A NOTE ON THE DISTRIBUTION OF THE STUDENTISED D^2-STATISTIC

By

S. N. ROY

Statistical Laboratory

Introduction

In an earlier paper[1] published by the author jointly with Mr. R. C. Bose the Studentised D^2-Statistic was defined, its statistical use and object were fully set forth and its sampling distribution was worked out. It is the object of the present note to considerably simplify the derivation of the distribution obtained there, by a twist in the geometrical arguments employed in that paper and a consequent change of procedure. This will incidentally throw fresh light on the use of hyperspace geometry in tackling this class of problems. To make this note intelligible even without reference to the earlier paper[1], or in other words, to make this note practically independent of that, we have to take over three small sections from that paper, where definitions and notations were laid down and certain preliminaries were also introduced which will suffice for our present purpose.

§1. Preliminaries.

Consider two samples Σ and Σ' of sizes n and n' from two multivariate normal populations Π and Π' with the same set of variances and covariances α_{ij} $(i, j = 1, 2 \ldots\ldots p)$ where $\alpha_{ij} = \rho_{ij} . \sigma_i . \sigma_j$, σ_i and σ_j being the standard deviations for the i-th and j-th characters respectively, and ρ_{ij} the correlation co-efficient between the i-th and j-th characters.[1] The matrix $\| \alpha_{ij} \|$ will be said to be the common dispersion matrix for the two populations. Let a_{ij}, a'_{ij} $(i, j = 1, 2, \ldots\ldots p)$ denote the respective variances and covariances of the samples Σ and Σ', so that $\| a_{ij} \|$ and $\| a'_{ij} \|$ are their respective dispersion matrices. Let α_i, α'_i $(i = 1, 2, \ldots\ldots p)$ be the means for the i-th character for the populations Π and Π' and let a_i, a'_i denote the corresponding quantities for the samples Σ and Σ'.

Let us set

$$c_{ij} = \frac{n a_{ij} + n' a'_{ij}}{n + n'} \quad \ldots \quad (1 \cdot 1)$$

Let c^{ij} as usual denote the minor of c_{ij} in the determinant $| c_{ij} |$ divided by the determinant itself. A like definition holds for a^{ij}. Then the Studentised D^2-statistic is defined by

$$p\, D^2 = c^{11}(a_1 - a'_1)^2 + c^{22}(a_2 - a'_2)^2 + \ldots\ldots c^{pp}(a_p - a'_p)^2 + 2c^{12}(a_1 - a'_1)(a_1 - a'_1)$$
$$+ \ldots\ldots\ldots 2c^{p-1\,p}(a_{p-1} - a'_{p-1})(a_p - a'_p) \quad \ldots \quad (1 \cdot 15)$$

Likewise if Δ^2 is the population value of D^2, then

$$p\,\Delta^2 = \alpha^{11}(\alpha_1-\alpha'_1)^2+\alpha^{22}(\alpha_2-\alpha_2)^2+\,\ldots\ldots\ldots\,\alpha^{pp}(\alpha_p-\alpha'_p)^2$$
$$+2\alpha^{12}(\alpha_1-\alpha'_1)\,(\alpha_2-\alpha'_2)\,+\,\ldots\ldots\ldots\,2\alpha^{p-1\,p}(\alpha_{p-1}-\alpha'_{p-1})\,(\alpha_p-\alpha'_p) \qquad \ldots \quad (1{\cdot}2)$$

We know that the joint distribution of the sample readings x_{ik}, $x'_{ik'}$ $(i=1, 2,\ldots\ldots\ldots p;\ k=1, 2\ldots\ldots\ldots n;\ k'=1, 2,\ldots\ldots\ldots n')$ is

$$\text{const. } e^{-\frac{1}{2}\sum\limits_{j=1}^{p}\sum\limits_{i=1}^{p}\alpha^{ij}\{n(a_i-\alpha_i)\,(a_j-\alpha_j)+n'(a'_i-\alpha'_i)\,(a'_j-\alpha'_j)+(na_{ij}+n'a'_{ij})\}}$$
$$\times\Pi\,dx_{ik}\,\Pi\,dx'_{ik'} \quad \ldots \quad (1{\cdot}25)$$

where $\Pi\,dx_{ik}$ stands for $dx_{11}\,dx_{12}\,dx_{22}\ldots\ldots dx_{pn}$ and a similar meaning attaches to $\Pi dx'_{ik'}$.

