PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

By R. C. Bose and K. R. Nair,
Statistical Laboratory, Calcutta.

§1. Introduction.

1. In agricultural field experimentation, the efficiency of Fisher's well known randomized block and Latin square designs, is found to deteriorate when the number of plots per block, or per row and column, goes above, say, ten or twelve. The advent of factorial (complex) experiments, where the number of treatments tried out in the same experiment became large, owing, either to the inclusion of several factors or of large number of levels of a small number of factors, or of both, brought in its wake large residual errors per plot, thereby lowering the accuracy of treatment comparisons. To overcome this growing difficulty, various devices were brought into practice, of which the most important and widely practised are the split-plot designs, which sacrifice information usually on certain main effects, and the confounded designs, which sacrifice information, partially or totally, on certain, usually unimportant, high-order interactions.

But if the experiment is non-factorial in structure (*i.e.* a single factor experiment) and yet having a large number of variants, it will be highly desirable to keep the size of a block (*i.e.* number of plots in a block) within efficient bounds. The need of such a large scale single factor experiment is felt most keenly when preliminary selection has to be done from a large number of new strains of a crop. Thus if we have 100 varieties to be tested, all in the same experiment, it is highly detrimental to the precision of varietal comparisons to have 100 plots per block. Here, the idea of confounding of high order interactions, which proved very fruitful in factorial experiments, has, apparently, no place, as we cannot reckon our comparisons in terms of main effects and interactions. On the other hand comparisons between every pair of verieties are the essential point of our study and not comparisons among several groups of them.

2. Yates came to the aid of the experimenter in two great advances. His first advance[6] brought him to a type of design which he called "quasi-factorial." This can work only if v, the number of varieties, is a factorisable number. Thus, let $v = p.\ q.\ r$ .... Yates now considers the v varieties as all combinations of a number of factors (non-existent) at levels p, q, r, etc. He can then confound the main effects and interactions of these quasi-factors, excepting the all-factor interactions, by arranging the v varieties in a hyper-dimensional lattice and assigning to various blocks the varieties occurring in lines parallel to the edges of this lattice. He thus gets v/p blocks of p plots, v/q blocks of q plots, v/r blocks of r plots, etc. The unequal size of blocks is a draw-back, as additional complications set in, owing to the necessity of getting weighted estimates of varietal effects. Higher block sizes can also be obtained confounding a lesser number of interactions, which will be pq, qr, etc. or, pqr, pqs, etc., etc., ..., by taking in a block varieties occurring in 2-flats, 3-flats, ... parallel to the bounding flats of his m-dimensional lattice. When $p = q = r =$ ......, so

that $v=p^m$, the analysis becomes simpler, as the block size remains constant, being either p, p^2, ... or p^{m-1} plots per block. If p in this case, happens to be a prime number or a power of a prime number, it will be possible to confound not only main effects and 2-, 3-, ... $(m-1)$-factor interactions but also the m-factor interactions. It will then lead to the symmetrical quasi-factorial design where every pair of varieties will occur together in an equal number of blocks, thus giving the same precision to comparisons between every pair of varieties. But if the confounding is confined to main effects only or main effects and certain of the interactions only, some varieties never occur together in the same blocks, so that comparisons between pairs of such varieties are less accurate than comparisons between pairs of varieties which occur together.

3. The next advance of Yates' led him to the discovery of a more general type of design, which, while not imposing the condition that v should be a factorisable number, shares the property of symmetry obtained in symmetrical quasi-factorial designs. These new designs came to be called "balanced incomplete block designs." Here the v varieties are replicated r times in b blocks of k plots such that every pair of varieties occurs together in $\lambda\,[=r\,(k-1)/(v-1)]$ blocks.

Fisher and Yates[4] have recently tabulated such among these designs as are likely to be of use in practical work (*i.e.* designs in which $k \leq r \leq 10$). A unified method of constructing these designs is discussed by one of the authors (R. C. Bose) in a forthcoming paper[2] in the *Annals of Eugenics.*

The balanced incomplete block designs are by far the best among all types of incomplete designs. But these designs are rather scarce. The only alternative incomplete designs hitherto available are the quasi-factorial designs. Here also if the block size should remain constant, the number of possible convenient designs becomes very limited. This comparative scarcity of designs with same block size forces on us the utilisation of quasi-factorial designs with unequal block size. The defect in the latter designs arises at the stage of analysing the data, owing to the need for making adequate allowance for possible inequalities in error variance, of plots belonging to blocks of different sizes. Theoretically, there is no difficulty in analysing the data, if this refinement is ignored, which can be permitted in practical work, only if the block sizes are nearly equal.

4. In this paper we are introducing a general class of designs with numerous possible and practically useful solutions, limiting ourselves to a constant block size. These designs will go a long way in putting out of use the quasi-factorial designs with unequal block sizes.

The balanced incomplete block designs, and the quasi-factorial designs of p^m varieties in blocks of p^{m-1} plots are special cases of our designs. The latter designs in blocks of $p^k (k<m-1)$ plots do not fit into our general class, but belong to another general class which we call "hyper-dimensional designs." It is proposed to discuss the latter class in another paper.

Only a small number of representative designs of the general class are included in the present paper, by way of bringing out the potentialities of this class of designs. A general enumeration of all arithmetically possible designs with the corresponding combinatorial solutions wherever available, which are likely to be of use in practical experimentation, has been taken in hand and will be published in a subsequent communication.

5. There are two problems involved in the analysis of data of any experimental design, namely, the problems of estimation and of tests of significance (collective and several). The method of getting efficient estimates of varietal effects has been indicated for the general case of designs of our type, and actual expressions obtained for two important special cases. For test of significance, collectively, of all varietal effects, the sum of squares due to varieties, has been obtained for the general case, in an elegant form. Similar expressions are given for sum of products due to varieties, when more than one character is the subject of study. Lastly, for testing significance of difference in estimated effects of any pair of varieties, the method of getting the variance of the difference has been indicated for the general case and actual expressions derived for the special cases. The expressions for the Efficiency Factor are also given for the latter cases.

§2. Relations between the Parameters of Partially Balanced Incomplete Block Designs.

1. Consider any v varieties or treatments, to be arranged in b blocks with k plots each (each plot being given one treatment, and no two plots in the same block receiving the same treatment). The arrangement will be called a partially balanced incomplete block design if the following conditions are satisfied.

(*i*) Every variety is replicated r times.

(*ii*) With respect to every given variety, the remaining ones fall into groups of $n_1, n_2, \dots n_m$ each, such that every variety of the i-th group occurs exactly λ_i times, with the given variety, the numbers λ_i and n_i being independent of the variety with which we start. Without loss of generality we will assume $\lambda_i > \lambda_{i+1}$. The set of numbers $\lambda_1, \lambda_2, \dots \lambda_m$ are all unequal and may include 0, but $n_1, n_2, \dots n_m$ must all be non-zero and may be equal or unequal. Two varieties occurring together λ_i times, may be called i-th associates. Each belongs of course to the i-th group with respect to the other.

(*iii*) Given any two varieties which are i-th associates, the number of varieties common to the j-th associates of one, and the k-th associates of the other, is independent of the pair of i-th associates with which we start. This number is denoted by p^i_{jk}. Clearly $p^i_{jk} = p^i_{kj}$.

In the particular case when $m=1$, our design reduces to the balanced incomplete block design of Yates. The quasi-factorial design of Yates, with $v=p^m$, $k=p^{m-1}$ is also a special case of our design, with

$$\lambda_i = m-i,\ n_i = {}_mC_i\,(p-1)^i,\ r=m,\ b=mp.$$

It will be seen, however, in §4–7, that there are numerous other designs, besides these, which belong to the class of partially balanced incomplete block designs.

2. The numbers $v, b, r, k ; \lambda_1, \lambda_2, \dots \lambda_m ; n_1, n_2, \dots n_m$ may be called the parameters of the *first kind*, and the numbers p^i_{jk} $(i, j, k = 1, 2, \dots m)$ the parameters of the *second kind*, belonging to our design. Thus there are $2m+4$ parameters of the first kind, and $m^2(m+1)/2$ parameters of the second kind (since p^i_{jk} and p^i_{kj} are identical).

From the condition (*i*) it is clear that

$$b\,k = v\,r \qquad \dots\ (2\cdot20)$$

Since with respect to any variety, the remaining $v-1$ fall into groups of $n_1, n_2, \ldots n_m$, it is clear that

$$v-1=n_1+n_2+\ldots\ldots\ldots+n_m \qquad \ldots \quad (2{\cdot}21)$$

Consider any particular variety θ. It occurs in r blocks. In each of these blocks there are $(k-1)$ other varieties. Hence any particular variety is a member of $r(k-1)$ pairs. But the n_i varieties which are the i-th associates to θ, each yield λ_i pairs. Hence we have

$$r(k-1)=n_1\lambda_1+n_2\lambda_2+\ldots+n_m\lambda_m \qquad \ldots \quad (2{\cdot}22)$$

The relations (2·20), (2·21), (2·22) show that only $2m+1$ of our parameters of the first kind are independent. We can conveniently take these independent parameters to be $\lambda_1, \lambda_2, \ldots \lambda_m$; $n_1, n_2, \ldots n_m$ and k. Of course the independent parameters cannot be chosen at will, due to the restriction that every parameter must be integral.

Each of the v varieties, has n_i i-th associates, so that we get vn_i pairs. But each pair is counted twice, once from each end. Hence the number of pairs of i-th associates is $vn_i/2$. The total number of pairs is then

$$\tfrac{1}{2}v(n_1+n_2+\ldots+n_m)=\tfrac{1}{2}v(v-1) \qquad \ldots \quad (2{\cdot}23)$$

as it should be.

3. Let us next consider identities involving both parameters of the first as well as the second kind. Let θ and ϕ be any two varieties which are i-th associates. Then ϕ is contained among the group of n_i varieties which are i-th associates to θ. Among the remaining n_i-1 varieties of this group, there are exactly p^i_{ik} varieties which are k-th associates of ϕ. Hence

$$\sum_{k=1}^{m} p^i_{ik}=p^i_{i1}+p^i_{i2}+\ldots\ldots\ldots\ldots+p^i_{im}=n_i-1 \qquad \ldots \quad (2{\cdot}30)$$

Again if $j\neq i$, then among the n_j varieties which are j-th associates to θ, there are exactly p^i_{jk} varieties which are at the same time k-th associates of ϕ. Hence

$$\sum_{k=1}^{m} p^i_{jk}=p^i_{j1}+p^i_{j2}+\ldots\ldots\ldots\ldots+p^i_{jm}=n_j\ (i\neq j) \qquad \ldots \quad (2{\cdot}31)$$

Taking together the relations (2·30) and (2·31) we have

$$\sum_{k=1}^{m} p^i_{jk}=n_j-1 \text{ or } n_j \text{ according as } i=j \text{ or } i\neq j \qquad \ldots \quad (2{\cdot}32)$$

Again consider the group G_i of n_i varieties which are i-th associates of a given variety θ, and the group G_j of the n_j varieties which are j-th associates of θ. Every variety belonging to G_i has got exactly p^i_{jk} k-th associates among the varieties of the group G_j. Again every variety belonging to G_j has exactly p^j_{ik} k-th associates among the varieties of the group G_i. Hence the number of pairs of k-th associates, which can be formed by taking one variety from the group G_i and one variety from the group G_j, is, on the one hand, $n_i\,p^i_{jk}$ and, on the other hand, $n_j\,p^j_{ik}$. Hence

$$n_i\,p^i_{jk}=n_j\,p^j_{ik} \qquad \ldots \quad (2{\cdot}33)$$

this equation being true for all values of i, j, k. Of course when $i=j$, the equation becomes automatic and gives no relation between the parameters of the first kind.

4. Our next problem is to determine the number of independent parameters of the second kind, the parameters of the first kind being given. The numbers p^i_{jk} for a fixed i, can be arranged in the form of a matrix of degree m

$$\left.\begin{matrix} p^i_{11} & p^i_{12} & p^i_{13} \ldots\ldots\ldots\ldots & p^i_{1m} \\ p^i_{21} & p^i_{22} & p^i_{23} \ldots\ldots\ldots\ldots & p^i_{2m} \\ p^i_{31} & p^i_{32} & p^i_{33} \ldots\ldots\ldots\ldots & p^i_{3m} \\ \vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\ \vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\ p^i_{m1} & p^i_{m2} & p^i_{m3} \ldots\ldots\ldots\ldots & p^i_{mm} \end{matrix}\right\} \quad .. \quad (2{\cdot}40)$$

This matrix is a symmetric matrix due to the relation $p^i_{jk}=p^i_{kj}$. Thus exactly $m\ (m+1)/2$ of the $m^2(m+1)/2$, p's are involved in this matrix. We may call the matrix (2·40), the matrix, (M_i). Giving i the m values 1, 2, ..., m we get the m matrices (M_1), (M_2), ..., (M_m).

The significance of the relations (2·32) then is that the marginal totals of the rows as well as of the columns of each of these m matrics is fixed. In fact the total of the j-th row or column of the matrix M_i is n_j-1 or n_j according as $i=j$ or $i\neq j$. Consider now the significance of the relation (2·33), with reference to these matrices. If the elements of the matrix (M_i) are all known, then the relations (2·33) fix the elements in the i-th row or column of all the other matrices (M_1), (M_2), ... (M_{i-1}) (M_{i+1}) ... (M_m). Since the marginal totals in (M_1) are fixed, the elements of the first row and column become fixed, so soon as the other elements are given. Taking into account the condition of symmetry, the number of independent p's in (M_1) is $m\ (m-1)/2$. The elements of (M_1) now being known, the elements in the first row or column of (M_2) are fixed. Also since the marginal totals are fixed, to completely fix every element of (M_2), we need only know the elements in the last $(m-2)$ rows and columns of (M_2). Thus the number of independent p's we get from (M_2) is $(m-1)(m-2)/2$. The elements of (M_1) and (M_2) now being fixed, the elements in the first and second rows of (M_3) are fixed. Since the marginal totals are also fixed, the number of independent p's obtained from (M_3) is $(m-2)(m-3)/2$. Proceeding on in this way we find that the number of independent parameters of the second kind, when the parameters of the first kind are given, is

$$\frac{m\ (m-1)}{2}+\frac{(m-1)\ (m-2)}{2}+\frac{(m-2)\ (m-3)}{2}+\ldots\ldots\ldots+1=\frac{m(m^2-1)}{6} \quad \ldots\ (2{\cdot}41)$$

We thus find that given the parameters of the first kind there are exactly $m(m^2-1)/6$ independent parameters of the second kind.

