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An Insurance Network -I : Nash Equilibrium

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AN INSURANCE NETWORK - I : NASH EQUILIBRIUM

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Abstract: We consider d insurance companies whose surplus processes are perturbed Levy processes. Suppose they have a treaty to diversify risk; accordingly, if one company needs a certain amount to prevent ruin, the other companies pitch in previously - agreed - upon fractions of the amount, and any shortfall is got from external sources. With each company trying to minimise its repayment liability, the situation is viewed upon as a d-person dynamic game with state space constraints and a Nash equilibrium is sought. Under certain natural conditions, it is shown that the Skorokhod problem of probability theory provides a (unique) Nash equilibrium.

Key words: d-person dynamic game, state space constraints, deterministic Skorokhod problem, orthant, half space, control, vague convergence, spectral radius, reinsurance model, reflection, drift.

1 Introduction

We consider d insurance companies operating under a treaty to diversify risk. Accordingly, if one company estimates at some instant of time that it needs a certain amount to prevent ruin, the other companies in the network pitch in previously - agreed - upon fractions of the amount. Any shortfall is to be obtained by the concerned company from "external" sources. The amount got from "internal" sources (viz. other members of the network) carry easier repayment terms, due to mutual obligations. The amount needed to prevent ruin is viewed upon as a control. Needless to say, each company will try to minimise its expected cost. The companies can possibly be in competition; and the objective of control is to keep the surplus of each company non-negative. Thus we are lead naturally to a d-person non-cooperative dynamic game with state space constraints, and we seek a Nash equilibrium. It turns out that under certain conditions a Nash equilibrium is provided by the solution to the appropriate Skorokhod problem in an orthant.

Skorokhod problem of probability theory has played a major role in the stochastic differential equation formulation of reflected (or regulated) Brownian motion / diffusions / Levy processes.

Thanks to the impetus from queueing networks, the problem in nonsmooth domains, like a quadrant or an orthant, has attracted a lot of attention. As sample path analysis is helpful in understanding the stochastic problem, the deterministic Skorokhod problem has also been extensively studied; see [HR], [Re], [MP], [Ra 1] and the references therein. In the present work also, the problem reduces to considering a deterministic game for similar reasons.

The surplus / risk processes in the absence of controls are independent one-dimensional perturbed Levy processes (see [RSST]) with time and space dependent drift. The optimally controlled process for the game is then the reflected / regulated Levy process in an orthant; (see [Ra 1], [DR] and the references therein for reflected Levy processes); the Nash equilibrium is given by the "pushing" part of the solution to Skorokhod problem.

In insurance mathematics, beginning perhaps with the works of Borch, game theoretic ideas have been used in the context of reinsurance by many authors; see the recent survey paper [Aa] and the references therein. However the flavour is quite different in our work. Besides being a continuous time model, our model has state space constraints, as the controlled process has to live in an orthant.

Optimality property of Skorokhod problem in one-dimension (in half line $[0, \infty)$) is well known; see [H]. This has also been used in [T] in the context of dividend payment.

Game theoretic aspects of Skorokhod problem in an orthant have recently been studied in [Ra 2], where the absolutely continuous case has been considered. Consequently the game becomes a differential game with state space constraints and the framework of HJB equations and viscosity solutions becomes available. Nevertheless it has some parallels with the present work. Incidentally optimal control problems with state space constraints were first studied by Soner, and one may see [FS], [BC] for accounts of this.

The paper is organized as follows. In Section 2 we present the basic network model, and the passage to the determinstic d-person game. As explained in Examples 2.1 - 2.4 our model with appropriate parameters can be considered as a sort of reinsurance model; the difference with conventional models being that the reinsurer helps out only when a demand is made by the cedent. Brief description of the Skorokhod problem in an orthant / a half space is given in Section 3.

In Section 4, the main body of the paper, the determinstic d-person game alluded to above and its connection to the Skorokhod problem are discussed. For this it is convenient to introduce a control problem in a half space, and show that the optimal solution is given by the solution to a Skorokhod problem in the half space. A priori bounds given in the context of Skorokhod problem help in fixing suitable compact sets where the controls lie; the topology is the topology of vague convergence of uniformly bounded measures (or sub-probability measures) on a finite interval $[0, T]$. We adopt a twin approximation procedure, one by solving a Skorokhod problem for 'known' function of time variable alone, and the other by solving the integral equations (4.1), (4.2). The hypotheses, which are natural in the context of Section 2, are rigged so that the cost does not increase at any stage of the approximation. Under additional assumptions, Skorokhod problem gives a unique Nash equilibrium for all $0 \leq t \leq T$.

It is well known that many parallels exists between insurance models and (one-dimensional) queueing theory; see [As] and the references therein. That the Skorokhod problem plays a central role in queueing networks has already been mentioned; see for example [CY] and the references therein. However, to our knowledge, no connection between insurance and queueing networks has been pointed out in the literature.