Sample Σ can be represented in the usual Fisherian space S_n of n dimensions, by the points with co-ordinates

$$(x_{i1},\,x_{i2},\ldots\ldots\ldots\ldots x_{in}),\ i=1,\,2,\ldots\ldots\ldots\,p. \qquad \ldots \quad (1{\cdot}3)$$

or what is the same thing, by p vectors $\mathbf{x}_i$ joining the points to the origin. We take another space $S'_{n'}$ of n' dimensions absolutely orthogonal to the former space, and represent in it the sample Σ' by p other similar vectors. Let

$$y_{ik}=x_{ik}-a_i,\quad y'_{ik'}=x'_{ik'}-a'_i, \qquad \ldots \quad (1{\cdot}35)$$

where $i=1, 2,\ldots\ldots\ldots p;\ k=1, 2\ldots\ \ldots\ldots n;\ k'=1. 2\ldots\ldots\ldots n$

Let $\mathbf{y}_i$, $\mathbf{y}'_i$ denote the vectors, with components $(y_{i1}, y_{i2},\ldots\ \ldots\ldots\ldots y_{in})$ and $(y'_{i1}, y'_{i2},\ldots\ldots\ldots\ldots y'_{in'})$ lying in the space S_n and $S'_{n'}$ respectively. Then the vectors $\mathbf{y}_i$ lie in a flat S_{n-1} of $n-1$ dimensions, perpendicular to the equiangular line in S_n. Similar considerations apply to the vector $\mathbf{y}'_i$.

Let O be the origin of co-ordinates and let M_i be the point on the equiangular line in S_n such that $OM_i=\dfrac{1}{\sqrt{n}}$ times the projection of $\mathbf{x}_i$ on the equiangular line. Then $OM_i=a_i$. In the same way we can find M'_i on the other equiangular line such that $OM'_i=a'_i$. Also if $\mathbf{y}_i\cdot\mathbf{y}_j$ is the scalar product of the vectors $\mathbf{y}_i$ and $\mathbf{y}_j$, then clearly

$$a_{ij}=(\mathbf{y}_i\cdot\mathbf{y}_j)/n,\quad a'_{ij}=(\mathbf{y}'_i\cdot\mathbf{y}'_j)/n' \qquad \ldots \quad (1{\cdot}4)$$

Let us now take a new set of $(n+n')p$ variable a_i, a'_i, z_{ik}, $z'_{ik'}$ $(i=1, 2,\ldots\ldots p;\ k=1, 2,\ldots\ldots n-1;\ k'=1, 2,\ldots\ldots n'-1)$

such that

$$(i)\quad a_i=\frac{1}{n}\sum_{k=1}^{n}x_{ik}\ \ (i=1,\,2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots p)$$

$$(ii)\quad a'_i=\frac{1}{n'}\sum_{k'=1}^{n'}x'_{ik'}\ \ (i=1,\,2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ p)$$

(iii) z_{ik} ($k = 1, 2, \ldots\ldots\ \ldots\ n-1$) are the components of y_i along any $n-1$ mutually orthogonal lines in S_{n-1}. Then z_{ik} is naturally a linear function of

$$x_{i1}, x_{i2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_{in}$$

(iv) Similar considerations apply to $z'_{ik'}$.

The distribution (1·25) can now be written in the form

$$\text{cen t} \times e^{-\frac{1}{2}\sum\limits_{i=1}^{p}\sum\limits_{j=1}^{p}\alpha^{ij}\{n(a_i-\alpha_i)(a_j-\alpha_j)\ +n'(a'_i-\alpha'_i)\ (a'_j-\alpha'_j)+(n\ a_{ij}+n'\ a'_{ij})\}}$$

$$\times \prod_{i=1}^{p} da_i \prod_{i=1}^{p} da'_i \prod dz_{ik} \prod dz'_{ik'} \quad \ldots \quad (1{\cdot}45)$$

It should be noted that a_{ij}'s are expressible purely in terms of z_{ik}'s and a'_{ij}'s are expressible purely in terms of $z'_{ik'}$'s