When $m=1$, *i.e.* when our design is a balanced incomplete block design, there is no independent parameter of the second kind. When $m=2$, there is one independent parameter of the second kind. When $m=3$, there are four independent parameters of the second kind.

5. For every arithmetically possible design with parameters of the first kind $b, v, r, k\ ;\ \lambda_1, \lambda_2, \ldots \lambda_m\ ;\ n_1, n_2, \ldots n_m$ and the associated parameters p^i_{jk} of the second kind, there is a complimentary design with same number of blocks and of varieties as before, but

having $v-k$ plots per block and $b-r$ replications of each variety. The λ's of this design will be $b-2r$ more than the λ's of the first design. The n's and p^i_{jk} will be the same for both designs. For given values of b and v it is sufficient to consider designs with $k \leqslant v/2$ as the complimentary designs can be obtained automatically from them.

§3. ANALYSIS OF THE DESIGNS.

1. It is now well recognised that the method of estimating block and varietal effects and of testing them for significance are closely related to the method of estimating partial regression coefficients and of testing their significance, in a sample of n observations of a normally distributed variable y and depending on p observed variables $x_1, x_2, \ldots\ldots x_p$ following any distribution law.

Supposing the variables to be all measured from their means, let $b_1, b_2, \ldots\ldots b_p$ be estimates of the unknown partial regression coefficients $\beta_1, \beta_2, \ldots\ldots\ldots \beta_p$ estimated on the assumption that y is distributed normally about Y, where

$$Y = \beta_1 x_1 + \beta_2 x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \beta_p x_p \quad \ldots \quad (3{\cdot}10)$$

The logarithm of the likelihood of any assigned values of $\beta_1, \beta_2, \ldots\ldots\ldots\ldots \beta_p$ is given by

$$-\Sigma\,(y - \beta_1 x_1 - \beta_2 x_2 - \ldots\ldots\ldots\ldots\ldots - \beta_p x_p)^2 \quad (3{\cdot}11)$$

Maximising the likelihood, which comes to minimising, with respect to $\beta_1, \beta_2, \ldots\ldots \beta_p$, the expression

$$\Sigma\,(y - \beta_1 x_1 - \beta_2 x_2 - \ldots\ldots\ldots\ldots\ldots - \beta_p x_p)^2 \quad \ldots \quad (3{\cdot}12)$$

we get the following normal equations to find out the maximal likelihood estimates of the β's namely, b_i $(i = 1, 2, \ldots\ldots p)$.

$$\left.\begin{array}{l}
b_1 S(x_1^2) + b_2 S(x_1 x_2) + \ldots\ldots\ldots + b_p S(x_1 x_p) = S(x_1 y) \\
b_1 S(x_1 x_2) + b_2 S(x_2^2) + \ldots\ldots\ldots + b_p S(x_2 x_p) = S(x_2 y) \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
b_1 S(x_1 x_p) + b_2 S(x_2 x_p) + \ldots\ldots\ldots + b_p S(x_p^2) = S(x_p y)
\end{array}\right\} \quad \ldots \quad (3{\cdot}13)$$

If, instead of solving equations (3·13) directly, we make, following Fisher[4], p such sets of p equations, by replacing $S(x_i y)$ by 1 and $S(x_j y)$ $(i \neq j)$ by 0, in the i-th set of equations, and denote the solutions of $b_1, b_2, \ldots\ldots b_p$ in the i-th set as $c_{i1}, c_{i2}, \ldots\ldots c_{ip}$, we get the following symmetrical matrix of solutions for $b_1, b_2, \ldots\ldots\ldots b_p$

$$\left.\begin{array}{ccccc}
c_{11} & c_{12} & c_{13} & \ldots\ldots\ldots\ldots\ldots & c_{1p} \\
c_{21} & c_{22} & c_{23} & \ldots\ldots\ldots\ldots\ldots & c_{2p} \\
c_{31} & c_{32} & c_{33} & \ldots\ldots\ldots\ldots\ldots & c_{3p} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{p1} & c_{p2} & c_{p3} & \ldots\ldots\ldots\ldots\ldots & c_{pp}
\end{array}\right\} \quad \ldots \quad (3{\cdot}14)$$

The values of $b_1, b_2, \ldots\ldots b_p$ that will satisfy equations (3·13) are given by

$$b_i = c_{i1}\ S(x_1\, y) + c_{i2}\ S(x_2\, y) + \ldots\ldots\ldots\ldots + c_{ip}\, S(x_p\, y) \qquad \ldots \quad (3{\cdot}15)$$

The importance of this auxiliary set of solutions represented by the c's lies in the fact that variance of any linear expression of the b's can be easily expressed in terms of the c's.

Thus

$$V\left(\sum_{i=1}^{p} l_i\, b_i\right) = \left(\sum_{i=1}^{p}\sum_{j=1}^{p} l_i\, l_j\, c_{ij}\right) V(y) \qquad \ldots \quad (3{\cdot}16)$$

We are more concerned in practice, with the following special cases.

$$V(b_i) = c_{ii}\, V(y) \qquad \ldots \quad (3{\cdot}17)$$

$$V(b_i - b_j) = (c_{ii} - 2c_{ij} + c_{jj})\, V(y) \qquad \ldots \quad (3{\cdot}18)$$

The sum of squares due to the fitted regression equation (*i.e.* due to $b_1, b_2, \ldots\ldots b_p$ collectively) is

$$\Sigma\,(b_1\, x_1 + b_2\, x_2 + \ldots\ldots\ldots\ldots\ldots + b_p\, x_p)^2 \qquad \ldots \quad (3{\cdot}19)$$

which can be simplified to

$$b_1\, S(x_1\; y) + b_2\, S(x_2\; y) + \ldots\ldots\ldots\ldots + b_p\, S(x_p\, y) \qquad \ldots \quad (3{\cdot}1{,}10)$$

(3·1, 10) cannot be split into separate sums of squares, attributable to each of the coefficients $b_1, b_2, \ldots\ldots b_p$, owing to their non-orthogonality. If it is necessary to test the significance of the coefficients, say, $b_{m+1}, b_{m+2} \ldots\ldots b_p$, the valid procedure is to find a regression equation between y and $x_1, x_2, \ldots\ldots x_m$. If $b_{1\cdot 1}, b_{2\cdot 1}, \ldots\ldots b_{m\cdot 1}$ be the partial regression coefficients obtained by solving

$$\left.\begin{array}{l} b_{1\cdot 1}\, S(x_1^2) + b_{2\cdot 1}\, S(x_1\, x_2) + \ldots\ldots\ldots + b_{m\cdot 1}\, S(x_1\, x_m) = S(x_1\, y) \\ b_{1\cdot 1}\, S(x_1\, x_2) + b_{2\cdot 1}\, S(x_2^2) + \ldots\ldots\ldots + b_{m\cdot 1}\, S(x_2\, x_m) = S(x_2\, y) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ b_{1\cdot 1}\, S(x_1\, x_m) + b_{2\cdot 1}\, S(x_2\, x_m) + \ldots\ldots\ldots + b_{m\cdot 1}\, S(x_m^2) = S(x_m\, y) \end{array}\right\} \ldots \quad (3{\cdot}1{,}11)$$

The sum of squares due to $b_{1\cdot 1}, b_{2\cdot 1}, \ldots\ldots b_{m\cdot 1}$ is

$$b_{1\cdot 1}\, S(x_1\, y) + b_{2\cdot 1}\, S(x_2\, y) + \ldots\ldots\ldots\ldots + b_{m\cdot 1}\, S(x_m\, y) \qquad \ldots \quad (3.1{,}12)$$

The difference between (3.1, 10) and (3·1, 12) is the appropriate measure of the sum of squares due to $b_{m+1}, b_{m+2}, \ldots\ldots b_p$.

If observations on another dependent variable y' are also taken simultaneously, for the same n individuals, we can fit partial regression coefficients $b'_1, b'_2, \ldots\ldots b'_p$, of y' on $x_1, x_2, \ldots\ldots x_p$.

The sum of products due to the fitted regressions will be

$$\begin{aligned} & S\,(b_1\, x_1 + b_2\, x_2 + \ldots\ldots + b_p\, x_p)\,(b'_1\, x_1 + b'_2\, x_2 + \ldots\ldots + b'_p\, x_p) \\ & = b_1\, S(x_1\, y') + b_2\, S(x_2\, y') + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + b_p\, S(x_p\, y') \\ & = b'_1\, S(x_1\, y) + b'_2\, S(x_2\, y) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + b'_p\, S(x_p\, y) \qquad \ldots \quad (3{\cdot}1{,}13) \end{aligned}$$

2. Coming now to our own problem, we have got observations y_{ij} of the yield, say, of the i-th variety in the j-th block. Let us postulate that m is the hypothetical mean, v_i the effect of the i-th variety, which we assume to remain the same whatever be the block, and b_j the effect of the j-th block on every one of its plots, irrespective of the variety occurring in any plot. On this postulate of additiveness of the block and varietal effects, we have

$$y_{ij} = m + b_j + v_i + e_{ij} \qquad \ldots \quad (3{\cdot}20)$$

where the b's and v's are subject to two limiting constraints $S(b)=0$, $S(v)=0$.

Our problem is to determine efficient estimates of m, b_j and v_i on the assumption that e_{ij} is distributed according to normal law about zero. To get maximal likelihood estimates of m, b_j and v_i, we have only to minimise $S(e^2_{ij})$ over all the observations. We have therefore to minimise

$$\Sigma\,\Sigma\,(y_{ij} - m - b_j - v_i)^2 \qquad \ldots \quad (3{\cdot}21)$$

subject to the following linear restraints on the b's and v's, namely,

$$\left.\begin{array}{l} b_1 + b_2 + \ldots\ldots\ldots\ldots + b_b = 0 \\ v_1 + v_2 + \ldots\ldots\ldots\ldots + v_v = 0 \end{array}\right\} \qquad \ldots \quad (3{\cdot}22)$$

limiting the degrees of freedom of blocks and varieties to $b-1$ and $v-1$ respectively.

The process of estimating m, b_1, b_2, ... b_b; v_1, v_2, ... v_v by minimising (3·21) subject to the conditions (3·22) may now be looked upon as a partial regression problem by introducing $b+v+1$ pseudo-variables x_1, x_2, ... x_{b+v+1} such that m, b_1, b_2, ... b_b, v_1, v_2 .... v_v will be the partial regression coefficients of y on this set of variables. These variables with the exception of x_1, will have the arbitrarily chosen values 1 or 0 in different plots. x_1 will always have value 1. Thus in the plot corresponding to y_{ij}

$$x_1 = 1 \qquad x_{j+1} = 1, \qquad x_{b+i+1} = 1 \qquad \ldots \quad (3{\cdot}23)$$

and all the other variables will have value 0.

If we denote by G the grand total of y in the $N(=bk=vr)$ plots of the experiment, B_1, B_2, ... B_b the block totals and V_1, V_2, ... V_v the variety totals of y, it can easily be seen that

$$\left.\begin{array}{l} S(x_1\, y) \;\;\; = G \\ S(x_{j+1}\, y) = B_j \\ S(x_{b+i+1}\, y) = V_i \end{array}\right\} \qquad \ldots \quad (3{\cdot}24)$$

Since m is the regression coefficient of y on x_1, b_j the partial regression coefficient of y on x_{j+1} and v_i that of y on x_{b+i+1}, the sum of squares due to the fitted constants m, b_1, b_2, ... b_b, v_1, v_2, ... v_v is, by (3.1,10),

$$m\,G + b_1\,B_1 + b_2\,B_2 + \ldots\ldots\ldots + b_b\,B_b + v_1\,V_1 + v_2\,V_2 + \ldots\ldots\ldots + v_v\,V_v \qquad (3{\cdot}25)$$

where $m, b_1, b_2, \ldots\ldots\ldots; v_1, v_2, \ldots\ldots\ldots$ are solutions of the following normal equations:

$$\left.\begin{array}{l} N\,m \quad \ldots \quad \ldots \quad \ldots = G_1 \\ k\,(m + b_1) + \text{sum of } v\text{'s of block } 1 = B_1 \\ k\,(m + b_2) + \quad \ldots \quad \ldots\ 2 = B_2 \\ \vdots \qquad\qquad \vdots \\ k\,(m + b_b) + \quad \ldots \quad \ldots\ b = B_b \\ r\,(m + v_1) + \text{sum of } b\text{'s of variety } 1 = V_1 \\ r\,(m + v_2) + \quad \ldots \quad \ldots\ 2 = V_2 \\ \vdots \qquad\qquad \vdots \\ r\,(m + v_v) + \quad \ldots \quad \ldots\ v = V_v \end{array}\right\} \quad \ldots \quad (3{\cdot}26)$$

and the restraining equations (3·22)

In practical problems we are interested in testing the significance of varietal effects only. To get the appropriate measure of the sum of squares due to $v_1, v_2, \ldots\ldots\ldots v_v$ we should now fit the partial regression of y on $x_1, x_2, \ldots\ldots\ldots x_{b+1}$ alone. If $m_{\cdot 1}, b_{1\cdot 1}, b_{2\cdot 1} \ldots\ldots\ldots b_{b\cdot 1}$ be the estimates of the partial regression coefficients, ignoring the variables $x_{b+2}, \ldots\ldots\ldots\ldots x_{b+v+1}$ the sum of squares due to these coefficients is

$$m_{\cdot 1}\, G + b_{1\cdot 1}\, B_1 + b_{2\cdot 1}\, B_2 + \ldots\ldots \ldots\ldots + b_{b\cdot 1}\, B_b \qquad (3{\cdot}27)$$

where $m_{\cdot 1}, b_{1\cdot 1} \ldots\ldots\ldots b_{b\cdot 1}$ are solutions of the normal equations:—

$$\left.\begin{array}{ll} N\,m_{\cdot 1} \quad \ldots & = G \\ k\,(m_{\cdot 1} + b_{1\cdot 1}) & = B_1 \\ k\,(m_{\cdot 1} + b_{2\cdot 1}) & = B_2 \\ \vdots & \vdots \\ k\,(m_{\cdot 1} + b_{b\cdot 1}) & = B_b \end{array}\right\} \quad \ldots \quad (3{\cdot}28)$$

and, as before,

$$b_{1\cdot 1} + b_{2\cdot 1} + \ldots\ldots\ldots\ldots\ldots\ldots + b_{b\cdot 1} = 0$$

(3·27) therefore reduces to

$$\frac{1}{k}\,(B^2_1 + B^2_2 + \ldots\ldots\ldots\ldots\ldots\ldots + B^2_b) \qquad (3{\cdot}29)$$

The appropriate measure of the sum of squares due to varieties is therefore (3·25) minus (3·29), which, on substituting for $b_1, b_2, \ldots\ldots b_b$ in terms of v's from equations (3·26), reduces to the elegant expression

$$\sum_{i=1}^{v} v_i\, Q_i \qquad (3.2,10)$$

where $Q_i = V_i -$ sum of the r block means of that variety, *i.e.* Q_i is sum of the r yields of variety i, each corrected by its block mean.

