both sides, i e. to show that the expressions (3·3) and (3·35) are equal. With a view to this we shall establish the general relation

$$\frac{\cos\phi_{ij}^{(m)} - \cos\phi_{im}^{(m)}\cos\phi_{jm}^{(m)}}{\kappa_i^{(m)}\kappa_j^{(m)}} = \frac{\cos\psi_{ij}^{(m-1)}}{\kappa_i^{(m-1)}\kappa_j^{(m-1)}} \quad \cdots \quad (3{\cdot}4)$$

where i and $j < m \leqslant p$. This is a fundamental relation which we shall repeatedly use. To prove (3·4) we proceed as follows. Making use of the notation developed so far in this paper we have from Fig. (2)

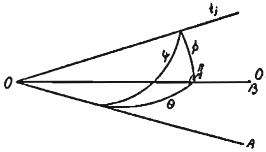

Fig. 2

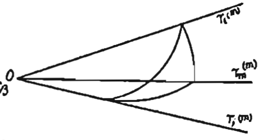

Fig. 3

$$\cos\psi_{ij}^{(m)} = \cos\psi_{im}^{(m)}\cos\psi_{jm}^{(m)} + \sin\psi_{im}^{(m)}\sin\psi_{jm}^{(m)}\cos(\tau_i^{(m)}\tau_m^{(m)}\tau_j^{(m)}) \quad \ldots \quad (3{\cdot}5)$$

Now $\cos\psi_{ij}^{(m-1)}$ = cosine of the angle between the spaces $(\tau_1,\ldots\tau_{i-1}, \tau_{i+1},\ldots\ldots\tau_{m-1})$ and $(\tau_1,\ldots\ldots\tau_{j-1}, \tau_{j+1},\ldots\ldots\tau_{m-1})$

Now, $\tau_i^{(m)}$ is perpendicular to the space $(\tau_1,\ldots\ldots\tau_{i-1}, \tau_{i+1},\ldots\tau_m)$,
$\tau_j^{(m)}$ is perpendicular to the space $(\tau_1,\ldots\ldots\tau_{j-1}, \tau_{j+1},\ldots\tau_m)$,
$\tau_m^{(m)}$ is perpendicular to the space $(\tau_1,\ldots\ldots\ldots\ldots\tau_{m-1})$,

all being in the space $(\tau_1, \tau_2, \ldots\ldots \tau_m)$.

Also $r_i^{(m-1)}$ is perpendicular to the space $(r_1, \ldots r_{i-1}, r_{i+1}, r_{m-1})$

$r_j^{(m-1)}$ is perpendicular to the space $(r_1, \ldots r_{j-1}, r_{j+1}, r_{m-1})$

all in the space $(r_1, \ldots \ldots r_{m-1})$

Now $r_m^{(m)}$ being perpendicular to the space $(r_1, \ldots r_{m-1})$, and $r_i^{(m-1)}$ and $r_j^{(m-1)}$ both lying in that space, $r_m^{(m)}$ is perpendicular to both $r_i^{(m-1)}$ and $r_j^{(m-1)}$. Also $r_i^{(m)}$, $r_i^{(m-1)}$ and $r_m^{(m)}$ being all perpendicular to the space $(r_1, \ldots\ldots r_{i-1}, r_{i+1} \ldots\ldots r_{m-1})$, and there being only one degree of freedom for the perpendicular to this space in the space $(r_1 \ldots\ldots\ldots r_m)$, we easily see that $r_i^{(m)}$, $r_i^{(m-1)}$, and $r_m^{(m)}$ are co-planar.

Similarly $r_j^{(m)}, r_j^{(m-1)}, r_m^{(m)}$ are coplanar; each being perpendicular to the space $(r_1, \ldots\ldots r_{j-1}, r_{j+1}, \ldots\ldots r_{m-1})$.