Next we introduce the new variables $a_i - a'_i$ and $a_i + a'_i$ in place of a_i and a'_i and integrating for $a_i + a'_i$ we get the distribution in the form

$$\text{const} \times e^{-\frac{1}{2}\sum\limits_{i=1}\sum\limits_{j=1}\alpha^{ij}\left[\frac{\bar{n}}{2}\{(a_i-a'_i)-(\alpha_i-\alpha'_i)\}\{(a_j-a'_j)-(\alpha_j-\alpha'_j)\}+Nc_{ij}\right]}$$

$$\times \prod_{\cdot 1}^{p} d(a_i - a'_i) \prod dz_{ik} \prod dz'_{ik'} \quad \ldots \quad (1{\cdot}5)$$

where c_{ij} is given by (1·1) and

$$\frac{2}{\bar{n}} = \frac{1}{n} + \frac{1}{n'},\ \ N = n + n' \qquad \ldots \quad (1{\cdot}55)$$

In the plane of the equiangular lines of S_n and $S'_{n'}$ let R_i be the point, whose projections on the equiangular lines coincide with M_i and M'_i. Then if Q_i is the projection of R_i on O Y, the external bisector of the equiangular lines

$$O\,Q_i = \frac{1}{\sqrt{2}}(a_i - a'_i) \qquad (i = 1, 2, \ldots\ldots\ldots\ldots p) \qquad \ldots \quad (1{\cdot}6)$$

Let the vector v_i be the resultant of the vectors y_i and y'_i. Then it is easily seen that

$$N\ c_{ij} = v_i \cdot v_j \qquad \ldots \quad (1{\cdot}65)$$

where the dot denotes the scalar product.

We may note that the spaces S_{n-1}, $S_{n'-1}$ containing the vectors y_i, y'_i respectively ($i = 1, 2, \ldots\ \ldots p$) are orthogonal to one another, as also to O Y. Hence the new vectors v_i are also orthogonal to O Y, and lie in the space $S_{n+n'-2}$ which comprises both S_{n-1} and $S'_{n'-1}$.

Let t_i be the resultant of the vectors u_i and the vector $\bar{n}^{\frac{1}{2}}$ Q Q_i, where $\bar{n}$ is given by (1·55).

Then

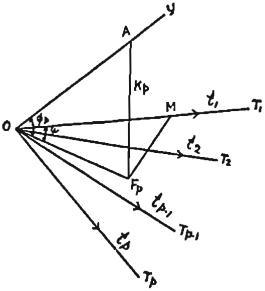
$$t_i . t_j = \frac{\bar{n}}{2} (a_i - a'_i)\ (a_j - a'_j) + (n a_{ij} + n' a'_{ij})$$

$$= \frac{\bar{n}}{2} (a_i - a'_i)\ (a_j - a'_j) + \mathrm{N}\ c_{ij} = g_{ij} \text{ (say)} \qquad (1·7)$$

The distribution (1·5) now takes the form

$$\text{const} \times e^{-\frac{1}{2} p \Delta^2 - \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} a^{ij} \{g_{ij} - \bar{n}(a_i - a'_i)\ (a_j - a'_j)\}}$$

$$\times \prod^{p} d(a_i - a'_i)\ \Pi\ dz_{ik}\ \Pi\ dz'_{ik}. \qquad (1·75)$$



Let T_i denote the extremity of the vector t_i.

Let OA be the unit vector along OY. We shall denote it by i. Let F_i denote the foot of the perpendicular from A to the space $(T_1\ T_2 ....T_i)$. Let the length AF_i be denoted by K_i

Let us now consider the length K_p. We shall show that it is very closely connected with D^2

$$\text{Now } K^2_p = \frac{\text{Vol } (i,\ t_1,\ t_2, ...t_p)}{\text{Vol } (t_1,\ t_2, ...t_p)} \qquad ... \quad (1·8)$$

But Vol $(i,\ t_1,\ t_2, ...t_p)$ is the same as the volume formed by the unit vector QA, and the projections of the vectors $t_1, t_2, ...t_p$ on the space perpendicular to OA, which we have earlier called $S_{n, n'-1}$. But these projections, from the way in which