The values of $v_1, v_2, \ldots\ldots\ldots v_v$ have to be obtained from equations (3·26). In the general case of m, λ's the explicit expressions for v_i cannot be easily obtained.

No convenient expression can therefore be given for the sum of squares of varieties, directly in terms of known yields. In every experiment, the adjusted varietal means $(G/N+v)$ have to be given for the individual comparisons among varieties. It will be presently seen that the solution of v_i will involve only Q's besides the parameters of the design. Once v_i is determined, the sum of squares follows immediately.

3. If another character is studied on each plot, say, y'_{ij}, and if v'_i, Q'_i be the varietal effects, etc., for that character, it can similarly be seen with the help of (3·1,13) that the sum of products of the two characters y and y', due to varieties, is

$$\Sigma\, v_i\, Q'_i = \Sigma\, v'_i\, Q_i \tag{3·30}$$

thus providing an independent check on the calculations.

4. Coming now to the solution of $v_1, v_2, \ldots\ldots\ldots v_v$ it will be easily seen that Q_i is independent of $m, b_1, b_2, \ldots\ldots\ldots b_b$. Let Σv_{i1} denote the sum of the n_1, v's which are first associates of v_i, Σv_{i2} the sum of the n_2, v's which are second associates of v_i, etc. Let ΣQ_{i1} be the sum of the Q's corresponding to the n_1 varieties comprised in Σv_{i1}, and so on. We then get,

$$k\, Q_i = r\,(k-1)\, v_i - \lambda_1\, \Sigma\, v_{i1} - \lambda_2\, \Sigma\, v_{i2} - \ldots\ldots\ldots\ldots - \lambda_m\, \Sigma\, v_{im} \tag{3·40}$$

(3·40) consists of v equations. In solving for v_i we impose, as condition for analysability of the design, that we should be able to eliminate $\Sigma v_{i1}, \Sigma v_{i2}, \ldots\ldots\ldots \Sigma v_{im}$ as such and not the individual v's within each summation. The parametric relations discussed in the previous section satisfy this condition of analysability.

Suppose n_j is the biggest among $n_1, n_2, \ldots\ldots\ldots n_m$. It is convenient then to eliminate $\Sigma\, v_{ij}$ immediately, using the property, $\sum_{i=1}^{v} v_i = 0$. (3·40) will then change to

$$\begin{aligned} k\, Q_i = {} & [r\,(k-1) + \lambda_j]\, v_i + (\lambda_j - \lambda_1)\, \Sigma\, v_{i1} + (\lambda_j - \lambda_2)\, \Sigma\, v_{i2} \\ & + \ldots\ldots\ldots + (\lambda_j - \lambda_{j-1})\, \Sigma v_{i,\,j-1} + (\lambda_j - \lambda_{j+1})\, \Sigma v_{i,\,j+1} \\ & + \ldots\ldots\ldots + (\lambda_j - \lambda_m)\, \Sigma v_{im} \end{aligned} \tag{3·41}$$

We find that

$$\begin{aligned} k\, \Sigma Q_{i1} = {} & [r\,(k-1) + \lambda_j]\, \Sigma v_{i1} \\ & + (\lambda_j - \lambda_1)\, (n_1\, v_i + p^1_{11}\, \Sigma v_{i1} + p^2_{11}\, \Sigma v_{i2} + \ldots \quad \ldots + p^j_{11}\, \Sigma v_{ij} + \ldots \ldots + p^m_{11}\, \Sigma v_{im}) \\ & + (\lambda_j - \lambda_2)\, (p^1_{12}\, \Sigma v_{i1} + p^2_{12}\, \Sigma v_{i2} + \ldots\ldots\ldots\ldots\ldots + p^j_{12}\, \Sigma v_{ij} + \ldots\ldots + p^m_{12}\, \Sigma v_{im}) \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ & + (\lambda_j - \lambda_m)\, (p^1_{1m}\, \Sigma v_{i1} + p^2_{1m}\, \Sigma v_{i2} + \ldots\ldots\ldots\ldots \ldots + p^j_{1m}\, \Sigma v_{ij} + \ldots\ldots + p^m_{1m}\, \Sigma v_{im}) \end{aligned} \tag{3·42}$$

We thus see that ΣQ_{i1} can be expressed as a function of the $m+1$ quantities, v_i, $\Sigma v_{i1}, \Sigma v_{i2}, \ldots\ldots\ldots \Sigma v_{ij}, \ldots\ldots \ldots \Sigma v_{im}$ and by eliminating Σv_{ij}, as before, it can be expressed as a function of the m quantities $v_i, \Sigma v_{i1}, \Sigma v_{i2}, \ldots\ldots \Sigma v_{i,\,j-1}, \Sigma v_{i,\,j+1} \ldots\ldots \Sigma v_{im}$. We can similarly express $\Sigma Q_{i2}, \Sigma Q_{i3}, \ldots\ldots \Sigma Q_{i,j-1}\ \Sigma Q_{i,j+1} \ldots\ldots \Sigma Q_{im}$ as functions of the above m quantities. We thus get m equations involving $v_i, \Sigma v_{i1}, \Sigma v_{i3}, \ldots\ldots \Sigma v_{i,j-1}, \Sigma v_{i,j+1}, \ldots\ldots \Sigma v_{im}$ from which v_i can be solved by the usual methods.

The more useful designs in practical work will be those with $m=2$ or 3. The detailed solutions of these two cases are worked out below.

5. $m=2$. If we eliminate Σv_{i2} which may be profitably done if $n_2 > n_1$, the expression for v_i, the effect of the i^{th} variety, is

$$v_i = \frac{k \begin{vmatrix} Q_i & B_{12} \\ \Sigma Q_{i1} & B_{22} \end{vmatrix}}{\begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix}} \qquad (3.50)$$

where

$$\left.\begin{aligned} A_{12} &= r(k-1)+\lambda_2 \\ A_{22} &= (\lambda_2-\lambda_1)p^2_{12} \\ B_{12} &= (\lambda_2-\lambda_1) \\ B_{22} &= r(k-1)+\lambda_2+(\lambda_2-\lambda_1)\ (p^1_{11}-p^2_{11}) \end{aligned}\right\} \qquad (3{\cdot}51)$$

If $n_1 > n_2$ it will be more convenient to express v_i as

$$v_i = \frac{k \begin{vmatrix} Q_i & B_{11} \\ \Sigma Q_{i2} & B_{21} \end{vmatrix}}{\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix}} \qquad \ldots\ (3{\cdot}52)$$

where

$$\left.\begin{aligned} A_{11} &= r(k-1)+\lambda_1 \\ A_{21} &= (\lambda_1-\lambda_2)\ p^1_{12} \\ B_{11} &= \lambda_1-\lambda_2 \\ B_{21} &= r(k-1)+\lambda_1+(\lambda_1-\lambda_2)\ (p^2_{22}-p^1_{22}) \end{aligned}\right\} \qquad \ldots\ (3{\cdot}53)$$

$$\text{Let } \Delta = \begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix} = \begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix}$$

6. We always need to test the significance of the difference between any two varietal effects, v_i and v_j. It is important therefore to get the variance of $v_i - v_j$. Varieties i and j may be either first or second associates. The expression for the variance will be different for each.

We now revert to the idea that v_j and v_i are of the nature of partial regression coefficients. Due to the constraining condition $\Sigma v_i = v$, the process explained in the opening paragraphs of the present section is applicable to the equations, (3·40) only after the following modification. In the right hand side of each of these equations, we introduce a pseudo-variate v_0, and to the system of v equations thus obtained, we add the constraining equation $\sum_{1}^{v} v_i = 0$. From the equations we now find c_{ii}, c_{jj}, c_{ij} in the usual manner.

If we use the expression (3·50) for v_i,

$$\left.\begin{aligned} c_{ii} = c_{jj} &= \frac{k}{\Delta}\begin{vmatrix} 1-\dfrac{1}{v} & B_{12} \\ -\dfrac{n_1}{v} & B_{22} \end{vmatrix} \\ c_{ij} &= \frac{k}{\Delta}\begin{vmatrix} -\dfrac{1}{v} & B_{12} \\ 1-\dfrac{n_1}{v} & B_{22} \end{vmatrix} \quad (i \text{ and } j, \text{ 1st. associate.}) \\ &= \frac{k}{\Delta}\begin{vmatrix} -\dfrac{1}{v} & B_{12} \\ -\dfrac{n_1}{v} & B_{22} \end{vmatrix} \quad (\;\text{,,} \quad \text{2nd} \quad \text{,,}\;) \end{aligned}\right\} \quad \dots \quad (3·60)$$

Corresponding to the expression (3·52) for v_i,

$$\left.\begin{aligned} c_{ii} = c_{jj} &= \frac{k}{\Delta}\begin{vmatrix} 1-\dfrac{1}{v} & B_{11} \\ -\dfrac{n_2}{v} & B_{21} \end{vmatrix} \\ C_{ij} &= \frac{k}{\Delta}\begin{vmatrix} -\dfrac{1}{v} & B_{11} \\ -\dfrac{n_2}{v} & B_{21} \end{vmatrix} \quad (i \text{ and } j, \text{ 1st. associate}) \\ &= \frac{k}{\Delta}\begin{vmatrix} -\dfrac{1}{v} & B_{11} \\ 1-\dfrac{n_2}{v} & B_{21} \end{vmatrix} \quad (\;\text{,,} \quad \text{, 2nd} \;) \end{aligned}\right\} \quad \dots \quad (3·61)$$

If σ^2 is the variance per plot, the variance of difference between estimated effects of two varieties which are first associates, follows from (3·18) as

$$V(v_i - v_j) = 2k\,(B_{22} + B_{12})\,\sigma^2/\Delta$$
$$= 2k\,B_{21}\,\sigma^2/\Delta \qquad \ldots \quad (3\cdot 62)$$

and for two second associates as

$$V(v_i - v_j) = 2k\,(B_{21} + B_{11})\,\sigma^2/\Delta$$
$$= 2k\,B_{22}\,\sigma^2/\Delta \qquad \ldots \quad (3\cdot 63)$$

Since $B_{22} = B_{21} + \lambda_1 - \lambda_2$ and $\lambda_1 > \lambda_2$ it will be obvious that (3·62) is smaller than (3·63). This is as it should be, as greater the association between two varieties the more precise is their comparison.

7. Since there are $vn_1/2$ comparisons of the first kind and $vn_2/2$ comparisons of the second kind, the mean variance of all comparisons is

$$V_m = \frac{2k\,\sigma^2}{v-1}\,\frac{\begin{vmatrix} v-1 & B_{12} \\ -n_1 & B_{22} \end{vmatrix}}{\begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix}} = \frac{2k\,\sigma^2}{v-1}\,\frac{\begin{vmatrix} v-1 & B_{11} \\ -n_2 & B_{21} \end{vmatrix}}{\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix}} = \frac{2c_{ii}\,\sigma^2}{1-\dfrac{1}{v}} \qquad \ldots \quad (3\cdot 70)$$

If the experiment was conducted in ordinary randomized blocks, utilising the same number of plots as here, the mean variance will be $2\sigma'^2/r$ where σ' is the standard erro. per plot in blocks of v plots. We should naturally expect $\sigma < \sigma'$ Ignoring this expected gain in precision, the relative efficiency (or the efficiency factor) of our design is

$$\text{E. F.} = \frac{(v-1)}{r\,k}\,\frac{\begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix}}{\begin{vmatrix} v-1 & B_{12} \\ -n_1 & B_{22} \end{vmatrix}} = \frac{(v-1)}{r\,k}\,\frac{\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix}}{\begin{vmatrix} v-1 & B_{11} \\ -n_2 & B_{21} \end{vmatrix}} = \frac{1-\dfrac{1}{v}}{r c_{ii}} \qquad \ldots \quad (3\cdot 71)$$

8. Putting $\lambda_1 = \lambda_2 = r(k-1)/(v-1)$ in (3·71) it reduces to $(1-1/k)/(1-1/v)$ which is the efficiency factor of Yates' balanced designs.

9. We shall also derive the efficiency factor for the most important among Yates' quasi-factorial designs, namely, of p^2 varieties in blocks of p plots. If s orthogonalised $p \times p$ squares are used in the design ($0 \leq s \leq p-1$, if p is a prime or power of a prime; $s = 0$ or 1 if $p = 4l+2$; $s = 0$, 1 or 2 for all values of p not of the form $4l+2$), the parameters become

$$v = p^2, \qquad b = p(s+2), \qquad r = (s+2), \qquad k = p$$
$$\lambda_1 = 1, \qquad n_1 = (p-1)(s+2), \qquad \lambda_2 = 0, \qquad n_2 = (p-1)(p-s-1)$$

$$p^1_{jk} = \begin{pmatrix} p + (s+2)(s-1) & (s+1)(p-s-1) \\ (s+1)(p-s-1) & (p-s-1)(p-s-2) \end{pmatrix}^*$$

$$p^2_{jk} = \begin{pmatrix} (s+1)(s+2) & (s+2)(p-s-2) \\ (s+2)(p-s-2) & s + (p-s-2)^2 \end{pmatrix}$$

The efficiency factor of this design follows, on substituting in (3·71), as

$$\text{E. F.} = \frac{(p+1)(s+1)}{(p+1)(s+1)+(s+2)} \qquad \ldots \quad (3\cdot 90)$$

*It is convenient to give the values of p^1_{ij} and p^2_{ij} in matrix form. Thus p^1_{12} is the element in the first row, and the second column, in the matrix for p^1_{ij}.

Yates has worked out the efficiencies for the designs corresponding to $s=0$, 1 and $p-1$, namely $\frac{p+1}{p+3}$, $\frac{p+1}{p+2\frac{1}{2}}$ and $\frac{p}{p+1}$. In the last case the design becomes completely balanced, with $\lambda=1$.