Therefore angle $(r_i^{(m)}\ r_m^{(m)}\ r_j^{(m)})$ which is the angle between the planes $(r_i^{(m)}, r_m^{(m)})$ and $(r_j^{(m)}, r_m^{(m)})$ = angle between normals to $r_m^{(m)}$ lying in the planes $r_i^{(m)}, r_m^{(m)}$ and $r_j^{(m)}, r_m^{(m)}$ = angle between $r_i^{(m-1)}$ and $r_j^{(m-1)}$, from what has been just proved $= \rho_{ij}^{(m-1)}$

Now to investigate the meaning of

$$\frac{\kappa_i^{(m)}}{\kappa_i^{(m-1)}} \text{ and } \frac{\kappa_j^{(m)}}{\kappa_j^{(m-1)}} \tag{3·6}$$

We notice that $\kappa_i^{(m)}$ s perpendicular from the end of r_i to the space $(r_1, \ldots r_{i-1}, r_{i+1} \ldots r_m)$

$$= |r_i|. \cos (r_i, r_i^{(m)})$$

Also $\kappa_i^{(m-1)}$ is perpendicular from the end of r_i to the space $(r_1 \ldots\ldots r_{i-1}, r_{i+1} \ldots r_{m-1})$

$$= |r_i|. \cos (r_i, r_i^{(m-1)})$$

Therefore,
$$\frac{\kappa_i^{(m)}}{\kappa_i^{(m-1)}} = \frac{\cos (r_i, r_i^{(m)})}{\cos r_i, r_i^{(m-1)}}$$

Now, $r_i^{(m)}$ is perpendicular to $(r_1, \ldots r_{i-1}, r_{i+1} \ldots r_{m-1}, r_m)$ and $r_i^{(m-1)}$ is perpendicular to $(r_1, \ldots r_{i-1}, r_{i+1} \ldots r_{m-1})$

Therefore, $(r^{(m)}\ r_i^{(m-1)})$ is a plane which is absolutely perpendicular to $(r_1 \ldots\ldots r_{i-1}, r_{i+1} \ldots\ldots\ldots r_{m-1})$

Any line lying on it is perpendicular to $(r_1, \ldots r_{i-1}, r_{i+1} \ldots r_{m-1})$. Therefore, in fig. 3, EF (perpendicular from E on OD) is perpendicular to both $r_i^{(m-1)}$ and $(r_1, \ldots r_{i-1}, r_{i+1}, \ldots r_{m-1})$, that is, perpendicular to the subspace $(r_1, \ldots\ldots r_i \ldots\ldots r_{m-1})$.

Therefore the plane $(r_i^{(m)}, r_i^{(m-1)})$ is perpendicular to the plane $(r_i, r_i^{(m-1)})$, as it passes through EF which is perpendicular to $(r_i, r_i^{(m-1)})$ [see Fig. 3].

Therefore, from Fig. 3, $\cos (r_i, r_i^{(m)}) = \cos (r_i^{(m)}, r_i^{(m-1)}) \cos (r_i^{(m-1)}\ r_i)$

That is $$\frac{\kappa_1^{(m)}}{\kappa_1^{(m-1)}} = \frac{\cos(r_1, r_1^{(m)})}{\cos(r_1, r_1^{(m-1)})} = \cos(r_1^{(m)}, r_1^{(m-1)}) = \sin(r_1^{(m)}, r_m^{(m)}) \qquad (3{\cdot}61)$$

because it has already been shown that $(r_1^{(m)}, r_1^{(m-1)}, r_m^{(m)})$ are co-planar and $r_m^{(m)}$ is perpendicular to $r_1^{(m-1)}$.

Similarly, $$\frac{\kappa_j^{(m)}}{\kappa_j^{(m-1)}} = \sin(r_j^{(m)}, r_m^{(m)}) \qquad \ldots \quad (3{\cdot}62)$$

Substituting in (3·5) from (3·6), (3·61), (3·62) we have the result (3·4).