$t_1, t_2, \ldots\ldots t_p$ have been derived, are at once seen to be $v_1, v_2, \ldots\ldots\ldots v_p$. Hence the numerator in (1:95) is numerically equal to vol $(v_1, v_2, \ldots\ldots\ldots v_p)$

$$\text{Therefore, } K^2_p = \frac{|\, v_i \cdot v_j \,|}{|\, t_i \cdot t_j \,|}$$

$$= \frac{|\, N\, c_{ij} \,|}{|\, N\, c_{ij} + \frac{\bar{n}}{2}\,(a_i - a'_i)\,(a_j - a'_j) \,|} \text{ from (1·65) and (1·7)}$$

$$= \frac{|\, N\, c_{ij} \,|}{|\, N\, c_{ij} \,| + \frac{\bar{n}}{2}\, N^{p-1} \Sigma\, C_{ij}(a_i - a'_i)\,(a_j - a'_j)} \qquad \ldots \quad (1·81)$$

C_{ij} denoting the minor of c_{ij} in $|\, c_{ij} \,|$

$$\text{Therefore from (1·15) } K^2_p = \frac{1}{1 + \frac{p\bar{n}}{2N}\, D^2} \qquad \ldots \quad (1·82)$$

Denoting by φ_p the angle AOF$_p$, *i.e.* the angle between the unit vector $\boldsymbol{i}$, and the flat $(t_1, t_2, \ldots\ldots\ldots t_p)$ we have, since $k_p = \sin\varphi_p$

$$D^2 = \frac{2N}{p\,\bar{n}}\ \cot^2\varphi_p \qquad \ldots \quad (1·83)$$

If a statistic similar to D^2 were constructed for the first i variates only, then we could denote it by D_i^2; we should then have had

$$K^2_i = \frac{1}{1 + \frac{i\bar{n}}{2N}\, D^2_i} \qquad \ldots \quad (1·84)$$

and

$$D^2_i = \frac{2N}{i\bar{n}}\ \cot^2\varphi_i \qquad \ldots \quad (1·85)$$

It should be noted that consistently with this notation D^2 should be preferably replaced by D^2_p. Let M be the foot of the perpendicular from F$_p$ to t_1 and let ψ be the angle F$_p$ OM.

§2. Derivation of the Sampling Distribution of D²

It was shown in section 2 of the earlier paper[1] that D² and Δ² defined in the present note by equations (1·15), (1·2), and $\sum\limits_{i=1}^{p}\ \sum\limits_{i=1}^{p}\ \alpha^{ij}\, a_{ij}$, $\sum\limits_{i=1}^{p}\ \sum\limits_{j=1}^{p}\ \alpha^{ij}\ (a_i - a_j)\,(\alpha_i - \alpha_j)$ occurring in the present note in the relation (1·75) are all invariant under any linear transformation. It was further shown in section 8 of the earlier paper[1] that a special linear transformation could be constructed such that

$$\left.\begin{array}{l} \alpha_{ij} = 0, \text{ if } i \neq j,\ \alpha_{ii} = 1 \\ (\alpha_1 - \alpha'_1) = p\,\Delta^2 \\ (\alpha_i - a_i) = 0 \text{ for } i = 2, 3, \ldots\ldots\ \ldots\ldots\ldots p \end{array}\right\} \qquad \ldots \quad (2·1)$$

$$\text{Therefore, } \alpha^{ij} = 0 \text{ if } i \neq j,\ \alpha^{ii} = 1 \qquad \ldots \quad (2·11)$$

We may consider Δ to be given by the positive root of Δ^2 and $\sqrt{p}$ the positive root of p. Then by suitably naming the populations as first and second we can write

$$a_1 - a'_1 = \Delta \sqrt{p} \qquad \dots \quad (2{\cdot}12)$$

Therefore without any loss of generality we can take the distribution in (1·75) as

$$\text{const} \times e^{-\frac{1}{2}\bar{n}\, p\Delta^2 - \frac{1}{2}\sum_{i=1}^{p} g_{ii} - \frac{\bar{n}}{2}\sqrt{p}.\ \Delta(a_1 - a'_1)} \ \Pi d(a_i - a'_i)\ \Pi dz_{ik}\ \Pi dz'_{ik}. \qquad \dots \quad (2{\cdot}13)$$