The question of non ruin of the network in the continuous case is taken up in a companion paper under preparation.

Although we have used problem from insurance to elucidate the model, other situations are also meaningful; some examples are potential demand/output lost in queueing networks due to buffer being empty, feasible production plans in input-output models, and allocation of funds/subsidies to various sectors (including welfare sectors) of an interdependent economy; see various comments in [Re], [CM], [Ra 1].

Finally some comments about notation. $D([0,\infty) : E), D([0,T] : E)$ denote respectively E-valued r.c.l.l. functions on $[0,\infty), [0,T]$.

$$\begin{aligned} D_{\uparrow}([0,T] : \mathbb{R}^d_+) &= \{y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot)) \in D([0,T] : \mathbb{R}^d) : \\ &\qquad y_i(\cdot) \geq 0, y_i(\cdot) \text{ nondecreasing}, 1 \leq i \leq d\}. \end{aligned}$$

For $1 \leq i \leq d, y \in \mathbb{R}^d$ we denote $y_{-i} = (y_1, \ldots, y_{i-1} y_{i+1}, \ldots, y_d)$; for $1 \leq i \leq d$, an $\mathbb{R}^d$-valued function $y(\cdot)$, similarly $y_{-i}(\cdot) = (y_1(\cdot), \ldots, y_{i-1}(\cdot), y_{i+1}(\cdot), \ldots, y_d(\cdot))$; for real $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d, \xi$, we denote $(\xi, y_{-i}) = (y_1, \ldots, y_{i-1}, \xi, y_{i+1}, \ldots, y_d)$.

2 A network model

Consider d insurance companies such that the surplus (or reserve) of Company i is given by, for $t \geq 0$,

$$S_i(t) = S_i(0) + B_i(t) + \int_0^t b_i(r, S_i(r))\, dr, \tag{2.1}$$

where $B_1(\cdot), \ldots, B_d(\cdot)$ are d independent one dimensional Levy processes, $b_i : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is the "drift" component for the i-th company, $S_i(0)$ is the initial reserve. The term $b_i(\cdot,\cdot)$ incorporates premium rate (> 0) of company i, interest rate (> 0) of bonds in which the company has invested part of its surplus, expected value/first order terms of claim payments

(< 0), drift term ($\leq$ or > 0) in behaviour of stocks in which the company has invested another part of its surplus. The Levy process $B_i(\cdot)$ incorporates volatility of stocks as well as variations in claim sizes. It can be seen that the classical Cramer-Lundberg model, or perturbation of such a model by a Brownian motion / a general Levy process, or diffusion approximation of such a model, etc. lead to equations of the form (2.1). Equation (2.1) describes the exogeneous evolution of Company i in the absence of any control. See [RSST] for detailed discussion.

Note that ruin of Company i is possible if Prob. ($S_i(t) \leq 0$ for some $t \geq 0$) > 0.

Assume that the d companies agree on a treaty along the following lines: Suppose Company i estimates (at some instant of time) that it needs an amount Δy_i to prevent ruin. Then for $j \neq i$, Company j gives $|R_{ji}|\Delta y_i$, where $R_{ji} \leq 0$ and $\sum\limits_{j \neq i} |R_{ji}| \leq 1$. Of course, the shortfall $(1 - \sum\limits_{j \neq i} |R_{ji}|)\Delta y_i$ has to be procured by Company i from "external" sources.

The rationale is: The amount $|R_{ji}|\Delta y_i$ that Company i gets from Company j may be considered a loan on soft interest terms, whereas the amount $(1 - \sum\limits_{j \neq i} |R_{ji}|)\Delta y_i$ may carry interest at market rates. This is reasonable as there is mutual obligation among the companies, to diversify the risk.

With the treaty in force, instead of (2.1), the system of equations giving the surplus of the companies becomes

$$\begin{aligned} Z_i(t) &= Z_i(0) + B_i(t) + \int_0^t b_i(r, Y(r), Z_i(r))\, dr \\ &\quad + Y_i(t) + \sum_{j \neq i} \int_0^t R_{ij}(r, Y(r-), Z_i(r-))\, dY_j(r), \end{aligned} \tag{2.2}$$

with the stipulation that

$$Z_i(t) \geq 0, \quad t \geq 0, \quad 1 \leq i \leq d \tag{2.3}$$

where $R_{ii}(\cdots) \equiv 1$ for all i, $R_{ij}(\cdots) \leq 0, i \neq j$. Here
$Z_i(t)$ = current surplus/reserve with Company i at time t;
$Y_i(t)$ = cumulative amount obtained by Company i over the period $[0, t]$, from both "internal" and "external" sources, specifically for the purpose of preventing ruin.

