Distance matrix and Laplacian of a tree with attached graphs

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Abstract

A tree with attached graphs is a tree, together with graphs defined on its partite sets. We introduce the notion of incidence matrix, Laplacian and distance matrix for a tree with attached graphs. Formulas are obtained for the minors of the incidence matrix and the Laplacian, and for the inverse and the determinant of the distance matrix. The case when the attached graphs themselves are trees is studied more closely. Several known results, including the Matrix Tree theorem, are special cases when the tree is a star. The case when the attached graphs are paths is also of interest since it is related to the transportation problem.

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1. Introduction

Minors of matrices associated with a graph has been an area of considerable interest, starting with the celebrated Matrix Tree theorem of Kirchhoff which asserts that any cofactor of the Laplacian matrix equals the number of spanning trees in the graph. Several papers have been devoted to the theme of Matrix Tree type theorems, see [19,10,3,4] for more information and further references.

Another matrix whose determinant and cofactors have been explicitly calculated is the distance matrix of a tree. An early, remarkable result for the distance matrix D of a tree on n vertices, due to Graham and Pollack [13], asserts that the determinant of D equals $(-1)^{n-1}(n-1)2^{n-2}$, and is thus a function of only the number of vertices. In subsequent work, Graham and Lovász [14] obtained a formula for D^{-1}, among other results.

The distance between two vertices in a graph is traditionally defined as the length of a shortest path between the two vertices. In contrast to this notion, the concept of *resistance distance*, introduced by Klein and Randić [17] arises naturally from several different considerations and is more amenable to mathematical treatment. The concept has also been of interest in the chemical literature, and in particular, an analog of the classical Wiener index based on the resistance distance has been proposed. We refer to [1,8,11,16,25,26] for more information on the resistance distance and for additional references. For a graph G, the matrix whose (i, j)-entry is the resistance distance between vertices i and j is called the resistance distance matrix (or simply the resistance matrix) of the graph. For a tree, the resistance distance coincides with the classical distance. Expressions for the determinant and the inverse of the resistance matrix are given in [26,2].

In this paper we introduce the concept of a tree with attached graphs, which is simply a tree with graphs defined on its partite sets. We introduce the notion of incidence matrix, Laplacian and distance matrix for a tree with attached graphs. Formulas are obtained for the minors of the incidence matrix and the Laplacian, and for the inverse and the determinant of the distance matrix. When the tree is a star, we recover some known results concerning the determinant and the inverse of the resistance distance matrix of a graph. On the other hand, when the attached graphs are paths, we obtain results for the distance matrix of the tree associated with a basic feasible solution of a transportation problem. We refer to Section 7 for details.

It is possible to state and prove the results of the present paper in the context of weighted graphs. However we restrict ourselves to unweighted graphs for the sake of clarity of presentation. For an illustration of results in the weighted case, see [5].

2. Preliminaries

We consider simple graphs, that is, graphs which have no loops or parallel edges. The vertex set and the edge set of the graph G are denoted by $V(G)$ and $E(G)$ respectively. By a directed graph we mean a graph in which each edge is assigned an orientation. It must be remarked that even when we consider a directed graph, we focus on the underlying undirected graph when defining paths, cycles, spanning trees, connectedness, etc. Thus by a “connected directed graph” we mean a directed graph whose underlying undirected graph is connected. The transpose of a matrix A

is denoted A'. If G is a directed graph with n vertices and m edges, then its incidence matrix Q is the $n \times m$ matrix defined as follows. The rows and the columns of Q are indexed by $V(G)$ and $E(G)$ respectively. If $i \in V(G)$ and $j \in E(G)$, then the (i, j)-entry of Q is 0 if vertex i and edge j are not incident and otherwise it is 1 or -1 according as j originates or terminates at i respectively. The Laplacian matrix L of G is defined as $L = QQ'$. The Laplacian does not depend on the orientation and thus is defined for an undirected graph. We assume familiarity with basic graph theory and with elementary properties of the incidence matrix and the Laplacian, see, for example, [8,19,24].

We now introduce some more definitions. If A is an $n \times m$ matrix, then an $m \times n$ matrix H is called a generalized inverse (or a g-inverse) of A if $AHA = A$. The Moore–Penrose inverse of A, is an $m \times n$ matrix H satisfying the equations $AHA = A$, $HAH = H$, $(AH)' = AH$ and $(HA)' = HA$. It is well-known that the Moore–Penrose inverse exists and is unique. We denote the Moore–Penrose inverse of A by A^+. For background material on generalized inverses, see [7,9].