The density factor in (2·13) does not contain any mean difference except the first. Therefore integrating out for all mean differences $(a_i - a'_i)$, $(i = 2, 3, \dots\dots p)$ we have the distribution in the form

$$\text{const} \times e^{-\frac{1}{2}\bar{n}\, p\Delta^2 - \frac{1}{2}\sum_{i=1} g_{ii} - \frac{\bar{n}}{2}\sqrt{p}\ \Delta(a_1 - a'_1)} \ d(a_1 - a'_1)\Pi dz_{ik}\ \Pi dz'_{ik}. \qquad \dots \quad (2{\cdot}14,$$

Now it is clear that the volume element $d(a_1 - a'_1)\ \Pi dz_{ik}\ \Pi dz'_{ik}$, is really the volume described by the points $(A, T_1, T_2, \dots\dots\dots T_p)$ in a space of $(n + n' - 1)$ dimensions, when the points are given whatever freedom we please but subject to the restriction that the variates will have to lie within infinitesimal ranges round about the values occurring in the density factor in (2·14). Let us denote the lengths $OT_1, OT_2, \dots\dots OT_p$ by $l_1, l_2, \dots \dots\dots l_p$. It is now clear from the definitions of $(a_1 - a'_1)$ and g_{ij} in (1·6) and (1·7) and from the figure in section 1 that

$$g_{ii} = l^2_i, \text{ and } (\sqrt{\bar{n}}/2)\ (a_1 - a'_1) = \text{projection of } OT_1 \text{ on } OY = l_1 \cos\phi_p \cos\psi \quad \dots \quad (2{\cdot}2)$$

The density factor in (2·14) now ʒeduces to

$$\text{const} \times e^{-\frac{1}{2}\bar{n}\, p\Delta^2 - \frac{1}{2}\sum_{i=1}^{p} l^2_i - (\bar{n}\, p/2)^{1/2}\ \Delta\ l_1 \cos\phi_p \cos\psi} \qquad \dots \quad (2{\cdot}21)$$

Give now the points $(A, T_1, T_2, \dots\dots\dots T_p)$ freedom to move subject to the restrictions that l_i lies between l_i and $l_i + dl_i$ $(i = 1, 2, \dots\dots\dots p)$ and ϕ_p and ψ lie respectively between ϕ_p and $\phi_p + d\phi_p$ and ψ and $\psi + d\psi$. Let this be done in two stages. It is clear that keeping the vectors $(l_1, l_2, \dots\dots l_p)$ fixed the volume element described by A is

$$\text{const.}\ (\sin\phi_p)^{N-p-2}\ (\cos\phi_p)^{p-1}\ (\sin\psi)^{p-2}\ d\phi_p\ d\psi \qquad \dots \quad (2{\cdot}22)$$

where N of course is $n + n'$

Relea͡se now the points $T_1, T_2, \dots\dots\dots T_p$ subject to the restrictions mentioned before; the volume element described by these points is

$$\text{const.}\ (l_1, l_2, \dots \dots\ l_p).^{N-2}\ dl_1, dl_2, \dots\dots\dots dl_p \qquad \dots \quad (2{\cdot}23)$$

Therefore, the joint volume element described by $(\Delta, T_1, T_2, \ldots\ldots\ldots T_p)$ i.

$$\text{const.}\ (\sin\varphi_p)^{N-p-2}\ (\cos\varphi_p)^{p-1}\ (\sin\psi)^{p-2}\ d\varphi_p\, d\psi\ (t_1, t_2, \ldots\ldots\ldots t_p)^{N-2}\ \Pi dt_i \qquad \ldots\ (2{\cdot}24)$$

If we restrict now each of φ_p and ψ to lie between 0 and $\pi/2$ we easily see irom (2·21) and (2·24) that the distribution (2·14) now reduces to

$$\text{const} \times e^{-\frac{1}{2}\bar{n}\, p\Delta^2 - \frac{1}{2}\sum\limits_{i=1}^{p} t_i^2} \cosh\,(c\Delta t_1 \cos\varphi_p \cos\psi)\ (\sin\varphi_p)^{N-p-2}\ (\cos\varphi_p)^{p-1}\ (\sin\psi)^{p-2}$$

$$\times\, d\varphi_p\, d\psi\ (t_1, t_2, \ldots\ .\ \ldots t_p)^{N-2} \prod_{i=1}^{p} dt_i \qquad \ldots\ (2{\cdot}25)$$

where $c^2 = \bar{n}\ p/2$ ... (2·255)