Note that $Y_j(\cdot)$ is nondecreasing and hence integration w.r.t. $dY_j(\cdot)$ makes sense as Stieltjes integral. Here and elsewhere $\int_0^t \cdots$ denotes integration over the closed interval $[0, t]$. We take $Z(0-) = 0 = Y(0-)$. Observe that for $i = 1, 2, \ldots, d$, the coefficients b_i, R_{ij}, though may depend on $Y(\cdot)$, are independent of $Z_l(\cdot), l \neq i$; this is to indicate that except for the "push" $Y(\cdot)$, the dynamics of one company does not depend on other companies.

Example 2.1 Let $d = 2; R_{11} \equiv R_{22} \equiv 1, R_{12} \equiv 0, R_{21} \equiv -1$. That is, if Company 1 needs an amount Δy_1 the entire amount is given by Company 2. In such a case it is natural that the drift component $b_2(\cdots)\,dr$ in the equation for Company 2 should be of the form

$$b_2(r,(y_1,y_2),z_2) = b_2^{(1)}(r,y_1) + \hat{b}_2(\cdots) \tag{2.4}$$

where $b_2^{(1)}(r,y_1)$ could be part of the premium income of Company 1. This corresponds to a "reinsurance" scheme. One can also have $R_{12} = 0, R_{21} = -\alpha, 0 < \alpha < 1$; this will be a sort of "proportional reinsurance". A difference from traditional reinsurance models is that Company 2 helps out Company 1 only when a "demand" is made.

Example 2.2 Let $d \geq 2; R_{ii} \equiv 1$ for all $i, R_{i+1,i} \equiv -1, 1 \leq i \leq d-1, R_{ij} \equiv 0$ otherwise. Similar to (2.4) let

$$b_{i+1}(r,y,z_{i+1}) = b_{i+1}^{(i)}(r,y_i) + \hat{b}_{i+1}(\cdots)$$

where $b_{i+1}^{(i)}(r,y_i)$ could be part of the premium income of Company i. This is easily seen to be a "reinsurance hierarchy" model; once again, Company $(i+1)$ pitches in the entire amount Δy_i, but only when a demand is made by Company i. (For another reinsurance hierarchy, see [RSST].)

Example 2.3 Let $d \geq 2; R_{ii} \equiv 1\ \forall\ i, R_{id} = 0, R_{di} = -\alpha_i, 0 < \alpha_i \leq 1, 1 \leq i \leq d-1$. Here Company d is a reinsurance company vis-a-vis the other companies; the other $(d-1)$ companies may or may not have a treaty among themselves. As before it is natural that

$$b_d(r,y,z_d) = \sum_{i=1}^{d-1} b_d^{(i)}(r,y_i) + \hat{b}_d(\cdots)$$

where $b_d^{(i)}(r,y_i)$ is part of the premium income of Company i. □

Example 2.4 Let $d \geq 2; R_{ii} \equiv 1\ \forall\ i, R_{1j} = 0, j \neq 1, R_{i1} = -\alpha_i, 0 \leq \alpha_i \leq 1, i \geq 2, \sum_{i\geq 2} \alpha_i = 1$. In this model Company 1, the primary insurer, could represent a high risk portfolio; the other companies are reinsurers. This may be considered a "reinsurance syndicate"; see [Aa] for another version of a reinsurance syndicate. As in the earlier examples, for $k \geq 2$

$$b_k(r,y,z_k) = b_k^{(1)}(r,y_1) + \hat{b}_k(\cdots)$$

with $b_k^{(1)}(r,y_1)$ denoting a part of the premium income of Company 1. □

Thus our model can be considered a model for diversifying risk, or in other words a reinsurance model.

Once the functions b_i, R_{ij} are known/agreed upon, how to determine $Y_i(\cdot)$? Both the times and the quanta of demands of each company are to be determined.

Assume that R_{ij} are constants; let $T > 0$ be fixed. Suppose the amounts taken by Company i to prevent ruin have to be repaid at time T with interest at respective rates a_1, a_2 for 'internal' and 'external' loans, where $0 \le a_1 < a_2$. So the expected total liability of Company i on account of borrowings to avoid ruin may be taken as

$$\begin{aligned} J_i(Y_i) &= E\left[\int_0^T e^{a_1(T-r)}\left(\sum_{j\neq i}|R_{ji}|\right)\, dY_i(r)\right. \\ &\quad \left. + \int_0^T e^{a_2(T-r)}\left(1-\sum_{j\neq i}|R_{ji}|\right)\, dY_i(r)\right]. \end{aligned} \tag{2.5}$$

Clearly Company i will try to minimise $J_i(Y_i)$ over all feasible $Y_i(\cdot)$. Since the companies can possibly be in competition, it is appropriate to consider the system as a noncooperative game and hence seek a Nash equilibrium.

If $Y_i(\cdot), \hat{Y}_i(\cdot)$ are feasible such that $Y_i(0-,\omega) = \hat{Y}(0-,\omega)$ and $Y_i(t,\omega) \le \hat{Y}_i(t,\omega)$ for all t, for a.a.ω then it is easily seen that $J_i(Y_i) \le J_i(\hat{Y}_i)$.