10. $m=3$. Suppose n_3 is the biggest among the n's so that it is profitable to get rid of Σv_{i3}. We have then the following three equations involving v_i, Σv_{i1} and Σv_{i2}.

$$\left.\begin{aligned}
A_{13}\, v_i + B_{13}\, \Sigma v_{i1} + C_{13}\, \Sigma v_{i2} &= Q_i \\
A_{23}\, v_i + B_{23}\, \Sigma v_{i1} + C_{23}\, \Sigma v_{i2} &= \Sigma Q_{i1} \\
A_{33}\, v_i + B_{33}\, \Sigma v_{i1} + C_{33}\, \Sigma v_{i2} &= \Sigma Q_{i2}
\end{aligned}\right\} \qquad (3\cdot101)$$

where

$$\left.\begin{aligned}
A_{13} &= r(k-1)+\lambda_3 \\
A_{23} &= (\lambda_3-\lambda_1)\,(n_1-p^{3}_{11})-(\lambda_3-\lambda_2)\,p^{3}_{12} \\
A_{33} &= (\lambda_3-\lambda_2)\,(n_2-p^{3}_{22})-(\lambda_3-\lambda_1)\,p^{3}_{12} \\
B_{13} &= \lambda_3-\lambda_1 \\
B_{23} &= r(k-1)+\lambda_3+(\lambda_3-\lambda_1)\,(p^{1}_{11}-p^{3}_{11})+(\lambda_3-\lambda_2)\,(p^{1}_{12}-p^{3}_{12}) \\
B_{33} &= (\lambda_3-\lambda_1)\,(p^{1}_{12}-p^{3}_{12})+(\lambda_3-\lambda_2)\,(p^{1}_{22}-p^{3}_{22}) \\
C_{13} &= \lambda_3-\lambda_2 \\
C_{23} &= (\lambda_3-\lambda_1)\,(p^{2}_{11}-p^{3}_{11})+(\lambda_3-\lambda_2)\,(p^{2}_{12}-p^{3}_{12}) \\
C_{33} &= r(k-1)+\lambda_3+(\lambda_3-\lambda_1)\,(p^{2}_{12}-p^{3}_{12})+(\lambda_3-\lambda_2)\,(p^{2}_{22}-p^{3}_{22})
\end{aligned}\right\} \quad \ldots \quad (3\cdot102)$$

If we eliminate Σv_{i2}, the coefficients there, may be called A_{12}, A_{22}, etc. and if Σv_{i1} is eliminated we may write A_{11}, A_{21}, etc., with values corresponding to (3·102)

The estimate v_i of the effect of the i-th variety, is

$$v_i = k \begin{vmatrix} Q_i & B_{13} & C_{13} \\ \Sigma Q_{i1} & B_{23} & C_{23} \\ \Sigma Q_{i2} & B_{33} & C_{33} \end{vmatrix} \div \begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{23} & B_{23} & C_{23} \\ A_{33} & B_{33} & C_{33} \end{vmatrix} \quad \ldots \quad (3\cdot103)$$

Two alternative expressions for v_i can be obtained if n_1 or n_2 happens to be the greatest of the n's.

Let us denote by Δ the determinant of the denominator of (3·103). To get the variance of the difference between the effects of two varieties i and j, we have the following auxiliary solutions:

$$\left.\begin{aligned}
c_{ii} = c_{jj} &= \frac{k}{\Delta} \begin{vmatrix} 1-\frac{1}{v} & B_{13} & C_{13} \\ -\frac{n_1}{v} & B_{23} & C_{23} \\ -\frac{n_2}{v} & B_{33} & C_{33} \end{vmatrix} \\
c_{ij} &= \frac{k}{\Delta} \begin{vmatrix} \alpha & B_{13} & C_{13} \\ \beta & B_{23} & C_{23} \\ \gamma & B_{33} & C_{33} \end{vmatrix}
\end{aligned}\right\} \quad \ldots \quad (3\cdot104)$$

where

$$\alpha = -\frac{1}{v},\ -\frac{1}{v},\ -\frac{1}{v}$$

$$\beta = 1-\frac{n_1}{v},\ -\frac{n_1}{v},\ -\frac{n_1}{v}$$

$$\gamma = -\frac{n_2}{v},\ 1-\frac{n_2}{v},\ -\frac{n_2}{v}$$

according as i and j represent 1st, 2nd, 3rd associates respectively.

For two first associates,

$$V(v_i - v_j) = \frac{2k\,\sigma^2}{\Delta}\Big[C_{23}(B_{33}+B_{13}) - B_{23}(C_{33}+C_{13})\Big] \qquad \ldots \quad (3{\cdot}105)$$

For two second associates,

$$V(v_i - v_j) = \frac{2k\,\sigma^2}{\Delta}\Big[B_{33}(C_{23}+C_{13}) - C_{33}(B_{23}+B_{13})\Big] \qquad \ldots \quad (3.106)$$

For two third associates,

$$V\,(v_i - v_j) = \frac{2k\,\sigma^2}{\Delta}\Big[B_{23}\,C_{33} - B_{33}\,C_{23}\Big] \qquad \ldots \quad (3{\cdot}517)$$

The mean variance of all comparisons is

$$V_m = \frac{2k\,\sigma^2}{(v-1)\,\Delta}\cdot \begin{vmatrix} v-1 & B_{12} & C_{13} \\ -n_1 & B_{22} & C_{23} \\ -n_2 & B_{32} & C_{33} \end{vmatrix} = \frac{2c_{11}\,\sigma^2}{1-\dfrac{1}{v}} \qquad \ldots \quad (3{\cdot}108)$$

and efficiency factor is

$$\text{E. F.} = \frac{(v-1)\,\Delta}{rk} \div \begin{vmatrix} v-1 & B_{12} & C_{13} \\ -n_1 & B_{22} & C_{23} \\ -n_2 & B_{32} & C_{33} \end{vmatrix} = \frac{1-\dfrac{1}{v}}{r\,c_{11}} \qquad \ldots \quad (3{\cdot}109)$$

§4. Construction of Partially balanced incomplete block designs. Simple geometrical configurations.

1. Partially balanced incomplete block designs can be obtained by a variety of different methods. The method and designs given in this and the following sections, are intended to be merely illustrative. A complete enumeration of all possible designs with $r \leq 10$, $k \leq 10$ and a unified theory of methods of constructing them will be attempted in a later paper.

Many interesting designs are obtainable by considering simple and well known geometrical configurations. In fact the two-dimensional quasifactorial design of Yates in two groups of sets, belongs to this class. We have simply to consider two groups of p parallel straight lines. Then straight lines form the blocks, and their finite intersections the varieties. We shall now proceed to consider some other configurations.

(*i*) Consider the Pappus configuration of nine points and nine lines, illustrated by the following Fig. 1. Considering the lines as blocks and points as varieties we have the following nine blocks.

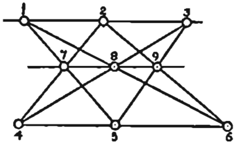

Fig-1.

(1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 7, 5), (2, 9, 6)
(1, 8, 6), (2, 7, 4), (3, 9, 5), (3, 8, 4).

The parameters are—

$$v=9, \quad b=9, \quad r=3, \quad k=3$$
$$\lambda_1=1, \quad n_1=6, \quad \lambda_2=0, \quad n_2=2$$
$$p^1_{ij}=\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}, \; p^2_{ij}=\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

E. F. = 8/11.

(*ii*) Consider the Desargue Configuration of ten points and ten lines, given by the following Fig. 2

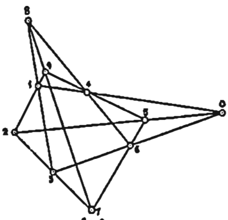

fig-2

We have the following ten blocks:

(0, 1, 4), (0, 2, 5), (0, 3, 6), (2 3, 7), (3, 1, 8), (1, 2, 9),
(5, 6, 7), (6, 4, 8), (4, 5, 9), (7, 8, 9).

The parameters are as follows :—

$$v=10, \quad b=10, \quad r=3, \quad k=3$$
$$\lambda_1=1, \quad n_1=6, \quad \lambda_2=0, \quad n_2=3$$
$$p^1_{ij}=\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$
$$\text{E. F.}=\frac{40}{57}$$

(iii) If ω is a complex cube root of unity, it is easy to see that the eight points with homogeneous coordinates (1, 0, 0) (0, 1, 1) (0, 1, 0) $(1, \omega, 0)$ $(1, 1, \omega^2)$ $(1, 0, \omega^2)$ (1, 1, 1) (0, 0, 1) lie three by three on eight straight lines. Calling these points 1, 2, 3, 4, 5, 6, 7, 8 respectively, and taking points for varieties and straight lines for blocks we have the following 8 blocks :—

(8, 1, 6), (1, 2, 7), (2, 3, 8), (3, 4, 1), (4, 5, 2), (5, 6, 3) (6, 7, 4), (7, 8, 1).

In this design the parameters are—

$$v=8, \quad b=8, \quad r=3, \quad k=3$$
$$\lambda_1=1, \quad n_1=6, \quad \lambda_2=0, \quad n_2=1$$
$$p^1_{ij}=\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\text{E. F.}=\frac{56}{75}$$

(iv) The simplest space configurations are provided by the regular polyhedra. We may get partially balanced incomplete block designs from these by considering the faces as blocks and points as varieties. We thus get the following six blocks using the simple configuration of Fig. 3 :—

(1, 2, 3, 4), (5, 6, 7, 8), (1, 4, 8, 5), (2, 3, 7, 6), (1, 2, 6, 5), (4, 3, 7, 8).

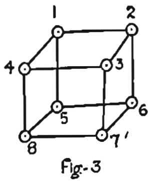

Fig-3

The parameters are as follows :—

$$v=8, \quad b=6, \quad r=3, \quad k=4$$
$$\lambda_1=2, \quad n_1=3, \quad \lambda_2=1, \quad n_2=3, \quad \lambda_3=0, \quad n_3=1$$
$$p^1_{ij}=\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad p^3_{ij}=\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\text{E. F.}=\frac{14}{17}$$

(*v*) Similarly by considering the octahedron of Fig. 4 we get the following blocks: (1, 2, 3), (4, 5, 6), (1, 3, 5), (4, 6, 2), (1, 5, 6), (4, 2, 3), (1, 6, 2), (4, 3, 5).

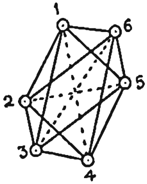

fig-4

The parameters are as follows:—

$$v=6, \quad b=8, \quad r=4, \quad k=3$$
$$\lambda_1=2, \quad n_1=4, \quad \lambda_2=0, \quad n_2=1$$
$$p^1_{ij}=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\text{E. F.} = \frac{10}{13}$$

(*vi*) Considering the icosahedron of Fig. 5 we get the following blocks:—

(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (1, 6, 2), (2, 3, 10), (3, 4 11), (4, 5, 7), (5, 6, 8), (6, 2, 9), (2, 9, 10), (3, 10, 11), (4, 11, 7), (5, 7, 8), (6, 8, 9), (7, 8, 12), (8, 9, 12), (9, 10, 12), (10, 11, 12), (11, 7, 12).

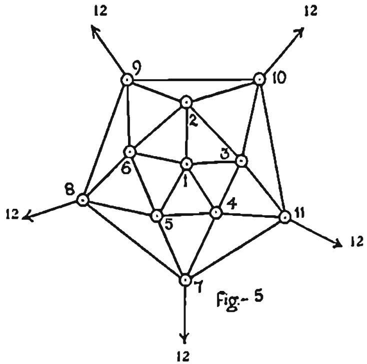

Fig.- 5

The parameters are as follows :—

$$v=12, \quad b=20, \quad r=5, \quad k=3$$

$$\lambda_1=2, \quad n_1=5, \quad \lambda_2=0, \quad n_2=6$$

$$p^1_{ij}=\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\text{E. F.} = \frac{242}{375}$$

(*vii*) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the faces of a tetrahedron with opposite vertices A_1, A_2, A_3, A_4 and similarly $\beta_1, \beta_2, \beta_3, \beta_4$ be the faces of another tetrahedron with vertices B_1, B_2, B_3, B_4. If A_1, A_2, A_3, A_4 lie on $\beta_1, \beta_2, \beta_3, \beta_4$ and B_1 lies on α_1, B_2 on α_2 and B_3 on α_3, then B_4 lies on α_4. Each of the two tetrahedra is both inscribed and circumscribed to the other. We thus get a configuration with 8 points and 8 planes. Take the points as varieties and the planes as blocks. We then get the following blocks :—

$(B_1\,A_2\,A_3\,A_4)$, $(A_1\,B_2\,A_3\,A_4)$, $(A_1\,A_2\,B_3\,A_4)$, $(A_1\,A_2\,A_3\,B_4)$
$(A_1\,B_2\,B_3\,B_4)$, $(B_1\,A_2\,B_3\,B_4)$, $(B_1\,B_2\,A_3\,B_4)$, $(B_1\,B_2\,B_3\,A_4)$

The parameters of this design are—

$$v=8, \quad b=8, \quad r=4, \quad k=4$$

$$\lambda_1=2, \quad n_1=6, \quad \lambda_2=0, \quad n_2=1$$

$$p^1_{ij}=\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{E. F.} = \frac{14}{17}$$

§5. Construction of Designs (continued). Application of Finite Geometry.

It has been seen in the last section that geometrical configurations in many instances lead to partially balanced incomplete block designs. The finite geometries $PG(N, p^n)$ and $EG(N, p^n)$ *i.e.*, the Projective and Euclidean N-dimensional geometries associated to the Galois field $GF(p^n)$ provide us with many interesting configurations leading to the desired type of designs[9].

The simplest example of a Galois field is provided by $GF(p)$ where p is prime. The elements are $0, 1, 2, \ldots\ldots\ldots\ldots p-1$. To add or multiply any two elements, we make the ordinary addition or multiplication and reduce the result (mod p).

Every element of $GF(p^n)$ for $n>1$ can be expressed in two forms, the additive and the multiplicative form. In the additive form the element is expressed as a polynomial of the $(n-1)$th degree, with integral coefficients less than p. In the multiplicative form every element other than the null element 0, is expressed in the form x^i $(i \leq p^n-2)$. To add two elements, we take them in the additive form, add as usual and reduce the coefficients (mod p). To multiply two elements we take them in the multiplicative form, multiply as usual and reduce by using the relation $x^{p^n-1}=1$. We give below the additive and multiplicative forms of the elements of GF (2^2), GF (2^3), GF (3^2), GF (2^4), GF(5^2) and GF (3^3) [see (1)].