Integrating out for $dt_2, dt_3, \ldots\ldots\ldots dt_p$ we have

$$\text{const} \times e^{-\frac{1}{2}\bar{n}\, p\Delta^2 - \frac{1}{2}t_1^2} \cosh\,(c\Delta t_1 \cos\varphi_p \cos\psi)\ (\sin\varphi_p)^{N-p-2}\ (\cos\varphi_p)^{p-1}\ (\sin\psi)^{p-2}$$

$$\times\, d\varphi_p\, d\psi\ t_1^{N-2} dt_1 \qquad \ldots\ (2{\cdot}26)$$

It should be noted that t_1 varies from O to ∞ and both φ_p and ψ vary from 0 to $\pi/2$

Using now the well known relation[1]

$$I_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^{\pi/2} \cosh\,(z\cos\theta)\ \sin^{2\nu}\theta\ d\theta \qquad \ldots\ (2{\cdot}3)$$

which is valid for $R(\nu+\frac{1}{2})>0$, for the special case $\nu=0$, and integrating (2·26) over ψ fr 0 to $\pi/2$ the distribution reduces to

$$\text{const} \times e^{-\frac{1}{2}t_1^2}\ (t_1)^{N-2-(p-2)/2}\ I_{\frac{p-2}{2}}\,(c\Delta t_1 \cos\varphi_p)\ (\sin\varphi_p)^{N-p-2}\ (\cos\varphi_p)^{p/2} dt_1\, d\varphi_p \qquad \ldots\ (2{\cdot}35)$$

Using now another well known relation[2]

$$\int_0^\infty I_\nu\ (at)\ e^{-b^2 t^2}\ t^{\mu-1} dt = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)}{2p^\mu\,\Gamma(\nu+1)} \cdot {}_1F_1\!\left(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu,\ \nu+1,\ \frac{a^2}{4b^2}\right) \qquad \ldots\ (2{\cdot}4)$$

for the special case $b^2=\frac{1}{2}$, $\nu=(p-2)/2$, $\mu=N-1-(p-2)/2$, $a=c\ \Delta\cos\varphi_p$, which do not violate the restrictions on the validity of the formula, and integrating (2·35) over t_1 from 0 to ∞, the distribution of φ_p is obtained in the form

$$\text{const} \times\ (\sin\varphi_p)^{N-p-2} (\cos\varphi_p)^{p-1}\ {}_1F_1\left(\frac{N-1}{2},\ \frac{p}{2}\quad \frac{c^2\,\Delta^2\cos^2\varphi_p}{2}\right) d\varphi_p \qquad \ldots\ (2{\cdot}45)$$

But remembering from (1·83) and (2·255) that

$$D^2 = \frac{N}{c^2} \cot^2 \varphi_p \qquad \dots \quad (2{\cdot}46)$$

and therefore,

$$\left.\begin{aligned} d\varphi_p &= \frac{c\sqrt{N}.\, d\,D^2}{2\,(D^2)^{\frac{1}{2}}\,(N+c^2\,D^2)} \\ \cos\varphi_p &= \frac{c(D^2)^{\frac{1}{2}}}{(N+c^2\,D^2)^{\frac{1}{2}}}, \quad \sin\varphi_p = \frac{N}{(N+c^2\,D^2)^{\frac{1}{2}}} \end{aligned}\right\} \qquad \dots \quad (2{\cdot}5)$$

we have the distribution of D² given in the form

$$\text{const.}\ \frac{(D^2)^{(p-2)/2}}{(N+c^2\,D^2)}\ \frac{N-p-2}{2}\ {}_1F_1\left(\frac{N-1}{2},\ \frac{p}{2},\ \frac{c^4\,\Delta^2}{2}\ \frac{D^2}{N+c^2D^2}\right) d(D^2) \qquad \dots \quad (2{\cdot}55)$$

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3. Loc. Cit., p. 100 and p. 393.

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