Therefore, as a first step, for any fixed ω, one can consider $Y_i(T,\omega)$ as the cost function for Company $i, 1 \le i \le d$, and seek a Nash equilibrium. Because $Z_i(\cdot,\omega) \ge 0, 1 \le i \le d$, this is basically a deterministic d-person dynamic game with state space constraints. It turns out that, under certain natural conditions, the solution to the Skorokhod problem in an orthant provides (unique) Nash equilibrium to the d-person game for every $0 \le t \le T$. This would mean that, in addition to (2.2), (2.3), we need to stipulate that

$$\int_0^t Z_i(s)\, dY_i(s) = 0, \quad 0 \le t \le T, \quad 1 \le i \le d; \tag{2.6}$$

that is, $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$. In other words, Company i can borrow only when its reserve is zero / it is in the red, and the amount borrowed should be just enough to keep it afloat.

3 Skorokhod problem

We now briefly describe the Skorokhod problem; the coefficients may be somewhat more general than needed for our purposes of Section 2.

Let $G = \{x \in I\!R^d : x_i > 0, 1 \le i \le d\}$ denote the d-dimensional positive orthant. Let $b : [0,\infty) \times I\!R^d \times I\!R^d \to I\!R^d, R : [0,\infty) \times I\!R^d \times I\!R^d \to I\!M_d(I\!R)$ respectively be the drift and reflection functions; here $I\!M_d(I\!R)$ denotes the space of all $d \times d$ matrices with real entries. Let $w \in D([0,\infty) : I\!R^d)$ with $w(0) \in \bar{G}$; note that $w(\cdot)$ is an $I\!R^d$-valued r.c.l.l. function on $[0,\infty)$. The deterministic Skorokhod problem $SP(w,b,R)$ consists in finding two r.c.l.l. functions $Yw(\cdot) = ((Yw)_1(\cdot), \cdots, (Yw)_d(\cdot)), Zw(\cdot) = ((Zw)_1(\cdot), \cdots, (Zw)_d(\cdot))$ such that the following are satisfied

(i) For any $t \geq 0$, the Skorokhod equation holds,

$$\begin{aligned} (Zw)(t) &= w(t) + \int_0^t b(r, Yw(r), Zw(r))\, dr \\ &\quad + \int_0^t R(r, Yw(r-), Zw(r-))\, d(Yw)(r), \end{aligned} \tag{3.1}$$

(ii) $Zw(t) \in \bar{G}$ for all $t \geq 0$;

(iii) $(Yw)_i(0) \geq 0, (Yw)_i(\cdot)$ is nondecreasing;

(iv) $(Yw)_i(\cdot)$ can increase only when $(Zw)_i(\cdot) = 0$, that is,

$$(Yw)_i(t) = \int_{[0,t]} 1_{\{0\}}((Zw)_i(r))\, d(Yw)_i(r). \tag{3.2}$$

The following hypotheses are assumed:

(A1): For $1 \leq i \leq d, b_i$ is bounded measurable; $(y,z) \mapsto b_i(t,y,z)$ is Lipschitz continuous, uniformly over t; $|b_i(t,y,z)| \leq \beta_i$. Denote $\beta = (\beta_1, \ldots, \beta_d)$.

(A2): For $1 \leq i,j \leq d, R_{ij}$ is bounded measurable; $(y,z) \mapsto R_{ij}(t,y,z)$ is Lipschitz continuous, uniformly over t. Also $R_{ii} \equiv 1\ \forall\ i$.

(A3): For $i \neq j$, there exists constant W_{ij} such that $|R_{ij}(t,y,z)| \leq W_{ij}$. Set $W = ((W_{ij}))$ with $W_{ii} = 0$ for all i. We assume $\sigma(W) < 1$, where $\sigma(W)$ denotes the spectral radius of W.

When $R(\cdots)$ is a constant matrix, jth column vector of R gives the direction of reflection on the j-th face of ∂G. The assumption $R_{ii} \equiv 1$ is a suitable normalization, indicating that the direction of reflection is bounded away from the tangential direction.

As $R_{ii} \equiv 1$ note that the Skorokhod equation (3.1) may be written as, for $1 \leq i \leq d, t \geq 0$

$$\begin{aligned} (Zw)_i(t) &= w_i(t) + \int_0^t b_i(r, Yw(r), Zw(r))\, dr + (Yw)_i(t) - (Yw)_i(0-) \\ &\quad + \sum_{j \neq i} \int_0^t R_{ij}(r, Yw(r-), Zw(r-))\, d(Yw)_j(r). \end{aligned} \tag{3.3}$$

As $(Yw)_j(\cdot)$ are required to be nondecreasing $d(Yw)_j(\cdot)$ integrals make sense.