If A is an $n \times n$ matrix, then for $i = 1, \dots, n$, $A(i|i)$ will denote the submatrix obtained by deleting row i and column i. Similarly for $i, j = 1, \dots, n$; $i \neq j$, $A(i, j|i, j)$ is the submatrix obtained by deleting rows i, j and columns i, j.

Let G be a graph with n vertices and let L be its Laplacian. For $i, j \in \{1, \dots, n\}$, the resistance distance between vertices i, j is defined as

$$d(i, j) = d_{ij} = \frac{\det L(i, j|i, j)}{\det L(i|i)}. \tag{1}$$

The matrix $D = [d_{ij}]$ is called the resistance distance matrix (or the resistance matrix) of G. For several equivalent definitions of the resistance distance see [1,17]. We denote the column vector of all ones by $\mathbf{1}$ and a matrix of all ones by J. The size of these is clear from the context.

Lemma 1. *Let A be a symmetric $n \times n$ matrix with zero row and column sums and of* rank $n - 1$. *If H is a g-inverse of A, then for any $i \neq j$,*

$$\frac{\det A(i, j|i, j)}{\det A(i|i)} = h_{ii} + h_{jj} - h_{ij} - h_{ji}. \tag{2}$$

Proof. Fix $i, j \in \{1, \dots, n\}$, $i \neq j$. Let x be the column vector with $x_i = 1$, $x_j = -1$ and with its remaining coordinates zero. Then x is orthogonal to $\mathbf{1}$. Since A has zero row and column sums, $\mathbf{1}$ is in the null-space of A. Also, since the rank of A is $n - 1$, any vector orthogonal to $\mathbf{1}$ is in the column space of A and thus $x = Ay$ for some y. If H is a g-inverse of A, then

$$h_{ii} + h_{jj} - h_{ij} - h_{ji} = x'Hx = y'AHAy = y'Ay$$

and hence $h_{ii} + h_{jj} - h_{ij} - h_{ji}$ is invariant with respect to the choice of g-inverse.

We now construct a specific g-inverse of A. Let $\widehat{H}$ be the $n \times n$ matrix defined as follows. Set $\widehat{H}(i|i) = A(i|i)^{-1}$ and set the entries in the ith row and column of $\widehat{H}$ to be zero. It is easily verified that $\widehat{H}$ is a g-inverse of A. In fact, the construction that we have described is a standard text-book method of computing a g-inverse, see, for example, [20, Chapter 1]. Note that since the ith row and column of $\widehat{H}$ are zero, $\hat{h}_{ii} = \hat{h}_{ij} = \hat{h}_{ji} = 0$, while, by the cofactor formula for the inverse of a matrix, $\hat{h}_{jj} = \frac{\det A(i,j|i,j)}{\det A(i|i)}$. Thus $\hat{h}_{ii} + \hat{h}_{jj} - \hat{h}_{ij} - \hat{h}_{ji} = \frac{\det A(i,j|i,j)}{\det A(i|i)}$. It follows that for any g-inverse H of A, (2) holds and the proof is complete. □

The hypotheses on A in Lemma 1 imply that the cofactors of A are all equal. This can be seen as follows. In $A(1|1)$, add all columns to the first column. This operation leaves the determinant unchanged. Since the row sums of A are zero, the resulting matrix is the same as $A(1|2)$, in which the first column is replaced by its negative. Thus we conclude $\det A(1|1) = -\det A(1|2)$. By a similar argument we can show that $(-1)^{i+j} \det A(i|j) = (-1)^{i+k} \det A(i|k)$ for any i, j, k. Again, since the column sums of A are zero as well, we can show that $(-1)^{i+j} \det A(i|j) = (-1)^{k+j} \det A(k|j)$ for any i, j, k and the claim is proved. Since the rank of A is $n-1$, the cofactors of A are in fact equal and nonzero. Thus the determinant $\det A(i|i)$ occurring in the statement of Lemma 1 does not depend on i.

The following notation will be used in the rest of the paper. Let T be a tree with $|V(T)| = n+1$. Let X_1 and X_2 be partite sets of T with $|X_1| = p_1, |X_2| = p_2, p_1 + p_2 = n+1$. Let G_i be a connected, directed graph with $V(G_i) = X_i$ and $|E(G_i)| = m_i, i = 1, 2$. We assume that the edges of T are directed as well and for convenience we take them to be oriented from X_1 to X_2. We think of the graph $T \cup G_1 \cup G_2$ as the tree T with attached graphs G_1, G_2.