We want now to establish the relation

$$\Sigma^p_{j=1} \Sigma^p_{i=1} (l_i \,.\, l_j) \,.\, \frac{\cos \phi_{ij}^{(p)}}{\kappa_i^{(p)} \,.\, \kappa_j^{(p)}} = l_1^2 + l_2^2 \ldots\ldots\ldots l_p^2 \qquad \ldots \quad (3{\cdot}7)$$

As noticed earlier we have only to show that the expressions (3·3) and (3·35) are equal.

Considering these two expressions, and having regard to (3·4) we have,

$$\frac{\cos \phi_{ij}^{(p)} - \cos \phi_{ip}^{(p)} \cos \phi_{jp}^{(p)}}{\kappa_i^{(p)} \kappa_j^{(p)}} = \frac{\cos \phi_{ij}^{(p-1)}}{\kappa_i^{(p-1)} \kappa_j^{(p-1)}}$$

Again, $$\frac{\cos \phi_{ij}^{(p-1)} - \cos \phi_{i,p-1}^{(p-1)} \cos \phi_{j,p-1}^{(p-1)}}{\kappa_i^{(p-1)} \kappa_j^{(p-1)}} = \frac{\cos \phi_{ij}^{(p-2)}}{\kappa_i^{(p-2)} \kappa_j^{(p-2)}}$$

and so on till we come to $\dfrac{\cos_{ij}^{(1)} - \cos \phi_{i1}^{(1)} \cos \phi_{j1}^{(1)}}{\kappa_i^{(1)} \kappa_j^{(1)}}$ which is zero since $\theta_{ij}^{(1)} = 0$

Therefore (3·54) = (3 55), which establishes the required identity (3·7). We have thus geometrically proved that the transformation (11·6), p. 26 of the *Sankhyā* paper reduces the fundamental quadric (11·11), p. 17 to a sum of squares given on the right hand side of (11·7), p. 20, all the references being to the *Sankhyā* paper. It should also be noticed that there is a very close parallelism between the geometrical proof of the reduction given here and the purely algebraic proof on pp. 18–20 of the *Sankhyā* paper. As a matter of fact the geometrical significance of practically every step there is brought out in the present investigation.

Section IV

The l-vectors are given in terms of the t-vectors by equation (11·6), p. 20 of the *Sankhyā* paper. In this section we shall calculate the t's in terms of the l's.

It is easily seen from (3·1) that the j-th vector t_j is given in terms of the l-vectors by

$$t_j = \begin{vmatrix} l_1 & l_2 & \ldots & l_{j-1} & l_j \\ \dfrac{1}{\kappa_1^{(1)}} & \dfrac{\cos \phi_{12}^{(2)}}{\kappa_1^{(2)}} & \ldots & \dfrac{\cos \phi_{1,j-1}^{(j-1)}}{\kappa_1^{(j-1)}} & \dfrac{\cos \phi_{1j}^{(j)}}{\kappa_1^{(j)}} \\ 0 & \dfrac{1}{\kappa_2^{(2)}} & \ldots & \dfrac{\cos \phi_{2,j-1}^{(j-1)}}{\kappa_2^{(j-1)}} & \dfrac{\cos \phi_{2,j}^{(j)}}{\kappa_2^{(j)}} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots & \ldots\ldots \\ 0 & 0 & \ldots & \dfrac{\cos \phi_{j-1,j-1}^{(j-1)}}{\kappa_{j-1}^{(j-1)}} & \dfrac{\cos \phi_{j-1,j}^{(j)}}{\kappa_{j-1}^{(j)}} \end{vmatrix} \times \kappa_1^{(1)} . \kappa_2^{(2)} \ldots \kappa_j^{(j)}$$

This should be compared with equation (12.2), p. 21, of the *Sankhyā* paper.