The condition (A3) is a popular hypothesis in queueing theory due to the pioneering paper of Harrison and Reiman [HR]. In economics also such a condition is known in the context of Leontief input-output models; in fact, Skorokhod problem can be considered a continuous time feedback analogue of open Leontief model [Ra 1]. It may also be noted that if $w(\cdot) \equiv$ constant (not necessarily in $\bar{G}$), b, R are constants then SP is just the linear complementarity problem of operations research; so SP is also called dynamic complementarity problem.

Note: Though for our purpose it suffices to assume $w(0) \in \bar{G}$, SP is well posed even wihtout such an assumption; see [Ra 1].

Remark 3.1 Suppose $R_{ij}(\cdots) \leq 0, i \neq j, R_{ii} \equiv 1$, and $\sum\limits_{j \neq i} W_{ij} < 1$ for all i, where $W_{ij} = \sup\{|R_{ij}(t,y,z)| : t \geq 0, y, z \in I\!R^d\}, j \neq i$. Set $W = ((W_{ij}))$ with $W_{ii} = 0$. Let λ_0 be the largest eigenvalue (in absolute value) of W and $x^{(0)}$ the corresponding eigenvector. By Perron-Frobenius theorem we may take $\lambda_0 \geq 0,\ x_i^{(0)} \geq 0\ \forall\ i$. Let $x_{i_0}^{(0)} \geq x_j^{(0)}$ for all j. Then

$$\lambda_0\ x_{i_0}^{(0)} = \sum_j W_{i_0 j}\ x_j^{(0)} = \sum_{j \neq i_0} W_{i_0 j}\ x_j^{(0)} < x_{i_0}^{(0)}$$

whence it follows that $\sigma(W) < 1$. In particular the situation described in Section 2 satisfies (A3) if $\sum\limits_{j \neq i} |R_{ij}| < 1$. The reinsurance schemes of Examples 2.1 and 2.2 satisfy (A3). In Example 2.3, (A3) is satisfied if the $(d-1) \times (d-1)$ matrix formed by the first $(d-1)$ indices satisfies (A3); a similar comment applies to Example 2.4.

Example 3.2 Take $d = 2; R_{11} = R_{22} = 1, R_{12} = R_{21} = -1$. In this case (A3) is not satisfied; nor is the deterministic SP well posed; see [Sh 2]. Moreover, in this case with $b \equiv 0, w(\cdot)$ coming from sample paths of Brownian motion, the reflected/regulated process cannot be realised as a semimartingale; see [W]. □

Because of the spectral radius condition (A3) observe that

$$(I - W)^{-1} = (I + W + W^2 + W^3 + \cdots) \tag{3.4}$$

is a well defined matrix of nonnegative entries. The following result is proved in [Ra 1]; see [HR], [Re], [CM], [MP], etc. for earlier results.

Theorem 3.3 *Assume (A1) - (A3). Then the determinstic Skorokhod problem $SP(w,b,R)$ is well posed; that is, for each $w(\cdot)$ there exists a unique pair $Yw(\cdot), Zw(\cdot)$ satisfying (i) - (iv) given above. Moreover for $1 \leq i \leq d, t \geq 0$*

$$0 \leq (Yw)_i(t) \leq ((I - W)^{-1}h)_i(t) \tag{3.5}$$

where

$$h_i(t) = \beta_i t + \sup_{0\le s\le t} \max\{0, -w_i(s)\}, \tag{3.6}$$

$h(\cdot) = (h_1(\cdot), \cdots, h_d(\cdot))$. *Also if* $w(\cdot)$ *is continuous, then so are* $Yw(\cdot), Zw(\cdot)$.

$\square$

For our purposes later, we need also to consider a Skorokhod problem in a half space. Though deterministic Skorokhod problem in a half space has been studied earlier, even with state dependent drift and reflection, see for example [Sh 1] and references therein, the version we present below does not seem to appear in the literature. As the arguments are similar to those in [Ra 1], [Ra 2] we give just the barest details.

Let $H \equiv H_1 = \{x \in I\!R^d : x_1 > 0\}$ denote the half space. Let $\ell : [0,\infty) \times I\!R \times I\!R^d \to I\!R^d, \Gamma : [0,\infty) \times I\!R \times I\!R^d \to I\!M_d(I\!R)$ be measurable functions; we denote $\ell(t,\xi,z) = (\ell_1(t,\xi,z), \cdots, \ell_d(t,\xi,z)), \Gamma(t,\xi,z) = ((\Gamma_{ij}(t,\xi,z)))_{1\le i,j\le d}$. We assume

(B1): For $1 \le i,j \le d, \ell_i, \Gamma_{ij}$ are bounded measurable, $(\xi,z) \mapsto \ell_i(t,\xi,z)$, $(\xi,z) \mapsto \Gamma_{ij}(t,\xi,z)$ are Lipschitz continuous, uniformly in t. Also $\Gamma_{ii} \equiv 1\ \forall\ i$; $|\ell_i(\cdot,\cdot,\cdot)| \le \beta_i,\ \beta = (\beta_1, \cdots, \beta_d)$.