(*i*) G F(2^2) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2=1+x$

(*ii*) G F(2^3) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2,\ \alpha_4=x^3=x^2+1,$
$\alpha_5=x^4=x^2+x+1,\ \alpha_6=x^5=x+1,\ \alpha_7=x^6=x^2+x.$

(*iii*) G F(3^2) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2=2x+1,\ \alpha_4=x^3=2x+2$
$\alpha_5=x^4=2,\ \alpha_6=x^5=2x,\ \alpha_7=x^6=x+2,\ \alpha_8=x^7=x+1$

(*iv*) G F(2^4) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2,\ \alpha_4=x^3,\ \alpha^5=x^4=x^3+1,$
$\alpha_6=x^5=x^4+x=x^3+x+1,\ \alpha_7=x^6=x^4+x^2+x=x^3+x^2+x+1,$
$\alpha_8=x^7=x^4+x^3+x^2+x=x^2+x+1,\ \alpha_9=x^8=x^3+x^2+x,$
$\alpha_{10}=x^9=x^4+x^3+x^2=x^2+1,\ \alpha_{11}=x^{10}=x^3+x,$
$\alpha_{12}=x^{11}=x^4+x^2=x^3+x^2+1,\ \alpha_{13}=x^{12}=x^4+x^3+x=x+1,$
$\alpha_{14}=x^{13}=x^2+x,\ \alpha_{15}=x^{14}=x^3+x^2.$

(*v*) G F(5^2) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2=3x+2,\ \alpha_4=x^3=3x^2+2x=x+1$
$\alpha_5=x^4=x^2+x=4x+2,\ \alpha_6=x^5=4x^2+2x=4x+3,$
$\alpha_7=x^6=4x^2+3x=3,\ \alpha_8=x^7=3x,\ \alpha_9=x^8=3x^2=4x+1,$
$\alpha_{10}=x^9=4x^2+x=3x+3,\ \alpha_{11}=x^{10}=3x^2+3x=2x+1,$
$\alpha_{12}=x^{11}=2x^2+x=2x+4,\ \alpha_{13}=x^{12}=2x^2+4x=4,$
$\alpha_{14}=x^{13}=4x,\ \alpha_{15}=x^{14}=4x^2=2x+3,$
$\alpha_{16}=x^{15}=2x^2+3x=4x+4,\ \alpha_{17}=x^{16}=4x^2+4x=x+3,$
$\alpha_{18}=x^{17}=x^2+3x=x+2,\ \alpha_{19}=x^{18}=x^2+2x=2,$
$\alpha_{20}=x^{19}=2x,\ \alpha_{21}=x^{20}=2x^2=x+4,$
$\alpha_{22}=x^{21}=x^2+4x=2x+2,\ \alpha_{23}=x^{22}=2x^2+2x=3x+4,$
$\alpha_{24}=x^{23}=3x^2+4x=3x+1.$

(*vi*) G F(3^3) :— $\alpha_0=0,\ \alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2,\ \alpha_4=x^3=x+2,\ \alpha_5=x^4=x^2+2x,$
$\alpha_6=x^5=x^3+2x^2=2x^2+x+2,\ \alpha_7=x^6=2x^3+x^2+2x=x^2+x+1,$
$\alpha_8=x^7=x^3+x^2+x=x^2+2x+2,\ \alpha_9=x^8=x^3+2x^2+2x=2x^2+2,$
$\alpha_{10}=x^9=2x^3+2x=x+1,\ \alpha_{11}=x^{10}=x^2+x,\ \alpha_{12}=x^{11}=x^3+x^2=x^2+x+2,$
$\alpha_{13}=x^{12}=x^3+x^2+2x=x^2+2,\ \alpha_{14}=x^{13}=x^3+2x=2,\ \alpha_{15}=x^{14}=2x,$
$\alpha_{16}=x^{15}=2x^2,\ \alpha_{17}=x^{16}=2x^3=2x+1,\ \alpha_{18}=x^{17}=2x^2+x,$
$\alpha_{19}=x^{18}=2x^3+x^2=x^2+2x+1,\ \alpha_{20}=x^{19}=x^3+2x^2+x=2x^2+2x+2,$
$\alpha_{21}=x^{20}=2x^3+2x^2+2x=2x^2+x+1,\ \alpha_{22}=x^{21}=2x^3+x^2+x=x^2+1,$
$\alpha_{23}=x^{22}=x^3+x=2x+2,\ \alpha_{24}=x^{23}=2x^2+2x,$
$\alpha_{25}=x^{24}=2x^3+2x^2=2x^2+2x+1,\ \alpha_{26}=x^{25}=2x^3+2x^2+x=2x^2+1.$

2. (*i*) Any ordered set of N elements

$$(x_1,\ x_2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ x_N)$$

belonging to GF (p^n) may be called a point of the finite N-dimensional Euclidean geometry EG (N, p^n). The number of points in EG (N, p^n) is clearly s^N where $s=p^n$.

All the points which satisfy a set of $N-m$, consistent and independent linear equations may be said to form an m-flat of EG (N, p^n) represented by these equations.

(*ii*) Again any ordered set of $N+1$ elements

$$(x_1, x_2, \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_{N+1})$$

where the x_i's belong to GF (p^n) and are not all simultaneously zero, may be called a point of the finite N-dimensional projective geometry PG (N, p^n), it being understood that the set $(x_1, x_2, \ldots\ldots\ldots x_{N+1})$ represents the same point as the set $(y_1, y_2, \ldots\ldots\ldots y_{N+1})$ when and only when there exists a non-zero element σ of GF (p^n) such that $y_i = \sigma x_i$ for $i = 1, 2, \ldots\ldots\ldots\ldots N+1$. The number of points in PG (N, p^n) is

$$s^N + s^{N-1} + \ldots\ldots\ldots\ldots\ldots\ldots s+1 = (s^{N+1}-1)/(s-1)$$

All the points which satisfy a set of $N-m$ independent linear homogeneous equations may be said to form an m-flat in PG (N, p^n) represented by these equations.

(*iii*) Whichever of the two geometries EG (N, p^n) or PG (N, p^n) we are considering, we may as usual call a 1-flat a line, and a 2-flat a plane. If we set

$$\phi(N, m, s) = \frac{(s^{N+1}-1)(s^N-1)(s^{N-1}-1)\ldots\ldots\ldots\ldots\ldots\ldots(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)(s^{m-1}-1)\ldots\ldots\ldots\ldots\ldots\ldots(s-1)}$$

then we can show that the number of m-flats in PG (N, p^n) is ϕ (N, m, s) and the number of m-flats in EG(N, p^n) is ϕ $(N, m, s) - \phi$ $(N-1, m, s)$.

From the space EG (N, p^n) let us cut out one point namely the origin $(0, 0, \ldots\ldots \ldots\ldots 0)$, and all the $(N-m)$-flats passing through this point. Let us take the retained $(N-m)$-flats as our blocks and the retained points as our varieties, a variety occurring in a block when and only when the corresponding point occurs on the corresponding $(N-m)$-flat.

(*a*) Consider in particular the case $N=2$, $m=1$. The number of retained points as well as of retained lines is s^2-1, where $s=p^n$. Hence $b=v=s^2-1$. On each of the retained lines there lie s points, and through each retained point there pass s retained lines, as the one joining the point to the origin is to be rejected. Thus $r=k=s$. Two points (varieties) are first or second associates according as the line joining them does not or does pass through the origin. To every retained point there are thus s^2-s first associates, and $s-2$ second associates. Thus $\lambda_1=1$, $n_1=s^2-s$, $\lambda_2=0$, $n_2=s-2$. Let O be the origin and P and Q be any two first associates. Then all points lying on lines other than P O and Q O are common first associates of P and Q. Thus $p^1_{11}=(s-1)^2$. In the same way we can find the values of other parameters of the second kind. We thus get a design in which

$$v=s^2-1, \quad b=s^2-1, \quad r=s, \quad k=s$$
$$\lambda_1=1, \; n_1=s^2-s, \; \lambda_2=0, \; n_2=s-2$$

$$p^1_{ij} = \begin{pmatrix} (s-1)^2 & s-2 \\ s-2 & 0 \end{pmatrix} \qquad p^2_{ij} = \begin{pmatrix} s^2-s & 0 \\ 0 & s-3 \end{pmatrix}$$

$$\text{E. F.} = \frac{(s^2-1)(s^2-2)}{(s^2-1)(s^2-2)+s^2-2(s+1)}$$

When $s=3$, we thus get a design which has been already otherwise obtained.

(*i*) Let $s=4$. We can then get a design with parameters—

$$v=15,\ b=15,\ r=4,\ k=4$$
$$\lambda_1=1,\ n_1=12,\ \lambda_2=0,\ n_2=2$$
$$p^1_{ij}=\begin{pmatrix} 9 & 2 \\ 2 & 0 \end{pmatrix},\qquad p^2_{ij}=\begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{E. F.}=\frac{35}{44}$$

We have now to consider the geometry EG $(2, 2^2)$. The co-ordinates of a point are of the form (x_1, x_2) where x_1, x_2 are elements $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ of GF (2^2). We have already seen how these elements are added and multiplied. The point $(\alpha_0=0,\ \alpha_0=0)$ has been cut out. The 15 retained lines then have the equations—

$$x_1=c,\ x_2=c,\ x_1=\alpha_1\,x_2+c,\ x_1=\alpha_2\,x_2+c,\ x_1=\alpha_3\,x_2+c\ \ (c=\alpha_1, \alpha_2, \alpha_3).$$

We thus get the following blocks, using for shortness the symbol *ij* for the point with co-ordinates (α_i, α_j).

(10,11,12,13),	(20,21,22,23),	(30,31,32,33)
(01,11,21,31),	(02,12,22,32),	(03,13,23,33)
(10,01,32,23),	(20,31,02,13),	(30,21,12,03)
(10,31,22,03),	(20,01,12,33),	(30,11,02,23)
(10,21,02,33),	(20,11,32,03),	(30,01,22,13)

(*ii*) Let $s=5$, we then get a design with parameters

$$v=24,\ b=24,\ r=5\ k=5$$
$$\lambda_1=1,\ n_1=20,\ \lambda_2=0,\ n_2=3$$
$$p^1_{ij}=\begin{pmatrix} 16 & 3 \\ 3 & 0 \end{pmatrix},\quad p^2_{ij}=\begin{pmatrix} 20 & 0 \\ 0 & 2 \end{pmatrix}$$
$$\text{E. F.}=\frac{552}{665}$$

We have then to consider the geometry EG (2,5). The co-ordinates of a point are of the form (x_1, x_2) where x_1, x_2 are the elements 0, 1, 2, 3, 4 of GF (5). The 24 retained lines have the equations—

$$x_1=c,\ x_2=c,\ x_1=x_2+c,\ x_1=2x_2+c,\ x_1=3x_2+c,\ x_1=4x_2+c\ \ (c=1, 2, 3, 4)$$

Using the symbol *ij* for the point (i, j) we therefore get the following blocks:

(10,11,12,13,14),	(20,21,22,23,24),	(30,31,32,33,34),	(40,41,42,43,44)
(01,11,21,31,41),	(02,12,22,32,42),	(03,13,23,33,43),	(04,14,24,34,44)
(10,21,32,43,04),	(20,31,42,03,14),	(30,41,02,13,24),	(40,01,12,23,34)
(10,31,02,23,44),	(20,41,12,33,04),	(30,01,22,43,14),	(40,11,32,03,24)
(10,41,22,03,34),	(20,01,32,13,44),	(30,11,42,23,04),	(40,21,02,33,14)
(10,01,42,33,24),	(20,11,02,43,34),	(30,21,12,03,44),	(40,31,22,13,04)

In the same way corresponding to the values $s=7,\ 8,\ 9$, we can construct h designs with parameters given below, by using the geometries EG (2, 7), EG $(2,\ 2^3)$, EG $(2,\ 3^2)$:—

(*iii*)

$$v=48,\ b=48,\ r=7,\ k=7$$
$$\lambda_1=1,\ n_1=42\ \lambda_2=0,\ n_2=5$$
$$p^1_{ij}=\begin{pmatrix} 36 & 5 \\ 5 & 6 \end{pmatrix},\quad p^2_{ij}=\begin{pmatrix} 42 & 0 \\ 3 & 4 \end{pmatrix};\ \text{E. F.}=\frac{753}{861}$$

(*iv*)

$$v=63,\ b=63,\ r=8,\ k=8$$
$$\lambda_1=1,\ n_1=56,\ \lambda_2=0,\ n_2=6$$
$$p^1_{ij}=\begin{pmatrix} 49 & 6 \\ 6 & 0 \end{pmatrix},\quad p^2_{ij}=\begin{pmatrix} 56 & 0 \\ 0 & 5 \end{pmatrix};\ \text{E. F.}=\frac{1953}{2200}$$

(*v*)

$$v=80,\ b=80,\ r=9,\ k=9$$
$$\lambda_1=1,\ n_1=72,\ \lambda_2=0,\ n_2=7$$
$$p^1_{ij}=\begin{pmatrix} 64 & 7 \\ 7 & 0 \end{pmatrix},\quad p^2_{ij}=\begin{pmatrix} 72 & 0 \\ 0 & 6 \end{pmatrix};\ \text{E. F.}=\frac{6320}{7029}$$

(*b*) Let us next consider the case $N=3$, $m=2$, *i.e.* we have to cut out from EG $(3,\ p^n)$ a point say (0, 0, 0) and planes passing through it, and then to identify our varieties with the retained points and blocks with the retained planes. For the design obtained in this manner, the parameters are (putting $s=p^n$) :—

$$v=s^3-1,\ b=s^3-1,\ r=s^2,\ k=s^2$$
$$\lambda_1=s,\ n_1=s^3-s,\ \lambda_2=0,\ n_2=s-2$$
$$p^1_{ij}=\begin{pmatrix} s^3-2s+1 & s-2 \\ s-2 & 0 \end{pmatrix},\qquad p^2_{ij}=\begin{pmatrix} s^3-s & 0 \\ 0 & s-3 \end{pmatrix}$$
$$\text{E. F.}=\frac{(s+1)\ (s^3-1)\ (s^3-2)}{(s+1)\ (s^3-1)\ (s^3-2)+s^3\ -2(s^2+s+1)}$$

As an example let us take $s=3$. The parameters become

$$v=26,\ b=26,\ r=9,\ k=9$$
$$\lambda_1=3,\ n_1=24,\ \lambda_2=0,\ n_2=1$$
$$p^1_{ij}=\begin{pmatrix} 22 & 1 \\ 1 & 0 \end{pmatrix},\qquad p^2_{ij}=\begin{pmatrix} 24 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\text{E. F.}=\frac{2600}{2817}$$