Let A be the $(n+1) \times n$ incidence matrix of T and let B_i be the $p_i \times m_i$ incidence matrix of $G_i, i = 1, 2$. We set

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Note that if $p_2 = 1$, then $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$.

In the following discussion we assume certain well-known properties of the incidence matrix of a tree. The columns of A are linearly independent. The column sums of A are zero and therefore $\mathbb{1}$ is orthogonal to any vector in the column space of A. Since the dimension of the column space of A is n, it is evident that the columns of A span the space of vectors in R^{n+1} that are orthogonal to $\mathbb{1}$. Thus there is a unique $n \times (m_1 + m_2)$ matrix Q satisfying $AQ = B$. We call Q, the incidence matrix and $L = QQ'$, the Laplacian of $T \cup G_1 \cup G_2$, viewed as a tree with attached graphs.

If any row of A is deleted, then the resulting matrix is known as the *reduced incidence matrix* of T and its determinant is ± 1. The matrix A is totally unimodular, i.e., all its minors are either 0 or ± 1. It follows from the well-known properties of linear systems with a totally unimodular coefficient matrix (see [23, Chapter 19]) that the entries of Q are in $\{0, 1, -1\}$.

There is a graphical interpretation of Q which is instructive. The rows and the columns of Q are indexed by $E(T)$ and $E(G_1) \cup E(G_2)$ respectively. Let e be an edge of $G_1 \cup G_2$. Then $T \cup \{e\}$ contains a unique circuit. Consider the incidence vector of the circuit. It is a vector indexed by $E(T)$. It contains a zero corresponding to any edge of T that does not feature in the circuit and otherwise the entries are -1 or 1 depending on whether the edge agrees with the orientation of e or otherwise respectively, as we traverse the circuit. The column of Q corresponding to e is given precisely by this incidence vector. In particular, since the edges of T are all oriented from X_1 to X_2, the sum of the entries in any column of Q is zero.

Since G_i is connected, B_i has full column rank, $i = 1, 2$ and hence B has full column rank as well; thus rank $B = n - 1$. Since $AQ = B$ and since the rank of a product cannot exceed that of either factor, we conclude that rank $Q \geqslant n - 1$. However Q has $n - 1$ columns and hence rank $Q = n - 1$.

We also note some elementary properties of L. Clearly L is $n \times n$ and symmetric. Since $L = QQ'$, it is positive semidefinite. The rank of L equals that of Q, which is $n - 1$. The entries of L are integers, but in contrast to the Laplacian of a graph, it does not necessarily have nonpositive off-diagonal entries.

We remark that $T \cup G_1 \cup G_2$, being a graph in its own right, has an incidence matrix and a Laplacian matrix defined in the usual way. However in the present paper we will not be concerned with these matrices as far as $T \cup G_1 \cup G_2$ is concerned. We view $T \cup G_1 \cup G_2$ as a tree with attached graphs and its incidence matrix Q and Laplcian matrix L will be as defined in the preceding discussion.

3. Minors of Q and L

We continue to work with the notation introduced in the previous section, the salient features of which are reproduced here for convenience. Thus let T be a tree with $|V(T)| = n + 1$. Let X_1 and X_2 be partite sets of T with $|X_1| = p_1$, $|X_2| = p_2$, $p_1 + p_2 = n + 1$. Let G_i be a connected, directed graph with $V(G_i) = X_i$ and $|E(G_i)| = m_i$, $i = 1, 2$. We assume that the edges of T are directed as well and for convenience we take them to be oriented from X_1 to X_2. We think of the graph $T \cup G_1 \cup G_2$ as the tree T with attached graphs G_1, G_2. Let Q and L be the incidence matrix and the Laplacian matrix of $T \cup G_1 \cup G_2$ as defined in Section 2. In the next result we describe the minors of the incidence matrix.

Lemma 2. *Consider the submatrix Q_1 of Q formed by the rows indexed by $F \subset E(T)$ and the columns indexed by $H \subset E(G_1 \cup G_2)$, $|F| = |H| = r$. Then Q_1 is nonsingular if and only if the graph induced by $(E(T) \setminus F) \cup H$ is a tree, in which case*, $\det Q = \pm 1$.