(B2): For $i \ne j$ there exists constant W_{ij} such that $|\Gamma_{ij}(t,\xi,z)| \le W_{ij}$. Set $W = ((W_{ij}))$ with $W_{ii} \equiv 0$. We assume that $\sigma(W) < 1$.

Note: If b, R are as earlier, and $y_{-1}(\cdot) = (y_2(\cdot), \cdots, y_d(\cdot))$ is such that $y_j(\cdot) \ge 0, y_j(\cdot)$ nondecreasing function on $[0,\infty)$ for each $j \ne 1$, then we need later to take $\ell(t,\xi,z) = b(t,(\xi,y_{-1}(t-)),z),\ \Gamma(t,\xi,z) = R(t,(\xi,y_{-1}(t-)),z),\ t \ge 0,\ \xi \in I\!R,\ z \in I\!R^d$. $\square$

Suppose $w \in D([0,\infty) : I\!R^d)$. Let $y_j(\cdot), j \ne 1$ be given such that,

$$0 \le y_j(t) \le ((I-W)^{-1}h)_j(t), \tag{3.7}$$

$y_j(\cdot)$ is r.c.l.l. and nondecreasing, where h is given by (3.6). The Skorokhod problem $SP(w,\ell,\Gamma;H_1,y_{-1}(\cdot))$ in the half space $\bar{H}_1$ consists in finding a real valued r.c.l.l. function $Y^{(1)}w$ and an $I\!R^d$-valued r.c.l.l. function $Z^{(1)}w$ such that the following hold:

(i)$'$ Skorokhod equation holds, viz.

$$\begin{aligned}(Z^{(1)}w)_1(t) &= w_1(t) + \int_0^t \ell_1(r, Y^{(1)}w(r-), Z^{(1)}w(r-))\, dr \\ &\quad + \sum_{j\ne 1} \int_0^t \Gamma_{1j}(r, Y^{(1)}w(r-), Z^{(1)}w(r-))\, dy_j(r) + Y^{(1)}w(t)\end{aligned} \tag{3.8}$$

and for $i = 2, \ldots, d$

$$\begin{aligned}
(Z^{(1)}w)_i(t) &= w_i(t) + \int_0^t \ell_i(r, Y^{(1)}w(r-), Z^{(1)}w(r-))\, dr \\
&\quad + \sum_{j\neq 1} \int_0^t \Gamma_{ij}(r, Y^{(1)}w(r-), Z^{(1)}w(r-))\, dy_j(r) \\
&\quad + \int_0^t \Gamma_{i1}(r, Y^{(1)}w(r-), Z^{(1)}w(r-))\, d(Y^{(1)}w)(r) \qquad (3.9)
\end{aligned}$$

(ii)$'$ $Z^{(1)}w(t) \in \bar{H}_1$ for all $t \geq 0$;

(iii)$'$ $Y^{(1)}w(0) \geq 0, Y^{(1)}w(\cdot)$ is nondecreasing;

(iv)$'$ $Y^{(1)}w(\cdot)$ can increase only when $(Z^{(1)}w)_1(\cdot) = 0$.

With $\ell,\ \Gamma,\ w(\cdot),\ y_j(\cdot),\ j \neq 1$ as above satisfying (B1), (B2), (3.7), if $Y^{(1)}w,\ Z^{(1)}w$ solve the above Skorokhod problem, then it can be proved as in Proposition 3.2 of [Ra 1] that for $t \geq 0$,

$$0 \leq Y^{(1)}w(t) \leq ((I - W)^{-1}h)_1(t). \qquad (3.10)$$

In view of the above a priori estimate it is enough to consider only those $y^{(1)}(\cdot) \in D_{\uparrow}[0,T]$ satisfying (3.10).

Let $t > 0$; for $(Y^{(1)}, Z^{(1)}), (U^{(1)}, V^{(1)}) \in D_{\uparrow}([0,t] : I\!R_+) \times D([0,t] : \bar{H}_1)$ define the metric d_t by

$$\begin{aligned}
& d_t((Y^{(1)}, Z^{(1)}), (U^{(1)}, V^{(1)})) \\
= \quad & \mathrm{Var}\ (Y^{(1)} - U^{(1)} : [0,t]) + \sup_{0\leq s\leq t} |Z^{(1)}(s) - V^{(1)}(s)| \qquad (3.11)
\end{aligned}$$

where $\mathrm{Var}(g : [0,t])$ denotes the total variation of the real valued function g over the interval $[0,t]$. Note that we get a complete metric space.