We have now to consider the geometry EG (3, 3). The co-ordinates of any point are of the form $(x_1,\ x_2,\ x_3)$ where $x_1,\ x_2,\ x_3$ are the elements of GF (3) *viz.* 0, 1, 2. The equations of the 26 retained planes can be written as

$$x_1=c,\ x_2=c,\ x_3=c,\ x_3=x_1+x_2+c,\ x_3=2x_1+x_2+c,$$
$$x_3=x_1+2x_2+c,\ x_3=2x_1+2x_2+c,\ x_1=x_2+c,\ x_1=2x_2+c,$$
$$x_2=x_3+c,\ x_2=2x_3+c,\ x_3=x_1+c,\ x_3=2x_1+c\ (c=1,2).$$

Using for shortness the symbol $i\ j\ k$ for the point whose co-ordinates are $(i,\ j,\ k)$ the 28 blocks come out as follows:—

(100,101,102,110,111,112,120,121,122),	(200,201,202,210,211,212,220,221,222)
(010,011,012,110,111,112,210,211,212),	(020,021,022,120,121,122,220,221,222)
(001,011,021,101,111,121,201,211,221),	(002,012,022,102,112,122,202,212,222)
(001,012,020,102,110,121,200,211,222),	(002,010,021,100,111,122,201,212,220)
(001,012,020,100,111,122,202,210,221),	(002,010,021,101,112,120,200,211,222)
(001,010,022,102,111,120,200,212,221),	(002,011,020,100,112,121,201,210,222)
(001,010,022,100,112,121,202,211,220),	(002,011,020,101,110,122,200,212,221)
(100,101,102,210,211,212,020,021,022),	(200,201,202,010,011,012,120,121,122)
(100,101,102,010,011,012,220,221,222),	(200,201,202,110,111,112,020,021,022)
(010,110,210,021,121,221,002,102,202),	(020,120,220,001,101,201,012,112,212)
(010,110,210,001,101,201,022,122,222),	(020,120,220,011,111,211,002,102,202)
(001,011,021,102,112,122,200,210,220),	(002,012,022,100,110,120,201,211,221)
(001,011,021,100,110,120,202,212,222),	(002,012,022,101,111,121,200,210,220)

3. Designs can also be formed by cutting out one point say (0, 0, ... 1) from the projective N-dimensional space PG $(N,\ p^n)$ and all $(N-m)$-flats passing through this point, and then taking the retained $(N-m)$-flats as our blocks and the retained points as our varieties. The designs thus arising in the particular case when $N=2,\ m=1$, shall arise otherwise in §7. We shall not consider them here.

(a) Let us consider the case $N=3,\ m=1$. The parameters of the design that we get, are (putting $s=p^n$):—

$$v=s^3+s^2+s,\ b=s^3,\ r=s^2,\ k=s^2+s+1$$

$$\lambda_1=s,\ n_1=s^3+s^2,\ \lambda_2=0,\ n_2=s-1$$

$$p^1_{ij}=\begin{pmatrix} s^3+s^2-s & s-1 \\ s-1 & 0 \end{pmatrix},\qquad p^2_{ij}=\begin{pmatrix} s^3+s^2 & 0 \\ 0 & s-2 \end{pmatrix}$$

$$\text{E. F.}=\frac{s(s+1)\ (s^3+s^2+s-1)}{s(s+1)\ (s^3+s^2+s-1)\ +(s^3-1)}$$

Thus if $s=2$, we have to cut out a single point (0, 0, 0, 1) from the projective 3-space PG (3, 2) and all planes through this point. The blocks are then given by retained planes, whose equations are

$$x_1+x_4=0,\ x_2+x_4=0,\ x_3+x_4=0,\ x_1+x_2+x_3+x_4=0$$

$$x_2+x_3+x_4=0,\ x_1+x_3+x_4=0,\ x_1+x_2+x_4=0\ \ x_4=0$$

Using the symbol $ijkl$ for the point with co-ordinates $(i,\ j,\ k,\ l)$ the 8 blocks can be written as

(0010,0100,0110,1001,1011,1101,1111),	(1000,0010,1010,0101,0111,1101,1111)
(0100,1000,1100,0011,0111,1011,1111),	(1100,1010,0110,0011,0101,1001,1111)
(0011,0101,0110,1011,1101,1110,1000),	(0011,1001,1010,0111,1101,1110,0100)
(0101,1001,1100,0111,1011,1110,0010),	(1110,0110,1010,1100,1000,0100,0010)

The parameters of this design are—

$$v=14,\ b=8,\ r=4,\ k=7$$
$$\lambda_1=2,\ n_1=12,\ \lambda_2=0\ n_2=1$$
$$p^1_{ij}=\begin{pmatrix}10 & 1\\ 1 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}12 & 0\\ 0 & 0\end{pmatrix}$$
$$\text{E. F.}=\frac{78}{85}$$

(*ii*) Let us consider the case $N=3$, $m=2$. The parameters of the design we then get are (putting $s=p^n$) :—

$$v=s^3+s^2+s,\ b=s^4+s^3+s^2,\ r=s^2+s,\ k=s+1$$
$$\lambda_1=1,\ n_1=s^3+s^2,\ \lambda_2=0,\ n_2=s-1$$
$$p^1_{ij}=\begin{pmatrix}s^3+s^2-s & s-1\\ s-1 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}s^3+s^2 & 0\\ 0 & s-2\end{pmatrix}$$
$$\text{E. F.}=\frac{(s^3+s^2+s)\,(s^3+s^2+s-1)}{(s^3+s^2+s)\,(s^3+s^2+s-1)+(2s^3-1)\,(s^2+s+1)-s^5}$$

Thus if $s=2$ we have to cut out a single point say (0, 0, 0, 1) from the 3-space PG (3, 2) and all the lines through this point. The blocks are then given by retained lines. We thus get the following blocks :—

(0100,0010,0110),	(1000,0100,1100),	(1000,0010,1010),	(0011,1101,1110)
(0101,1011,1110),	(0110,1101,1011),	(1001,1110,0111),	(1100,1011,0111)
(1010,1101,0111),	(0010,0101,0111),	(0100,0011,0111),	(1000,0011,1011)
(0010,1001,1011),	(1000,0101,1101),	(0100,1001,1101),	(0010,1100,1110)
(1000,0110,1110),	(0100,1010,1110),	(0011,1100,1111),	(0101,1010,1111)
(0110,1001,1111),	(1000,0111,1111),	(0100,1011,1111),	(0010,1101,1111)
(0011,0101,0110),	(0011,1001,1010),	(1100,1001,0101),	(1100,1010,0110)

The parameters of this design are—

$$v=14,\ b=28,\ r=6,\ k=3$$
$$\lambda_1=1,\ n_1=12,\ \lambda_2=0,\ n_2=1$$
$$p^1_{ij}=\begin{pmatrix}10 & 1\\ 1 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}12 & 0\\ 0 & 0\end{pmatrix}$$
$$\text{E. F.}=\frac{182}{255}$$

4. (*i*) From PG (3, p^n), let us cut out all points lying on a line, and all planes passing through this line. Let our varieties be identified with the retained points, and our blocks with the retained planes. Then we get a design with parameters (putting $s=p^n$) :—

$$v=s^3+s^2,\ b=s^3+s^2,\ r=s^2+s,\ k=s^2+s$$
$$\lambda_1=s+1,\ n_1=s^3,\ \lambda_2=s,\ n_2=s^2-1$$
$$p^1_{ij}=\begin{pmatrix}s^3-s^2 & s^2-1\\ s^2-1 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}s^3 & 0\\ 0 & s^2-2\end{pmatrix}$$
$$\text{E. F.}=\frac{s(s+2)\,(s^3+s^2-1)}{s(s+2)\,(s^3+s^2-1)+(s+1)^2\,(s-1)}$$

Take in particular $s=2$. Let us cut out points of the line $x_1=x_2=0$, and the planes passing through this line. The equations of the retained planes can then be written as—

$$x_1+x_4=0,\ x_3+x_4=0,\ x_2+x_4=0,\ x_1+x_2+x_3+x_4=0$$
$$x_2+x_3+x_4=0,\ x_1+x_3+x_4=0,\ x_1+x_2+x_4=0,\ x_4=0$$
$$x_1+x_2+x_3=0,\ x_2+x_3=0,\ x_1+x_3=0,\ x_3=0.$$

The 12 blocks are then given by

(0100,0110,1001,1011,1101,1111),	(1000,1010,0101,0111,1101,1111)
(0100,1000,1100,0111,1011,1111),	(1100,1010,0110,0101,1001,1111)
(0101,0110,1011,1101,1110,1000),	(1001,1010,0111,1101,1110,0100)
(0101,1001,1100,0111,1011,1110),	(1110,0110,1010,1100,1000,0100)
(0110,1010,1100,0111,1011,1101),	(1000,1001,0110,0111,1110,1111)
(0100,0101,1010,1011,1110,1111),	(1000,0100,0101,1001,1100,1101)

The parameters of the above design are—

$$v=12,\ b=12,\ r=6,\ k=6$$
$$\lambda_1=3,\ n_1=8,\ \lambda_2=2,\ n_2=3$$
$$p^1_{ij}=\begin{pmatrix}4 & 3\\ 3 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}8 & 0\\ 0 & 2\end{pmatrix}$$
$$\text{E. F.}=\frac{88}{97}$$

(*ii*) Again from PG (3, p^n), let us cut out all points lying on a line, and all lines passing through points of this line. Let us identify the varieties with points and blocks with straight lines.

$$v=s^3+s^2,\ b=s^4,\ r=s^2,\ k=s+1$$
$$\lambda_1=1,\ n_1=s^3,\ \lambda_2=0,\ n_2=s^2-1$$
$$p^1_{ij}=\begin{pmatrix}s^3-s^2 & s^2-1\\ s^2-1 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}s^3 & 0\\ 0 & s^2-2\end{pmatrix}$$
$$\text{E. F.}=\frac{s(s^3+s^2-1)}{s(s^3+s^2-1)+(s-1)\ (s+1)^2}$$

where, as before, $s=p^n$.

(*a*) Taking in particular $s=2$, and cutting out the points of the line $x_1=x_2=0$, and all lines passing through the points of this line, we get the following 16 blocks.

(1000,0100,1100),	(0101,1011,1110),	(0110,1101,1011),	(1001,1110,0111)
(1100,1011,0111),	(1010,1101,0111),	(1000,0101,1101),	(0100,1001,1101)
(1000,0110,1110),	(0100,1010,1110),	(0101,1010,1111),	(0110,1001,1111)
(1000,0111,1111),	(0100,1011,1111),	(1100,1001,0101),	(1100,1010,0110)

The parameters of the above design are—

$$v=12,\ b=16,\ r=4,\ k=3$$
$$\lambda_1=1,\ n_1=8,\ \lambda_2=0,\ n_2=3$$
$$p^1_{ij}=\begin{pmatrix}4 & 3\\ 3 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}8 & 0\\ 0 & 2\end{pmatrix}$$
$$\text{E. F.}=\frac{22}{31}$$

(*b*) In the same way if we take $s=3$, we can by using the projective space PG (3, 3) construct a design with the following parameters:

$$v=36,\ b=81;\ r=9,\ k=4$$
$$\lambda_1=1,\ n_1=27,\ \lambda_2=0,\ n_2=8$$
$$p^1_{ij}=\begin{pmatrix}18 & 8\\ 8 & 0\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}27 & 0\\ 0 & 7\end{pmatrix}$$
$$\text{E. F.}=\frac{105}{137}$$

5. From PG $(2, p^n)$ let us cut out all points on three non-concurrent lines, and all lines through the points of intersection of these lines two by two. Let us identify the blocks with retained straight lines and the varieties with retained points. We then get a design in which

$$v=(s-1)^2,\ b=(s-1)^2,\ r=s-2,\ k=s-2$$
$$\lambda_1=1,\ n_1=(s-2)(s-3),\ \lambda_2=0,\ n_2=3(s-2)$$
$$p^1_{ij}=\begin{pmatrix}s^2-8s+17 & 3(s-4)\\ 3(s-4) & 6\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}(s-3)(s-4) & 2(s-3)\\ 2(s-3) & s-1\end{pmatrix}$$
$$\text{E. F.}=\frac{s(s-1)^2(s-3)(s-4)}{(s-2)^2\,[s^2(s-4)+3]}$$

(*i*) As a particular case let us take $s=5$. We then get a design with the parameters

$$v=16,\ b=16,\ r=3,\ k=3,$$
$$\lambda_1=1,\ n_1=6,\ \lambda_2=0,\ n_2=9$$
$$p^1_{ij}=\begin{pmatrix}2 & 3\\ 3 & 6\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}2 & 4\\ 4 & 4\end{pmatrix}$$
$$\text{E. F.}=\frac{40}{63}$$

We have to consider the geometry PG (2, 5). The co-ordinates of every point are of the form (x_1, x_2, x_3) where x_1, x_2, x_3 are elements of GF (5) *viz.* 0, 1, 2, 3, 4. Let us cut out the three lines $x_1=0$, $x_2=0$, $x_3=0$, all points on these three lines, and all lines through the points (1, 0, 0), (0, 1, 0), (0, 0, 1). The equations of the 16 retained lines can then be put in the form

$$x_3=a\,x_1+b\,x_2 \qquad a,\ b=1, 2, 3, 4$$

We must get the following 16 blocks—

(112,123,134),	(113,132,144),	(114,122,143),	(124,133,142)
(113,124,141),	(114,121,133),	(123,131,144),	(111,134,143)
(114,131,142),	(122,134,141),	(111,124,132),	(112,121,142)
(121,132,143),	(111,123,142),	(112,133,141),	(113,122,131)

In the same way putting $s=7$, 8, 9, 11 respectively we can derive designs with parameters given below.

(*ii*)
$$v=36,\ b=36,\ r=5,\ k=5$$
$$\lambda_1=1,\ n_1=20,\ \lambda_2=0,\ n_2=15$$
$$p^1_{ij}=\begin{pmatrix}10 & 9\\ 9 & 6\end{pmatrix}\quad p^2_{ij}=\begin{pmatrix}12 & 8\\ 8 & 6\end{pmatrix};\ \text{E. F.}=\frac{504}{625}$$

(iii)

$$v=49,\ b=49,\ r=6,\ k=6$$
$$\lambda_1=1,\ n_1=30,\ \lambda_2=0,\ n_2=18$$
$$p^1_{ij}=\begin{pmatrix}17 & 12\\ 12 & 6\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}20 & 10\\ 10 & 7\end{pmatrix};\ \text{E. F.}=\frac{1960}{2331}$$

(iv)

$$v=64,\ b=64,\ r=7,\ k=7$$
$$\lambda_1=1,\ n_1=42,\ \lambda_2=0,\ n_2=21$$
$$p^1_{ij}=\begin{pmatrix}26 & 15\\ 15 & 6\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}30 & 12\\ 12 & 8\end{pmatrix};\ \text{E. F.}=\frac{720}{833}$$

(v)

$$v=100,\ b=100,\ r=9,\ k=9$$
$$\lambda_1=1,\ n_1=72,\ \lambda_2=0,\ n_2=27$$
$$p^1_{ij}=\begin{pmatrix}50 & 21\\ 21 & 6\end{pmatrix},\quad p^2_{ij}=\begin{pmatrix}56 & 16\\ 16 & 10\end{pmatrix};\ \text{E. F.}=\frac{1232}{1377}$$

§6. CONSTRUCTION OF DESIGNS (CONTINUED).