Proof. We assume, without loss of generality, that

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},$$

where Q_1 is $r \times r$. Then

$$A\begin{bmatrix} Q_1 & 0 \\ Q_3 & I_{n-r} \end{bmatrix} = \begin{bmatrix} B_1 & A_2 \end{bmatrix}, \tag{3}$$

where B_1 denotes the columns of B indexed by H and A_2 denotes the columns of A indexed by $E(T) \setminus F$. Let $A(n+1, .)$ and $[B_1, A_2](n+1, .)$ denote the matrices obtained by deleting row $n+1$ of A and of $[B_1, A_2]$ respectively. It follows from (3) that

$$A(n+1, .)\begin{bmatrix} Q_1 & 0 \\ Q_3 & I_{n-r} \end{bmatrix} = \begin{bmatrix} B_1 & A_2 \end{bmatrix}(n+1, .). \tag{4}$$

The three matrices in (4) are all $n \times n$. Taking determinants,

$$\det A(n+1, .) \det Q_1 = \det\begin{bmatrix} B_1 & A_2 \end{bmatrix}(n+1, .). \tag{5}$$

Clearly, as remarked in Section 2, $A(n+1, .)$, being the reduced incidence matrix, is nonsingular and furthermore, $\det A(n+1, .) = \pm 1$. Thus, in view of (5), Q_1 is nonsingular if and only if $[B_1, A_2](n+1, .)$ is nonsingular. Note that $[B_1, A_2]$ is the incidence matrix of the graph with vertex set $V(T)$ and edge set $(E(T) \setminus F) \cup H$. It follows from well-known properties of incidence matrices (see, for example, [18, Chapter 6]) that $[B_1, A_2]$ has full column rank if and only if the corresponding graph is a tree, in which case any minor of $[B_1, A_2]$ of order $n-1$ is ± 1. Thus Q_1 is nonsingular if and only if the graph induced by $(E(T) \setminus F) \cup H$ is a tree, in which case, $\det[B_1, A_2](n+1, .) = \pm 1$, and then, from (5), $\det Q = \pm 1$. That completes the proof. □

According to Lemma 2, Q is totally unimodular. This also follows from the observation that Q is a network matrix in the sense of Tutte, see [23, p. 276].

Theorem 3. *Consider the principal submatrix $\hat{L}$ of L formed by the rows and the columns indexed by $F \subset E(T)$. Then $\det \hat{L}$ equals the number of spanning trees of $T \cup G_1 \cup G_2$ whose edge set, intersected with $E(T)$, equals $E(T) \setminus F$.*

Proof. Since $L = QQ'$, $\hat{L}$ is the product of the submatrix of Q formed by its rows indexed by F, and the transpose of the same submatrix. The result follows by a standard application of the Cauchy–Binet formula, using Lemma 2. The argument parallels the usual proof of the Matrix Tree theorem using Cauchy–Binet formula (see, for example, [24]) and is omitted. □

Corollary 4. (i) *Any cofactor of L equals $t_1 t_2$, where t_i denotes the number of spanning trees in G_i, $i = 1, 2$.*

(ii) *Let $e = ik$ and $f = j\ell$ be edges of G_1 and G_2 respectively, where $i, j \in X_1$ and $k, \ell \in X_2$. Let α_{ij} be the number of spanning forests of G_1 with two components, one containing i and the other containing j. Similarly, let $\beta_{k\ell}$ be the number of spanning forests of G_2 with two components, one containing k and the other containing ℓ. Then* $\det L(e, f|e, f) = t_1\beta_{k\ell} + t_2\alpha_{ij}$.

Proof. It may be remarked that the notation $e = ik$ means that i and k are the end-vertices of the edge e, and furthermore, e is oriented from i to k. Assertions (i) and (ii) follow by applying Theorem 3 to the case of principal minors of L of order $n-1$ and $n-2$ respectively. □

4. A distance on $T \cup G_1 \cup G_2$

We continue to work under the setup introduced in Section 2 and restated at the beginning of Section 3.

Theorem 5. *Let $d_i(\cdot,\cdot)$ denote the (resistance) distance in G_i, $i = 1, 2$. Let $e = ik$ and $f = j\ell$ be edges of T where $i, j \in X_1$ and $k, \ell \in X_2$. Then*

$$\frac{\det L(e, f|e, f)}{\det L(e|e)} = d_1(i, j) + d_2(k, \ell). \tag{6}$$

Proof. Let L_1 and L_2 be the Laplacians of G_1 and G_2 respectively. By definition (1),

$$d_1(i, j) = \frac{\det L_1(i, j|i, j)}{\det L_1(i|i)}, \quad d_2(k, \ell) = \frac{\det L_2(k, \ell|k, \ell)}{\det L_2(k|k)}. \tag{7}$$