The co-efficient of l_j in the above equation is easily seen to be equal to $\kappa_j^{(j)}$. The co-efficient of l_i (where $i < j$)

$$= \kappa_1^{(1)} \cdot \kappa_2^{(2)} \ldots\ldots \kappa_j^{(j)} \times$$

$$\left| \begin{array}{cccccccccc}
\frac{1}{\kappa_1^{(1)}} & 0 & 0 \ldots & 0 & 0 & 0 & 0 & .. & 0 \\
\frac{\cos_{12}^{(2)}}{\kappa_1^{(2)}} & \frac{1}{\kappa_2^{(2)}} & 0 \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & .. & \ldots \\
\frac{\cos\phi_{1,i-1}^{(i-1)}}{\kappa_1^{(i-1)}} & \frac{\cos\phi_{2,i-1}^{(i-1)}}{\kappa_2^{(i-1)}} & \ldots & \frac{\cos\phi_{i-1,i-1}^{(i-1)}}{\kappa_{i-1}^{(i-1)}} & 0 & 0 & 0 & \ldots & 0 \\
\frac{\cos\phi_{1,i+1}^{(i+1)}}{\kappa_1^{(i+1)}} & \frac{\cos\phi_{2,i+1}^{(i+1)}}{\kappa_2^{(i+1)}} & \ldots & \frac{\cos\phi_{i-1,i+1}^{(i+1)}}{\kappa_{i-1}^{(i+1)}} & \frac{\cos\phi_{i,i+1}^{(i+1)}}{\kappa_i^{(i+1)}} & \frac{\cos\phi_{i+1,i+1}^{(i+1)}}{\kappa_{i+1}^{(i+1)}} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\cos\phi_{1,j-1}^{(j-1)}}{\kappa_1^{(j-1)}} & \frac{\cos\phi_{2,j-1}^{(j-1)}}{\kappa_2^{(j-1)}} & \ldots & & \ldots & \ldots & \ldots & \ldots & \frac{\cos\phi_{j-1,j-1}^{(j-1)}}{\kappa_{j-1}^{(j-1)}} \\
\frac{\cos\phi_{1j}^{(j)}}{\kappa_1^{(j)}} & \frac{\cos\phi_{2j}^{(j)}}{\kappa_2^{(j)}} & \ldots & & \ldots & \ldots & \ldots & \ldots & \frac{\cos\phi_{j-1,j}^{(j)}}{\kappa_{j-1}^{(j)}}
\end{array} \right|$$

$$= \kappa_i^{(i)} \kappa_{i+1}^{(i+1)} \ldots \kappa_j^{(j)} \left| \begin{array}{cccccc}
\frac{\cos\phi_{i,i+1}^{(i+1)}}{\kappa_i^{(i+1)}} & \frac{\cos\phi_{i+1,i+1}^{(i+1)}}{\kappa_{i+1}^{(i+1)}} & 0 & 0 & \ldots & 0 \\
\frac{\cos\phi_{i,i+2}^{(i+2)}}{\kappa_i^{(i+2)}} & \frac{\cos\phi_{i+1,i+2}^{(i+2)}}{\kappa_{i+1}^{(i+2)}} & \frac{\cos\phi_{i+2,i+2}^{(i+2)}}{\kappa_{i+2}^{(i+2)}} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\cos\phi_{i,j-1}^{(j-1)}}{\kappa_i^{(j-1)}} & \frac{\cos\phi_{i+1,j-1}^{(j-1)}}{\kappa_{i+1}^{(j-1)}} & \ldots & \ldots & \ldots & \frac{\cos\phi_{j-1,j-1}^{(j-1)}}{\kappa_{j-1}^{(j-1)}} \\
\frac{\cos\phi_{ij}^{(j)}}{\kappa_i^{(j)}} & \frac{\cos\phi_{i+1,j}^{(j)}}{\kappa_{i+1}^{(j)}} & \ldots & \ldots & \ldots & \frac{\cos\phi_{j-1,j}^{(j)}}{\kappa_{j-1}^{(j)}}
\end{array} \right| \quad (4\cdot2)$$

The determinant is to be calculated by remembering that

$$\frac{\cos\phi_{ij}^{(m-1)}}{\kappa_i^{m-1}} = \frac{\kappa_j^{(m-1)}}{\kappa_j^{(m)}} \left[\frac{\cos\phi_{ij}^{(m)} - \cos\phi_{im}^{(m)} \cos\phi_{jm}^{(m)}}{\kappa_i^{(m)}} \right] \quad \cdots \quad (4\cdot3)$$

which follows directly from (3·4).