THE METHOD OF DIFFERENCES.

1. The method of differences has been extensively used by one of the authors (R. C. Bose) to obtain balanced incomplete block designs[8]. We shall give here the application of this method to the construction of partially balanced incomplete block designs, for the simplest case, namely, $b=v$, $k=r$. Other applications of this method will be discussed in a subsequent communication.

A set of elements is said to form a modul M, when there exists a law of composition, *viz.* the addition, denoted by $+$, satisfying the following axioms:—

(i) To any two elements a and b of M, there exists a unique element s of M defined by $a+b=s$.

(ii) $a+b=b+a$.

(iii) $a+(b+c)=(a+b)+c$.

(iv) To any two elements a and b of M there exists an element x belonging to M, satisfying $a+x=b$.

On the basis of these axioms we can prove that the element x in (iv) is unique. Also there exists a unique element 0, with the property that c being any element of M, $c+0=c$. If $c+d=0$, we denote d by $-c$. $a+(-c)$ may be denoted by $a-c$. The element x in (iv) is then equal to $b-a$, and may be said to be the difference of b and a.

2. Consider a finite modul with exactly v elements. Suppose it is possible to find k different elements,

$$x_1,\ x_2,\ \ldots\ldots\ldots\ x_k$$

out of the v elements of M satisfying the following conditions:—

(i) Among the $k\ (k-1)$ differences x_i-x_j $(i,\ j=1,\ 2,\ \ldots\ldots\ k,\ i\neq j)$, just n_i of the non-zero elements of M are repeated λ_i times $(i=1,\ 2,\ \ldots\ m)$. Clearly in this case

$$n_1+n_2+\ldots\ldots\ldots\ldots\ldots+n_m=v-1$$
$$n_1\lambda_1+n_2\lambda_2+\ldots\ldots\ldots\ldots\ldots+n_m\lambda_m=k(k-1)$$

(*ii*) Denote by $\alpha^i_1, \alpha^i_2 \ldots\ldots \alpha^i_{n_i}$, the n_i elements of M, which occur just λ_i times among the differences $x_i - x_j$ ($i, j = 1, 2 \ldots\ldots k$, $i \neq j$). Then among the $n_i(n_i - 1)$ differences $\alpha^i_u - \alpha^i_w$ ($u, w = 1, 2 \ldots\ldots n_i$, $w \neq u$) every number of the set $\alpha_1^l, \alpha_2^l \ldots\ldots \alpha^l_{n_l}$ should be repeated exactly p^l_{ii} times ($l = 1, 2 \ldots\ldots m$). Also among the $n_i n_j$ differences $\alpha^i_u - \alpha^j_w$ ($u = 1, 2 \ldots\ldots n_i$, $w = 1, 2 \ldots\ldots n_j$), the numbers of the set $\alpha^l_1\ \alpha^l_2 \ldots\ldots \alpha^l_{n_l}$ occur exactly p^l_{ij} times ($l = 1, 2 \ldots\ldots m$; $i, j = 1, 2 \ldots\ldots m$, $i \neq j$).

When these conditions are satisfied, we shall show that the design in which the v varieties are v elements of M, and the v blocks are

$$x_1 + \theta,\ x_2 + \theta, \ldots\ldots\ldots\ldots x_k + \theta \qquad \ldots \quad (6\cdot21)$$

where θ is anyone of the elements of M, is a partially balanced incomplete block design with v, $b = v$, k, $r = k$, n_i, λ_i as the parameters of the first kind and p^l_{ij} as the parameters of the second kind.

Since to each element θ of M, there corresponds one block, we may call (6·21), the block θ. A variety c occurs in a block θ, if and only if we can find an i such that $x_i + \theta = c$. Given i this equation uniquely fixes θ. Hence c occurs in the blocks $c - x_1, c - x_2, \ldots\ c - x_k$, and in these blocks only. Hence every variety is replicated exactly k times.

Two varieties c and d will occur together in the same block θ, if we can find u, v and θ satisfying

$$x_u + \theta = c, \qquad x_v + \theta = d \qquad \ldots \quad (6\cdot22)$$

Then $x_u - x_v = c - d$. The number $c - d$ belongs to one and only one of the sets $(\alpha^1_1, \alpha^1_2 \ldots \ldots \alpha^1_{n_1})$, $(\alpha^2_1, \alpha^2_2 \ldots\ldots \alpha^2_{n_2}) \ldots\ldots (\alpha^m_1, \alpha^m_2, \ldots\ldots \alpha^m_{n_m})$. Let it belong to $(\alpha^i_1, \alpha^i_2, \ldots\ldots \alpha^i_{n_i})$. Then from the condition (*i*) we shall be able to find exactly λ_i pairs of numbers (u, v) such that $x_u - x_v = c - d$. When u and v are fixed, θ is fixed and is given by $\theta = c - x_u = d - x_v$. Thus there are exactly λ_i blocks in which the varieties c and d occur together. We thus see that the varieties c and d are i-associates if $c - d$ belongs to the set $(\alpha^i_1, \alpha^i_2, \ldots\ldots \alpha^i_{n_i})$

Given that c and d are i-associates, let us find the number of varieties which are i-associates to c and j-associates to d, ($i \neq j$). If t is an i-associate to c and j-associate to d then

$$c - t = \alpha^i, \qquad d - t = \alpha^j \qquad \ldots \quad (6\cdot23)$$

where α^i is some number of the set $(\alpha^i_1, \alpha^i_2, \ldots\ldots \alpha^i_{n_i})$ and α^j is some number of the set $(\alpha^j_1, \alpha^j_2, \ldots\ldots \alpha^j_{n_j})$. Then $\alpha^i - \alpha^j = c - d =$ a number of the set $(\alpha^l_1, \alpha^l_2, \ldots\ldots \alpha^l_{n_l})$ Hence from the condition (*ii*), we can find the pair (α^i, α^j) in exactly p^l_{ij} ways. Then t is determined by any one of the equations (6·23). Thus if two varieties are l-associates, then the number of varieties common to the i-associates of one and the j-associates of the other, is exactly p^l_{ij}. In the same way we can prove that if two varieties are l-associates, then the number of varieties common to the i-associates of the two respective varieties is exactly p^l_{ii}.

3. The simplest example of a modul is the following. Let v be any positive integer. Then the elements of the modul are the integers

$$0, 1, 2, \ldots\ldots\ldots\ldots v - 1$$

To add any two elements we proceed as usual, but always reduce the result (mod v). Thus we say that $a+b=s$ when and only when $a+b\equiv s$ (mod v). This modul may be called the modul of the classes of residues (mod v).

(*i*) Let $v=15$. Consider the modul of the classes of residues (mod 15). Thus our 15 varieties are 0, 1, 2, 3, . . . 14. Let

$$x_1=1,\ x_2=2,\ x_3=4,\ x_4=8$$

Then the 12 differences x_i-x_j ($i, j=1, 2, 3, 4, i\neq j$) are 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14. Denote these by $\alpha^1_1, \alpha^1_2 \ldots \alpha^1_{12}$. Denote 5 and 10 by α^2_1, α^2_2 respectively. Thus the numbers of the set ($\alpha^1_1, \alpha^1_2, \ldots\ldots \alpha^1_{12}$) occur once and the numbers of the set (α^2_1, α^2_2) occur zero times in the differences x_i-x_j. Call these sets, the sets I and II respectively. Hence $\lambda_1=1,\ \lambda_2=0,\ n_1=12,\ n_2=2$.

Now among the 132 differences $\alpha^1_u-\alpha^1_w$ ($u, w=1, 2,\ldots\ldots 12, u\neq w$), the numbers of the set I each occurs 0 times, and the numbers of the set II each occurs 12 times. Among the 2 differences $\alpha^2_u-\alpha^2_w$ ($u, w, =1, 2, u\neq w$), each number of the set I occurs 0 times, and each number of the set II occurs once. Finally in the 24 differences $\alpha^1_u-\alpha^2_w$ ($u=1, 2, \ldots\ldots\ldots 12, w=1, 2$) each number of the set I occurs twice, and each number of the set II occurs zero times. Thus by taking the 15 blocks

$$1+\theta, \quad 2+\theta, \quad 4+\theta, \quad 8+\theta$$

$\theta=0, 1,\ldots\ldots\ldots 14$, we get a design with the parameters

$$v=15,\ b=15,\ r=4,\ k=4$$
$$\lambda_1=1,\ n_1=12,\ \lambda_2=0,\ n_2=2$$
$$p^1_{ij}=\begin{pmatrix} 9 & 2 \\ 2 & 0 \end{pmatrix}, \quad p^2_{ij}=\begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{E. F.} = \frac{35}{44}$$

The complete design can be written as follows :—

(1,2,4,8), (2,3,5,9), (3,4,6,10) (4,5,7,11), (5,6,8,12), (6,7,9,13), (7,8,10,14), (8,9,11,0), (9,10, 12,1), (10,11,13,2), (11,12,14,3), (12,13,0,4), (13,14,1,5), (14,0,2,6), (0,1,3,7).

(*ii*) Let $v=31$. Consider the modul of the classes of residues (mod 31). Then our 31 varieties are 0, 1, 2, 3,............30. Let $x_1=1,\ x_2=2,\ x_3=4,\ x_4=8,\ x_5=15,\ x_6=16,\ x_7=23,\ x_8=27,\ x_9=29,\ x_{10}=30$.

Then among the 90 differences x_i-x_j ($i, j=1, 2,\ldots\ldots\ldots 10, i\neq j$), the numbers of the set (3, 6, 7, 12, 14, 17, 19, 24, 25, 28) each occurs 4 times, the numbers of the set (1, 2, 4, 8, 15, 16, 23, 27, 29, 30) each occurs 3 times and the numbers of the set (5, 9, 10, 11, 13, 18, 20, 21, 22, 26) each occurs twice. Call these sets, the sets I, II, III, respectively.

Among the 90 differences $\alpha-\alpha'$ where α and α' are any two elements of the set I, every number of the set I occurs thrice, every number of the set II occurs twice, and every number of the set III occurs 4 times. Again among the 100 differences $\beta-\gamma$ where β is any number of the set II, and γ any number of the set III, the numbers of the sets I and III occur 4 times each, while the numbers of the set II occur twice. Similar other results hold for differences of other types. Hence by taking the 31 blocks

$$1+\theta,\ 2+\theta,\ 4+\theta,\ 8+\theta,\ 15+\theta,\ 16+\theta,\ 23+\theta,\ 27+\theta,\ 29+\theta,\ 30+\theta, \qquad \text{(mod.31)}$$

where $\theta=0, 1, 2, \ldots\ldots\ldots\ldots 30$, we get a design with the parameters

$$v=31,\ b=31,\ r=10,\ k=10$$
$$\lambda_1=4,\ n_1=10,\ \lambda_2=3,\ n_2=10,\ \lambda_3=2,\ n_3=10$$

$$p^1{}_{ij}=\begin{vmatrix} 3 & 2 & 4 \\ 2 & 4 & 4 \\ 4 & 4 & 2 \end{vmatrix} \quad p^2{}_{ij}=\begin{vmatrix} 2 & 4 & 4 \\ 4 & 3 & 2 \\ 4 & 2 & 4 \end{vmatrix} \quad p^3{}_{ij}=\begin{vmatrix} 4 & 4 & 2 \\ 4 & 2 & 4 \\ 2 & 4 & 3 \end{vmatrix}$$

$$\text{E. F.} = \frac{100102}{108005}$$

(*iii*) Let $v=29$. Consider the modul of residue classes (mod 29).

Then our varieties are $0, 1, 2, \ldots\ldots\ldots\ldots\ldots 28$. Let

$$x_1=0,\ x_2=1,\ x_3=7,\ x_4=16,\ x_5=20,\ x_6=23,\ x_7=24,\ x_8=25$$

Then among the 56 differences x_i-x_j $(i, j,=1, 2, \ldots\ldots\ldots 8,\ i\neq j)$, the numbers of the set (1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28) occur three times each, while of the numbers of the set (2, 3, 8, 10, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27) each occurs once. Call these sets the sets I and II respectively. Then among the 182 differences of the type $\alpha-\alpha'$, where α and α' are any two different numbers of the set I, each number of the set I occurs 6 times, and each number of the set II occurs 7 times. Among the 182 differences of the type $\beta-\beta'$, where β and β' are any two different numbers of the set II, each number of the set I occurs 7 times, while each number of the set II occurs 6 times. Finally among the 196 differences of the type $\alpha-\beta$, where α is a number of the set I and β is a number of the set II, each number of the set I, as well as each number of the set II, occurs exactly 7 times. Thus by taking the 29 blocks

$$\theta,\ 1+\theta,\ 7+\theta,\ 16+\theta,\ 20+\theta,\ 23+\theta,\ 24+\theta,\ 25+\theta, \qquad \text{(mod 29)}$$

$\theta=0, 1, 2, 3, \ldots\ldots\ldots\ldots 28$, we get a design with parameters

$$v=29,\ b=29,\ r=8,\ k=8$$
$$\lambda_1=3,\ n_1=14,\ \lambda_2=1,\ n_2=14$$
$$p^1{}_{ij}=\begin{pmatrix} 6 & 7 \\ 7 & 7 \end{pmatrix}, \quad p^2{}_{ij}=\begin{pmatrix} 7 & 7 \\ 7 & 6 \end{pmatrix}$$
$$\text{E. F.} = \frac{3335}{3712}$$

§7. Construction of Designs (Continued). Miscellaneous Methods.