Let t_1, t_2, α_{ij} and $\beta_{k\ell}$ be defined as in Corollary 4. It is well-known [12,10,3] that $\det L_1(i|i) = t_1$, $\det L_2(k|k) = t_2$, $\det L_1(i, j|i, j) = \alpha_{ij}$ and $\det L_2(k, \ell|k, \ell) = \beta_{k\ell}$. Therefore, using (7)

$$\begin{aligned} d_1(i, j) + d_2(k, \ell) &= \frac{\alpha_{ij}}{t_1} + \frac{\beta_{k\ell}}{t_2} \\ &= \frac{t_2\alpha_{ij} + t_1\beta_{k\ell}}{t_1 t_2} \\ &= \frac{\det L(e, f|e, f)}{\det L(e|e)}, \end{aligned}$$

where the last equality follows by Corollary 4. Thus (6) is proved. □

We introduce a distance function on the edges of $E(T)$ as follows. If $e, f \in E(T)$, then set

$$d(e, f) = \frac{\det L(e, f|e, f)}{\det L(e|e)}. \tag{8}$$

As in Theorem 5, let $d_i(\cdot,\cdot)$ denote the distance in G_i, $i = 1, 2$. By Theorem 5, if $e = ik$, $f = j\ell$, where $i, j \in X_1$ and $k, \ell \in X_2$, then $d(e, f) = d_1(i, j) + d_2(k, \ell)$.

We define the $n \times n$ matrix, with its rows and columns indexed by $E(T)$, given by $D = [d(e, f)]$ to be the (edge) distance matrix of $T \cup G_1 \cup G_2$.

If $L^+ = [\ell_{ij}^+]$ is the Moore–Penrose inverse of L, then L^+ is symmetric and by Lemma 1,

$$d(e, f) = \ell_{ee}^+ + \ell_{ff}^+ - 2\ell_{ef}^+. \tag{9}$$

As noted in Section 2, L has rank $n-1$, and since $\mathbf{1}$ is in the null-space of L, $L + \frac{1}{n}J$ is nonsingular. Let $X = (L + \frac{1}{n}J)^{-1}$. Then $L^+ = X - \frac{1}{n}J$ and it follows from (9) that

$$d(e, f) = x_{ee} + x_{ff} - 2x_{ef}. \tag{10}$$

Note that X is a positive definite matrix and thus (10) implies that D is a classical distance matrix in the sense of Schoenberg (see [6, Chapter 4]) as well. In particular, D is nonsingular and has exactly one positive eigenvalue.

Our next objective is to obtain a formula for the inverse of D and then derive an expression for $\det D$.

5. Inverse and determinant of D

Let $\widetilde{X} = \operatorname{diag}(x_{11}, \ldots, x_{nn})$, the diagonal matrix with $x_{11}, \ldots, x_{nn}$ along the diagonal.

Theorem 6. *Let* $\tau = L\widetilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}$. *Then*

$$D^{-1} = -\frac{1}{2}L + \frac{1}{\tau' D\tau}\tau\tau'. \tag{11}$$

Proof. By (10) we can write

$$D = \widetilde{X}J + J\widetilde{X} - 2X. \tag{12}$$

Then

$$LD = L\widetilde{X}J - 2LX, \tag{13}$$

since $L\mathbf{1} = 0$. Now $(L + \frac{1}{n}J)X = I$ and hence $LX = I - \frac{1}{n}JX$. Since $L + \frac{1}{n}J$ has all row sums equal to 1, X has all row sums equal to 1 as well. It follows from (13) that

$$LD = L\widetilde{X}J - 2\left(I - \frac{1}{n}JX\right) = L\widetilde{X}J - 2I + \frac{2}{n}J.$$

Thus

$$LD + 2I = L\widetilde{X}J + \frac{2}{n}J = \tau\mathbf{1}'. \tag{14}$$

Note that $\mathbf{1}'\tau = \mathbf{1}'(L\widetilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}) = 2$ and hence from (14), $(LD + 2I)\tau = 2\tau$. Therefore $LD\tau = 0$. As remarked earlier, D is nonsingular, and since $\tau \neq 0$, then $D\tau \neq 0$

as well. As the null-space of L is one-dimensional, it follows that $D\tau = \alpha\mathbf{1}$ for some $\alpha \neq 0$. Then $\tau' D\tau = \alpha\tau'\mathbf{1} = 2\alpha$ and hence $\alpha = \frac{\tau' D\tau}{2}$.

Now

$$\begin{aligned}\left(-\frac{1}{2}L + \frac{1}{\tau' D\tau}\tau\tau'\right)D &= -\frac{1}{2}LD + \frac{1}{\tau' D\tau}\tau\tau' D\\ &= -\frac{1}{2}LD + \frac{1}{\tau' D\tau}\tau\frac{\tau' D\tau}{2}\mathbf{1}'\\ &= I,\end{aligned}$$

where the last equality follows by (14). That completes the proof. □

We remark that, as noted in (14), $\tau\mathbf{1}' = LD + 2I$ and hence $\tau = \mathrm{diag}(LD + 2I)\mathbf{1}$ is another expression for τ.