To solve $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$, if

$$\begin{aligned}
& w_1(0) - w_1(0-) + \sum_{j\neq 1} \Gamma_{1j}(0, Y^{(1)}w(0-), Z^{(1)}w(0-))[y_j(0) - y_j(0-)] \\
:= \quad & w_1(0) + \sum_{j\neq 1} \Gamma_{1j}(0,0,0) y_j(0) < 0, \qquad (3.12)
\end{aligned}$$

take $Y^{(1)}w(0) = -[w_1(0) + \sum_{j\neq 1} \Gamma_{1j}(0,0,0)y_j(0)], Z^{(1)}w(0) = 0$. So we may take without loss of generality that l.h.s. of (3.12) ≥ 0.

For $(Y^{(1)}, Z^{(1)}) \in D_{\uparrow}([0,T] : I\!R_+) \times D([0,T] : \bar{H}_1)$ define $(\hat{Y}^{(1)}, \hat{Z}^{(1)}) \in D_{\uparrow}([0,T] : I\!R_+) \times D([0,T] : \bar{H}_1)$ by stipulating that $(\hat{Y}^{(1)}, \hat{Z}^{(1)})$ solves the Skorokhod problem for the function

$$t \quad \mapsto \quad w_i(t) + \int_0^t \ell_i(r, Y^{(1)}(r-), Z^{(1)}(r-))\, dr$$

$$+\sum_{j\neq 1}\int_0^t \Gamma_{ij}(r, Y^{(1)}(r-), Z^{(1)}(r-))\ dy_j(r), 1 \leq i \leq d.$$

Note that the above is a *known* $\mathbb{R}^d$-valued function. Getting $\hat{Y}^{(1)}, \hat{Z}^{(1)}$ is done by solving the one dimensional Skorokhod problem for

$$\begin{aligned} t \quad \mapsto \quad & w_1(t) + \int_0^t \ell_1(r, Y^{(1)}(r-), Z^{(1)}(r-))\ dr \\ & + \sum_{j\neq 1}\int_0^t \Gamma_{1j}(r, Y^{(1)}(r-), Z^{(1)}(r-))\ dy_j(r). \end{aligned}$$

Put $S(Y^{(1)}, Z^{(1)}) = (\hat{Y}^{(1)}, \hat{Z}^{(1)})$. If $y_j(\cdot), j \neq 1$ satisfy (3.7) it follows that $\hat{Y}^{(1)}$ satisfies (3.10). Using Shashiashvili's estimate [Sh 2] one can show that there exists $t_1 > 0$ such that S is a strict contraction on $(D_{\uparrow}([0,t] : \mathbb{R}_+) \times D([0,t] : \bar{H}_1), d_t)$ for all $t < t_1$. So $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$ is well posed on $[0, t_1)$ and hence $Y^{(1)}w(t_1-), Z^{(1)}w(t_1-)$ are defined. Well posedness on $[0, t_1]$ follows by defining

$$\begin{aligned} & Y^{(1)}w(t_1) - Y^{(1)}w(t_1-) \\ = \quad & \max\{0, -[(Z^{(1)}w)_1(t_1-) + w_1(t_1) - w_1(t_1-) \\ & + \sum_{j\neq 1}\Gamma_{1j}(t_1, Y^{(1)}w(t_1-), Z^{(1)}w(t_1-)) \cdot [y_j(t_1) - y_j(t_1-)]]\} \end{aligned}$$

and then defining $Z^{(1)}w(t_1)$ accordingly; in particular $(Z^{(1)}w)_1(t_1) = 0$ if $Y^{(1)}w(t_1) > Y^{(1)}w(t_1-)$.

Repeating the above as in [Ra 1] one can get $0 < t_1 < t_2 < \ldots < t_n < \ldots$ such that $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$ is well posed on $[0,t]$ for all $t \leq t_n, n = 1, 2, \ldots$ Now use an argument as in Step 7 of Section 3 of [Ra 1] to conclude that $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$ is well posed on $[0, T]$ for any $T > 0$. Thus we have

Theorem 3.4 *Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ satisfy (B1), (B2), (3.7). Then $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$ is well posed on $[0, T]$ for any $T > 0$. Moreover $Y^{(1)}w(\cdot)$ satisfies (3.10) for all t.* □

4 A Nash equilibrium

We begin with a brief description of the deterministic dynamic d-person game with state space constraints, alluded to in Section 2.