1. If the number of varieties be a factorisable number, it can be expressed in the form $v=pq$. Form a rectangular lattice with these varieties, having p rows and q columns. A design can be obtained with pq blocks. Every block has a variety associated with it and

will comprise that variety and all varieties placed in the same row and column as that variety. The following will be the parameters of the design. We assume that $p>q>2$.

$$v=b=pq, \qquad r=k=p+q-1$$
$$\lambda_1=p \qquad n_1=p-1$$
$$\lambda_2=q \qquad n_2=q-1$$
$$\lambda_3=2 \qquad n_3=(p-1)(q-1)$$

$$p^1_{jk}=\begin{vmatrix} p-2 & 0 & 0 \\ 0 & 0 & q-1 \\ 0 & q-1 & (p-2)(q-1) \end{vmatrix} \qquad p^2_{jk}=\begin{vmatrix} 0 & 0 & p-1 \\ 0 & q-2 & 0 \\ p-1 & 0 & (p-1)(q-2) \end{vmatrix}$$

$$p^3_{jk}=\begin{vmatrix} 0 & 1 & p-2 \\ 1 & 0 & q-2 \\ p-2 & q-2 & (p-2)(q-2) \end{vmatrix}$$

If $q=2$, this degenerates into a design with only two associate-classes. The parameters are:

$$v=b=2p, \qquad r=k=p+1$$
$$\lambda_1=p \qquad n_1=p-1$$
$$\lambda_2=2 \qquad n_2=p$$

$$p^1_{jk}=\begin{pmatrix} p-2 & 0 \\ 0 & p \end{pmatrix} \qquad p^2_{jk}=\begin{pmatrix} 0 & p-1 \\ p-1 & 0 \end{pmatrix}$$

$$\text{E. F.} = \frac{4p\,(p+2)\,(2p-1)}{3\,(p+1)^2\,(3p-2)}$$

2. If in the above (three λ) designs, we had formed blocks by taking all varieties in the same row and column as that variety, excepting itself, the parameters will be

$$v=b=pq, \qquad r=k=p+q-2$$
$$\lambda_1=p-2 \qquad n_1=p-1$$
$$\lambda_2=q-2 \qquad n_2=q-1$$
$$\lambda_3=2 \qquad n_3=(p-1)(q-1)$$

The parameters of the second kind will be the same as in the previous design.

When p or $q=4$, the design degenerates into a design with two λ's.

Thus, let $v=4p$ ($p>4$). The parameters then are:

$$v=b=4p, \qquad r=k=p+2$$
$$\lambda_1=p-2 \qquad n_1=p-1$$
$$\lambda_2=2 \qquad n_2=3p$$

$$p^1_{jk}=\begin{pmatrix} p-2 & 0 \\ 0 & 3p \end{pmatrix}, \qquad p^2_{jk}=\begin{pmatrix} 0 & p-1 \\ p-1 & 2p \end{pmatrix}$$

$$\text{E. F.} = \frac{8p\,(p+4)\,(4p-1)}{5\,(p+2)^2\,(7p-4)}$$

Let $v=4p$ $(p<4)$. Then the parameters are:

$$v=b=4p, \qquad r=k=p+2$$
$$\lambda_1=2 \qquad n_1=3p$$
$$\lambda_2=p-2 \qquad n_2=p-1$$

$$p^1_{jk}=\begin{pmatrix} 2p & p-1 \\ p-1 & 0 \end{pmatrix} \qquad p^2_{jk}=\begin{pmatrix} 3p & 0 \\ 0 & p-2 \end{pmatrix}$$

E.F. is the same as for $p>4$.

There are only 2 designs in this series, as p can take only two values, namely, 2 and 3.

3. If v is a perfect square $(=p^2)$ we can get designs by forming blocks such that with respect to every variety we form a block with all the varieties occurring in the same row, column and having the same Latin letter as itself in each of s orthogonalised squares $(s=0, 1, 2, \ldots\ldots\ldots\ldots p-1$, according to the properties of $p)$. If each variety is included in the block associated with it, the parameters of the design are:—

$$v=b=p^2 \qquad r=k=(s+2)\,p-(s+1)$$
$$\lambda_1=p+s\,(s+1) \qquad n_1=(s+2)\,(p-1)$$
$$\lambda_2=(s+1)\,(s+2) \qquad n_2=(p-1)\,(p-s-1)$$

$$p^1_{jk}=\begin{pmatrix} p+(s+2)\,(s-1), & (s+1)\,(p-s-1) \\ (s+1)\,(p-s-1), & (p-s-1)\,(p-s-2) \end{pmatrix}, \quad p^2_{jk}=\begin{pmatrix} (s+1)\,(s+2), & (s+2)\,(p-s-2) \\ (s+2)\,(p-s-2), & (p-s-2)^2+s \end{pmatrix}$$

The E. F. takes a complicated form, but, in special cases, is easily calculated. Thus for $p=3$, $s=1$, E. F. $=2880/2989$.

This design will degenerate into a series of Yates' balanced incomplete designs, if $p=2(s+1)$ *i.e.*, $p=2^n$, $s=2^{n-1}-1$. For other values of p and s satisfying this condition, no design is known to exist. Thus $s=2$, $p=6$ satisfies the condition $p=2(s+1)$, but the 6×6 square has no Græco-Latin square and so the design does not exist.

4. If in the above design, the variety associated with each block is cut out from it, the parameters become—

$$v=b=p^2, \qquad r=k=(s+2)\,(p-1)$$
$$\lambda_1=p-2+s\,(s+1) \qquad n_1=(s+2)\,(p-1)$$
$$\lambda_2=(s+1)\,(s+2) \qquad n_2=(p-1)\,(p-s-1)$$

and the parameters of the second kind remain unaltered.

The E. F. takes a complicated form, but, in special cases, is easily calculated. Thus for $p=3$, $s=1$, E. F. $=108/117$.

This will degenerate into a series of Yates' balanced incomplete designs if

$$p=2(s+2).$$

Designs are possible only when $s=1$, $p=6$ and when $s=2^{n-1}-2$ and $p=2^n$.

When $s=0$, the designs for $v=p^2$ discussed in §7·3 and §7·4 follow as special cases of the designs for $v=pq$ discussed in §7·1 and §7·2, by putting $p=q$.

5. Let $v=pq$, where $p>q>1$ and also p should be such that q orthogonalised Latin squares exist for $p \times p$ squares.

Let the varieties be arranged as follows:—

$$\begin{array}{lll} 1, & 2, \ldots\ldots\ldots & \ldots\ldots\ldots p \\ p+1, & p+2, \ldots\ldots\ldots\ldots\ldots & 2p \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ (q-1)\,p+1, & (q-1)\,p+2 \ldots\ldots\ldots\ldots & qp \end{array}$$

In the first orthogonalised Latin square replace the p letters by the varieties $1, 2, \ldots\ldots\ldots p$, in the second square by the varieties $p+1$, $p+2$, ... $2p$, etc., and lastly in the q^{th} square by the varieties $(q-1)\,p+1$, $(q-1)\,p+2$, ..., qp. Now superimpose on the first square, the remaining $q-1$ squares. Each cell will now contain q varieties which will be a block of our design, with the following parameters:—

$$v=pq,\ b=p^2,\ r=p,\ k=q$$

$$\lambda_1=1 \qquad n_1=p\,(q-1)$$

$$\lambda_2=0 \qquad n_2=p-1$$

$$p^1_{jk}=\begin{pmatrix} p\,(q-2) & p-1 \\ p-1 & 0 \end{pmatrix}, \quad p^2_{jk}=\begin{pmatrix} p\,(q-1) & 0 \\ 0 & p-2 \end{pmatrix}$$

$$\text{E. F.} = \frac{(q-1)\,(pq-1)}{(q-1)\,(pq-1)+q(p-1)}$$

6. Let $v=pq$ $(p>2,\ q>1)$. Keep the v varieties in p sets of q varieties each. Label the sets by p letters. Arrange the letters in the form of a $p \times p$ Latin square. Strike off any row (or column) of this square. Take as blocks the p columns (or rows) each containing $p-1$ letters, and so $q(p-1)$ varieties. The parameters are:

$$v=pq,\ b=p,\ r=p-1,\ k=q(p-1)$$

$$\lambda_1=p-1 \qquad n_1=q-1$$

$$\lambda_2=p-2 \qquad n_2=q\,(p-1)$$

$$p^1_{jk}=\begin{pmatrix} q-2 & 0 \\ 0 & q(p-1) \end{pmatrix}, \quad p^2_{jk}=\begin{pmatrix} 0 & q-1 \\ q-1 & q(p-2) \end{pmatrix}$$

$$\text{E. F.} = \frac{p(p-2)\,(pq-1)}{p(p-2)\,(pq-1)+(p-1)}$$

If p is a prime or power of a prime and if we write the $p \times p$ Latin square in the standard form, it is possible in some cases to get designs with 2 or more λ's, even if more than one row or column is struck off from the square.

7. Many designs can be obtained by the *principle of duality* between blocks and varieties. In a known design of our type, or, of Yates' type, let us number the varieties as

1, 2, ... v and the blocks as 1, 2, ... b. Now, calling variety 1, as block 1, etc., and *vice versa* we can in some cases get a design with v blocks, b varieties, r plots per block and k replications of each variety.

Designs with $b=v$ are self-dual with respect to blocks and varieties, so that we will not get new designs from them.

The designs obtained by inverting some of the published designs of Yates, with $b \neq v$, are given below.

	Yates' design	Our design, on inversion
(1)	$v=6,\ b=10,\ r=5,\ k=3$ $\lambda=2$	$v=10,\ b=6,\ r=3\ \ k=5$ $\lambda_1=2,\ \lambda_2=1,\ n_1=3,\ n_2=6$ $p^1_{jk}=\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix},\ p^2_{jk}=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$; E. F. $=\frac{36}{41}$
(2)	$v=10,\ b=15,\ r=6,\ k=4$ $\lambda=2$	$v=15,\ b=10,\ r=4,\ k=6$ $\lambda_1=2,\ \lambda_2=1,\ n_1=6,\ n_2=8$ $p^1_{jk}=\begin{pmatrix}1 & 4\\ 4 & 4\end{pmatrix},\ p^2_{jk}=\begin{pmatrix}3 & 3\\ 3 & 4\end{pmatrix}$; E. F. $=\frac{70}{79}$
(3)	$v=13,\ b=26\ \ r=6,\ k=3$ $\lambda=1$	$v=26,\ b=13,\ r=3,\ k=6$ $\lambda_1=1,\ \lambda_2=0,\ n_1=15,\ n_2=10$ $p^1_{jk}=\begin{pmatrix}8 & 6\\ 6 & 4\end{pmatrix},\ p^2_{jk}=\begin{pmatrix}9 & 6\\ 6 & 3\end{pmatrix}$; E. F. $=\frac{65}{77}$
(4)	$v=19,\ b=57,\ r=9\ \ k=3$ $\lambda=1$	$v=57,\ b=19,\ r=3,\ k=9$ $\lambda_1=1,\ \lambda_2=0,\ n_1=24,\ n_2=32$ $p^1_{jk}=\begin{pmatrix}11 & 12\\ 12 & 20\end{pmatrix},\ p^2_{jk}=\begin{pmatrix}9 & 15\\ 15 & 16\end{pmatrix}$; E. F. $=\frac{133}{151}$
(5)	$v=21,\ b=70,\ r=10,\ k=3$ $\lambda=1$	$v=70,\ b=21,\ r=3,\ k=10$ $\lambda_1=1,\ \lambda_2=0,\ n_1=27,\ n_2=42$ $p^1_{jk}=\begin{pmatrix}12 & 14\\ 14 & 28\end{pmatrix},\ p^2_{jk}=\begin{pmatrix}9 & 18\\ 18 & 23\end{pmatrix}$; E. F. $=\frac{161}{208}$

Though the five designs of our type obtained above will, on inversion, give designs of Yates' type, the generality of our designs, on inversion, may give designs of the same type as the original. There will be many cases where no dual design of any sort exists. As an example of our design having a dual design of the same sort, consider the following.

In §7·5 we considered the design

$$v=pq,\ b=p^2,\ r=p,\ k=q \qquad (p>q) \qquad (1$$

which was obtained by a method which restricts that $q=2$ only if p is not of the form $4l+2$ and that $q>2$ only if p is a prime or power of a prime. It can be easily seen that this is a dual of the quasi-factorial design

$$v=p^2,\ b=pq,\ r=q,\ k=p \qquad (2)$$

where $q-2$ orthogonalised $p \times p$ squares are used to get the design. Thus for $q=2$ or 3, $p=6$, designs of the class (1) can now be obtained, since designs of class (2) exist for those values of p and q, though our general method of getting the former class of designs precluded it.

SUMMARY.

Recently Yates introduced two important types of non-orthogonal designs, namely, the quasi-factorial and the balanced incomplete block designs. The second was the generalisation, of a special case of the first type of designs, namely, of the symmetrical quasi-factorial design with p^m varieties (treatments) in blocks of p^k $(k<m)$ plots, where p is a prime or a power of a prime.

In the present paper we have generalised another special case of the first type of designs, namely, the unsymmetrical quasi-factorial designs, with p^m varieties in blocks of p^{m-1} plots, where p can be any integer. Unsymmetrical designs with p^m varieties in blocks of p^k $(k<m-1)$ plots are also capable of generalisation, which will be attempted in a subsequent paper. Yates' balanced incomplete block designs now appear to be the simplest and the most efficient sub-system within a general system of incomplete block designs with equal block size.

Besides indicating the method of analysis of data appropriate for our partially balanced incomplete block designs and of calculating their Efficiency Factor, three major and a number of miscellaneous methods of constructing the designs are discussed, for the sake of illustration only. It is hoped to completely enumerate those designs of our type, which are likely to prove practically useful, in a later communication.

REFERENCES.

1. BOSE, R. C.: On the Application of the Properties of Galois Fields to the Problem of the Construction of Hyper-Græco Latin Squares. *Sankhyā: The Indian Journal of Statistics*, Vol. 3, Part 4, 1938, pp. 323—338.

2. BOSE, R. C.: On the Construction of Balanced Incomplete Block Designs. *Annals of Eugenics*, Vol. IX, Pt. 3, 1939 (in the Press).

3. CARMICHAEL, ROBERT, D.: *Introduction to the Theory of Groups of Finite Order.* Boston, U.S.A. and London: Ginn & Co., 1937.

4. FISHER, R. A.: *Statistical Methods for Research Workers.* Edinburgh: Oliver and Boyd, 1925—1938.

5. FISHER, R. A. AND YATES, F.: *Statistical Tables.* Edinburgh: Oliver and Boyd, 1938.

6. YATES, F.: A new method of Arranging Variety Trials Involving a Large Number of Varieties. *Jour. Agri. Sc.*, Vol. XXVI, Part III, 1936, pp. 424—455.

7. YATES, F.: Incomplete Randomized Blocks. *Annals of Eugenics*, Vol. VII, Part II, 1936, pp. 121—140.

[*Paper Received, 31st March, 1939*]