Theorem 7. $\det D = (-1)^{n-1}2^{n-3}\frac{\tau' D\tau}{t_1 t_2}$, *where t_i denotes the number of spanning trees in G_i, $i = 1, 2$.*

Proof. By (11), $D^{-1} = -\frac{1}{2}L + \frac{1}{\tau' D\tau}\tau\tau'$. Also, by Corollary 4, any cofactor of L equals $t_1 t_2$. It follows, using the multilinearity of the determinant, that

$$\begin{aligned}\det D^{-1} &= \left(-\frac{1}{2}\right)^{n-1}\frac{t_1 t_2}{\tau' D\tau}\sum_i\sum_j \tau_i\tau_j\\ &= \left(-\frac{1}{2}\right)^{n-1}\frac{t_1 t_2}{\tau' D\tau}\left(\sum_i \tau_i\right)^2.\end{aligned}$$

Now the result follows since $\sum_i \tau_i = 2$. □

6. The case when G_1 and G_2 are trees

We now consider the case when G_1 and G_2 are trees. Thus the present setup can be summarized as follows. Let T be a tree with $|V(T)| = n + 1$, let X_1 and X_2 be partite sets of T, $|X_i| = p_i$, $i = 1, 2$; $p_1 + p_2 = n + 1$ and let T_i be a directed tree on X_i, $i = 1, 2$. We assume that the edges of T are directed from X_1 to X_2. Let matrices A, B and Q be defined as before. Recall that the orders of these matrices are $(n + 1) \times n$, $(n + 1) \times (n - 1)$ and $n \times (n - 1)$ respectively. The rows and columns of Q are indexed by $E(T)$ and $E(T_1 \cup T_2)$ respectively.

The resistance distance reduces to the classical distance (length of a shortest path) when the graph is a tree. Thus $D_1 = [d_1(i, j)]$ and $D_2 = [d_2(i, j)]$ will now be the usual (classical) distance matrices of T_1 and T_2 respectively. Let $D = [d_{ij}]$ be the edge distance matrix of $T \cup T_1 \cup T_2$, as defined in Section 4. The rows and columns of D are indexed by $E(T)$. For $i, j \in E(T)$, d_{ij} is simply the sum of the distances between the endvertices of i, j in X_1 and the endvertices of i, j in X_2.

Given the edge i and the vertex j of a directed tree, we say that i is directed towards j if the distance of j from the head of i is less than its distance from the tail of i. Otherwise we say that i is directed away from j.

In the next result we describe the structure of $Q'D$. Note that the rows and columns of $Q'D$ are indexed by $E(T_1 \cup T_2)$ and $E(T)$ respectively.

Theorem 8. *Let $i \in E(T_1 \cup T_2)$, $j \in E(T)$. Then the (i, j)-entry of $Q'D$ is 1 if, either $i \in E(T_1)$ and i is directed towards the endvertex of j in X_1, or, $i \in E(T_2)$ and i is directed away from the endvertex of j in X_2. Otherwise the (i, j)-entry of $Q'D$ is -1.*

Proof. Let $i \in E(T_1 \cup T_2)$, $j \in E(T)$. First suppose $i \in E(T_1)$, $i = uv$, $j = xy$, where $u, v, x \in X_1$, $y \in X_2$. Let the unique (u, v)-path in T be composed of the edges $w_1, \ldots, w_t$ in $E(T)$. From the definition of Q it follows that the (i, j)-entry of $Q'D$ is given by

$$\sum_{k=1}^{n} q_{ki} d_{kj} = d_{w_1 j} - d_{w_2 j} + d_{w_3 j} - \cdots - d_{w_t j},$$

which equals $d_1(u, x) - d_1(v, x)$ due to pairwise cancellations. Now observe that $d_1(u, x) - d_1(v, x)$ equals 1 if i is directed towards x and -1 if i is directed away from x. Thus the result is proved in this case. The case when $i \in E(T_2)$ is treated similarly. □

Theorem 9. $DLD + 2D = (n - 1)J$.

Proof. The rows and columns of DLD are indexed by $E(T)$. Let $i, j \in E(T)$, $i = uv$, $j = wz$, $u, w \in X_1$, $v, z \in X_2$. Since $DLD = DQQ'D = (Q'D)'Q'D$, the (i, j)-entry of DLD is given by the inner product of the ith and the jth columns of $Q'D$.