Now take for simplicity $i = 2$, $j = 5$, (the method, of course, is perfectly general). We find in this case from (4·2) by repeated application of (4·3) and of the property of determinants that the coefficient of l_2 in the equation giving l's in terms of l's is given by

$$= \kappa_2^{(2)}.\kappa_3^{(3)}.\kappa_4^{(4)}.\kappa_5^{(5)}. \times \begin{vmatrix} \dfrac{\cos\varphi_{23}^{(3)}}{\kappa_2^{(3)}} & \dfrac{\cos\varphi_{33}^{(3)}}{\kappa_3^{(3)}} & 0 \\ \dfrac{\cos\varphi_{24}^{(4)}}{\kappa_2^{(4)}} & \dfrac{\cos\varphi_{34}^{(4)}}{\kappa_3^{(4)}} & \dfrac{\cos\varphi_{44}^{(4)}}{\kappa_4^{(4)}} \\ \dfrac{\cos\varphi_{25}^{(5)}}{\kappa_2^{(5)}} & \dfrac{\cos\varphi_{35}^{(5)}}{\kappa_3^{(5)}} & \dfrac{\cos\varphi_{45}^{(5)}}{\kappa_4^{(5)}} \end{vmatrix}$$

$$= \kappa_2^{(2)}.\kappa_3^{(3)}.\kappa_4^{(4)}.\kappa_5^{(5)} . \frac{\kappa_3^{(3)}}{\kappa_3^{(4)}} \times \begin{vmatrix} \dfrac{\cos\varphi_{23}^{(4)}}{\kappa_2^{(4)}} & \dfrac{\cos\varphi_{33}^{(4)}}{\kappa_3^{(4)}} & \dfrac{\cos\varphi_{43}^{(4)}}{\kappa_4^{(4)}} \\ \dfrac{\cos\varphi_{24}^{(4)}}{\kappa_2^{(4)}} & \dfrac{\cos\varphi_{34}^{(4)}}{\kappa_3^{(4)}} & \dfrac{\cos\varphi_{44}^{(4)}}{\kappa_4^{(4)}} \\ \dfrac{\cos\varphi_{25}^{(5)}}{\kappa_2^{(5)}} & \dfrac{\cos\varphi_{35}^{(5)}}{\kappa_3^{(5)}} & \dfrac{\cos\varphi_{45}^{(5)}}{\kappa_4^{(5)}} \end{vmatrix}$$

$$= \frac{\kappa_2^{(2)}.\kappa_3^{(3)}.\kappa_4^{(4)}.\kappa_5^{(5)}.\kappa_3^{(3)}.\kappa_4^{(4)}.\kappa_3^{(4)}}{\kappa_3^{(4)}.\kappa_4^{(5)}.\kappa_3^{(5)}} \times \begin{vmatrix} \dfrac{\cos\varphi_{23}^{(5)}}{\kappa_2^{(5)}} & \dfrac{\cos\varphi_{33}^{(5)}}{\kappa_3^{(5)}} & \dfrac{\cos\varphi_{43}^{(5)}}{\kappa_4^{(5)}} \\ \dfrac{\cos\varphi_{24}^{(5)}}{\kappa_2^{(5)}} & \dfrac{\cos\varphi_{34}^{(5)}}{\kappa_3^{(5)}} & \dfrac{\cos\varphi_{44}^{(5)}}{\kappa_4^{(5)}} \\ \dfrac{\cos\varphi_{25}^{(5)}}{\kappa_2^{(5)}} & \dfrac{\cos\varphi_{35}^{(5)}}{\kappa_3^{(5)}} & \dfrac{\cos\varphi_{45}^{(5)}}{\kappa_4^{(5)}} \end{vmatrix}$$