Let γ_1 and γ_2 denote the (u, w)-path in T_1 and the (v, z)-path in T_2 respectively, and let $\ell(\gamma_1)$ and $\ell(\gamma_2)$ be their respective lengths. We will make use of the structure of $Q'D$ developed in Theorem 8. Recall that the entries of $Q'D$ are ± 1. If $k \in E(T_1)$, then by Theorem 8, the (i, k)-entry of $(Q'D)'$ and the (k, j)-entry of $Q'D$ both have the same sign unless k is on γ_1. Similarly, if $k \in E(T_2)$, then the (i, k)-entry of $(Q'D)'$ and the (k, j)-entry of $Q'D$ both have the same sign unless k is on γ_2. Therefore the inner product of the ith and the jth columns of $Q'D$ is $n - 1 - 2(\ell(\gamma_1) + \ell(\gamma_2))$. Observe that $d(i, j) = \ell(\gamma_1) + \ell(\gamma_2)$. Thus the (i, j)-entry of $DLD + 2D$ is $n - 1$ for any i, j and the proof is complete. □

Corollary 10. *Let* τ *be as defined in Theorem 6. Then*

(i) $D\tau = (n-1)\mathbf{1}$.
(ii) $\tau' D\tau = 2(n-1)$.

Proof. By Theorem 9,

$$D(LD + 2I) = (n-1)J, \tag{15}$$

while by (14),

$$LD + 2I = \tau\mathbf{1}'. \tag{16}$$

The proof of (i) is complete in view of (15) and (16). The second part follows from (i) since $\mathbf{1}'\tau = 2$. □

The expressions for the inverse and the determinant of D, obtained in Theorems 6 and 7, can be made more precise in the present situation, when the attached graphs are trees, using Corollary 10. These are given in the next result. The proof is easy and is omitted.

Theorem 11

(i) $D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau'$.
(ii) $\det D = (-1)^{n-1}2^{n-2}(n-1)$.

7. Special cases

Two special cases of the general setup considered in Sections 2–5 are of interest. The first is the case when T is a star. Thus suppose $|X_2| = 1$. Then there is a one-to-one correspondence between the edges of T and $X_1 = V(G_1)$. Thus the edge distance matrix of $T \cup G_1 \cup G_2$ can indeed be regarded as the resistance matrix of G_1. Similarly, the Laplacian of $T \cup G_1 \cup G_2$ coincides with the (classical) Laplacian of G_1. Theorem 3 and Corollary 4 are then seen as the classical Matrix Tree theorem and the Matrix Tree theorem for principal minors of the Laplacian, see [10,3,4,12,19]. For the special case of T being a star, Theorem 6 has been proved in [2], and Theorem 7 in [26]. Both these results in turn, are extensions of earlier work of Graham and Pollack [13] and Graham and Lovász [14] on the determinant and the inverse of the distance matrix of a tree.

The second special case of interest arises when G_1 and G_2 are paths. We first introduce some notation. Consider a transportation problem with a set of sources $\mathscr{S}$, with $|\mathscr{S}| = p$, and a set of destinations, $\mathscr{D}$, with $|\mathscr{D}| = q$. To any feasible solution of the problem we may associate a bipartite graph. The partite sets of the graph are $\mathscr{S}$ and $\mathscr{D}$. We assume that the elements of $\mathscr{S}$ and $\mathscr{D}$ are numbered $1, \ldots, p$

Table 1
A transportation tableau

1		2	
		3	4
	5	6	

and $1, \ldots, q$ respectively. If $i \in \mathscr{S}$ and $j \in \mathscr{D}$, then there is an edge from i to j if and only if a positive quantity is shipped from source i to destination j. It is well-known (see, for example, [15]) that such a bipartite graph is a tree if and only if the corresponding feasible solution is a basic feasible solution. From now onwards we assume that T is a tree corresponding to a basic feasible solution with $\mathscr{S}$ and $\mathscr{D}$ as its partite sets. There is a natural distance on the edges of T. If $e = ik$ and $f = j\ell$ are edges of T with $i, j \in \mathscr{S}$ and $k, \ell \in \mathscr{D}$, then the distance between e and f is $|i - j| + |k - \ell|$. This distance is also known as the Manhattan distance or the taxicab distance. The $(p + q - 1) \times (p + q - 1)$ distance matrix afforded by this distance has been considered in the literature in the context of some problems in numerical analysis [21,22]. The determinant of the matrix was obtained in [5].

If P_1 and P_2 denote paths with vertex sets $\mathscr{S}$ and $\mathscr{D}$ respectively then $T \cup P_1 \cup P_2$ is a tree with attached graphs. The edge distance matrix of this tree as introduced in Section 4 is the same as the distance matrix considered in the previous paragraph. Thus we can write a formula for the inverse and the determinant of the edge distance matrix using Theorem 11.

We conclude with an example. Consider a transportation problem with three sources and four destinations. A tableau corresponding to a basic feasible solution and the associated tree with attached graphs (which are paths) are shown in Table 1 and Fig. 1 respectively.

The matrices A, B, Q, D for $T \cup P_1 \cup P_2$, as introduced in Sections 2 and 4, are given by

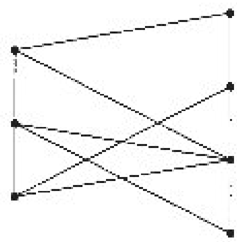

Fig. 1. $T \cup P_1 \cup P_2$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 & 3 & 4 & 3 & 4 \\ 2 & 0 & 1 & 2 & 3 & 2 \\ 3 & 1 & 0 & 1 & 2 & 1 \\ 4 & 2 & 1 & 0 & 3 & 2 \\ 3 & 3 & 2 & 3 & 0 & 1 \\ 4 & 2 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

We compute $\tau = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}'$, $\tau' D \tau = 10$, $\det D = 80$. The formula for D^{-1} asserted in Theorem 11 is easily verified, though we omit the details.

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References

[1] R.B. Bapat, Resistance distance in graphs, Math. Student 68 (1999) 87–98.

[2] R.B. Bapat, Resistance matrix of a weighted graph, MATCH Commun. Math. Comput. Chem. 50 (2004) 73–82.

[3] R.B. Bapat, J.W. Grossman, D.M. Kulkarni, Generalized matrix tree theorem for mixed graphs, Linear and Multilinear Algebra 46 (1999) 299–312.

[4] R.B. Bapat, D.M. Kulkarni, Minors of some matrices associated with a tree, in: Algebra and its Applications, in: D.V. Huynh, S.K. Jain, S.R. Lopez-Permouth (Eds.), Contemporary Mathematics, vol. 259, American Math Society, 2000, pp. 45–66.

[5] R.B. Bapat, S.J. Kirkland, M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl., in press (doi:10.1016/j.laa.2004.05.011).

[6] R.B. Bapat, T.E.S. Raghavan, Nonnegative Matrices and Applications, Encyclopedia of Mathematics and its Applications, vol. 64, Cambridge University Press, 1997.
[7] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, Wiley–Interscience, 1974.
[8] B. Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998.
[9] S.L. Campbell, C.D. Meyer Jr., Generalized Inverses of Linear Transformations, Pitman, 1979.
[10] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, SIAM J. Algebraic Discrete Methods 3 (1982) 319–329.
[11] P.G. Doyle, J.L. Snell, Random Walks and Electrical Networks, Math. Assoc. Am., Washington, 1984.
[12] M. Fiedler, Aggregation in graphs, Colloq. Math. Soc. János Bolyai 18 (1978) 315–330.
[13] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.
[14] R.L. Graham, L. Lovász, Distance matrix polynomials of trees, Adv. in Math. 29 (1) (1978) 60–88.
[15] E.L. Kaplan, Mathematical Programming and Games, John Wiley, 1982.
[16] D.J. Klein, Graph geometry, graph metrics and Weiner, MATCH Commun. Math. Comput. Chem. 35 (1997) 7–27.
[17] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81–95.
[18] W. Mayeda, Graph Theory, Wiley–Interscience, 1972.
[19] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197,198 (1994) 143–176.
[20] C.R. Rao, Linear Statistical Inference and its Applications, John Wiley, New York, 1973.
[21] L. Reid, X. Sun, Distance matrices and ridge function interpolation, Can. J. Math. 45 (6) (1993) 1313–1323.
[22] X. Sun, Solvability of multivariate interpolation by radial or related functions, J. Approx. Theory 72 (3) (1993) 252–267.
[23] A. Schrijver, Theory of Linear and Integer Programming, John Wiley, 1986.
[24] D.B. West, Introduction to Graph Theory, second ed., Prentice-Hall Inc., 2001.
[25] W. Xiao, I. Gutman, On resistance matrices, MATCH Commun. Math. Comput. Chem. 49 (2003) 67–81.
[26] W. Xiao, I. Gutman, Resistance distance and Laplacian spectrum, Theor. Chem. Acc. 110 (2003) 284–289.