$$= \kappa_2^{(2)}.(\kappa_3^{(3)})^2.(\kappa_4^{(4)})^2.(\kappa_5^{(5)})^2 . \times \begin{vmatrix} \dfrac{\cos\varphi_{23}^{(5)}}{\kappa_2^{(5)}\,\kappa_3^{(5)}} & \dfrac{\cos\varphi_{33}^{(5)}}{\kappa_3^{(5)}\,\kappa_3^{(5)}} & \dfrac{\cos\varphi_{43}^{(5)}}{\kappa_4^{(5)}\,\kappa_3^{(5)}} \\ \dfrac{\cos\varphi_{24}^{(5)}}{\kappa_2^{(5)}\,\kappa_4^{(5)}} & \dfrac{\cos\varphi_{34}^{(5)}}{\kappa_3^{(5)}\,\kappa_4^{(5)}} & \dfrac{\cos\varphi_{44}^{(5)}}{\kappa_4^{(5)}\,\kappa_4^{(5)}} \\ \dfrac{\cos\varphi_{25}^{(5)}}{\kappa_2^{(5)}\,\kappa_5^{(5)}} & \dfrac{\cos\varphi_{35}^{(5)}}{\kappa_3^{(5)}\,\kappa_5^{(5)}} & \dfrac{\cos\varphi_{45}^{(5)}}{\kappa_4^{(5)}\,\kappa_5^{(5)}} \end{vmatrix}$$

From (1·8) and the theory of determinants the above determinant is easily seen to be equal to the leading minor of the second order in the determinant $|\alpha_{ij}^{(5)}|$ divided by the determinant itself.

Therefore the above expression

$$= \kappa_2^{(2)}(\kappa_3^{(3)})^2.(\kappa_4^{(4)})^2.(\kappa_5^{(5)})^2 \times \frac{\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{15} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{51} & \alpha_{52} & \cdots & \alpha_{55} \end{vmatrix}}$$

where $||\alpha_{ij}||$ is the dispersion matrix for the population.

Remembering that

$$(s_r^{(r)})^2 = \frac{v(r_1, r_2, \ldots\ldots, r_r)}{v(r_1, r_2, \ldots\ldots, r_{r-1})} \qquad \ldots \quad (4\cdot4)$$

we now easily see that the above expression

$$= \frac{\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}}{\alpha_{11}^{\frac{1}{2}} \cdot \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}^{\frac{1}{2}}} \qquad \ldots \quad (4\cdot5)$$

Hence the general expression (4·1) can be similarly shown to reduce to

$$\frac{\begin{vmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1,j-1} & \vdots & \alpha_{1j} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2,j-1} & \vdots & \alpha_{2j} \\ \ldots & \ldots & \ldots & \ldots & & \ldots \\ \alpha_{j-1,1} & \alpha_{j-1,2} & \ldots & \alpha_{j-1,j-1} & \vdots & \alpha_{j-1,j} \\ \alpha_{i1} & \alpha_{i2} & \ldots & \alpha_{i,j-1} & \vdots & \alpha_{ij} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{1,j-1} \\ \ldots & \ldots \\ \alpha_{j-1,1} & \alpha_{j-1,j-1} \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} \alpha_{11} & \ldots & \alpha_{1i} \\ \ldots & \ldots & \ldots \\ \alpha_{i1} & \ldots & \alpha_{ii} \end{vmatrix}^{\frac{1}{2}}} \qquad \ldots \quad (4.6)$$

which again $= r_{ij}$ ... (4·7)

from the formula (2·1) previously deduced. Therefore the right hand side of (4·7) is equal to the right hand side of equation (12·5), p. 21 of the *Sankhyā* paper.

Therefore,

$$l_j = \Sigma^j_{i=1} \, r_{ij} \, l_i \qquad \ldots \quad (4\cdot8)$$

regard being paid to the convention of signs.

October, 1937.
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