Let $w(\cdot) \in D([0,\infty) : \mathbb{R}^d)$. For $y(\cdot) = (y_1(\cdot), \cdots, y_d(\cdot))$ with $y_i(\cdot) \geq 0, y_i(\cdot)$ nondecreasing r.c.l.l., consider the integral equation, called the *state equation*,

$$z(t) \quad := \quad z(t; w(\cdot), y(\cdot))$$

$$= \quad w(t) + \int_0^t b(r, y(r-)), z(r-))\ dr + \int_0^t R(r, y(r-)), z(r-))\ dy(r) \tag{4.1}$$

where $\int_0^t \cdots = \int_{[0,t]} \cdots$, and we take $w(0-) = y(0-) = z(0-) = 0$; clearly $\int_0^t b(r, y(r-), z(r-))\ dr = \int_0^t b(r, y(r), z(r))\ dr$. Note that (4.1) can be written, for $i = 1, 2, \ldots, d$

$$\begin{aligned} z_i(t) &= w_i(t) + \int_0^t b_i(r, y(r-), z(r-))\ dr + y_i(t) \\ &\quad + \sum_{j \neq i} \int_{[0,t]} R_{ij}(r, y(r-), z(r-))\ dy_j(r). \end{aligned} \tag{4.2}$$

Proposition 4.1 *Under (A1), (A2) for given $w(\cdot), y(\cdot)$ as above, equation (4.1) has a unique solution $z(\cdot) \in D([0, \infty) : \mathbb{R}^d)$.*

Proof: We just indicate the key steps of the proof. By (4.1) it is clear that $z(0)$ should be given by

$$z(0) = z(0-) + [w(0) - w(0-)] + R(0, 0, 0)\ [y(0) - y(0-)].$$

With $z(0)$ uniquely determined, putting $z(0) = z_0$, note that (4.1) may be written as

$$\begin{aligned} z(t) &= z_0 + [w(t) - w(0)] + \int_{(0,t]} b(r, y(r-), z(r-))\ dr \\ &\quad + \int_{(0,t]} R(r, y(r-), z(r-))\ dy(r). \end{aligned}$$

For $t > 0$, set $D_{z_0,t} = \{z \in D([0,t] : \mathbb{R}^d) : z(0) = z_0\}$; note that $D_{z_0,t}$ is a complete metric space under the uniform metric. For r.c.l.l. z with $z(0) = z_0$ define $(Sz)(\cdot)$ by

$$\begin{aligned} (Sz)(t) &= z_0 + [w(t) - w(0)] + \int_{(0,t]} b(r, y(r-), z(r-))\ dr \\ &\quad + \int_{(0,t]} R(r, y(r-), z(r-))\ dy(r). \end{aligned}$$

By Lipschitz continuity of b, R in the z-variables we get

$$\sup_{0 \le s \le t} |Sz(s) - S\hat{z}(s)| \le K \left[t + \sum_{j=1}^{d} (y_j(t) - y_j(0)) \right] \cdot \sup_{0 \le s \le t} |z(s) - \hat{z}(s)|$$

for r.c.l.l. $z(\cdot), \hat{z}(\cdot)$ with $z(0) = \hat{z}(\cdot) = z_0$. By right continuity of y_j's, one can get $\hat{t} > 0$ such that S is a strict contraction on $D_{z_0, \hat{t}}$ and hence the state equation (4.1) is well posed on $[0, \hat{t}]$. If $\hat{t}$ is a point of continuity of all the y_j's, repeat the above procedure with $\hat{t}, z(\hat{t})$ playing the roles of $0, z_0$. Iterating, one can get $t_1 > 0$ such that the problem is well posed on $[0, t]$ for every $t < t_1$. Thus $z(t_1-)$ is uniquely determined.

It is clear that $z(t_1)$ should be given by

$$\begin{aligned} z(t_1) &= z(t_1-) + [w(t_1) - w(t_1-)] \\ &\quad + R(t_1, y(t_1-), z(t_1-)) \ [y(t_1) - y(t_1-)]; \end{aligned}$$

thus $z(\cdot)$ is uniquely determined on $[0, t_1]$.

Repeat the above steps with $t_1, z(t_1)$ playing respectively the roles of $0, z_0$. Iterate. One can get $0 < t_1 < t_2 < \cdots < t_n < \cdots$ such that $z(\cdot)$ is uniquely determined on $[0, t_n]$ for every n. Apply an argument as in Step 7, Section 3 of [Ra 1] to conclude that $z(\cdot)$ is uniquely determined on $[0, T]$ for any $T > 0$. □

We consider a d-person game whose state equation is given by (4.1) or (4.2). The nondecreasing function $y_i(\cdot)$ represents the control for the ith player, $1 \leq i \leq d$. However we consider only those controls that ensure that $z(\cdot)$ lives in $\bar{G}$; so it is a *d-person dynamic game with state space constaints*; cf. see [Ra 2].

For $T > 0, w(\cdot) \in D([0, T] : I\!R^d)$ let $\mathcal{A}(w(\cdot), T) = \{y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot)) \in D([0, T] : I\!R^d) : y_i(\cdot) \geq 0, y_i(\cdot)$ nondecreasing and $z(t) \in \bar{G}$ for all $0 \leq t \leq T$, where $z(\cdot)$ is given by (4.1)$\}$ denote the set of *feasible controls*. By Theorem 3.3, under (A1) - (A3), $\mathcal{A}(w(\cdot), T) \neq \phi$. In view of the discussion in Section 2, we consider the *cost function J_i for the i-th player* given by, for $T > 0, w(\cdot) \in D([0, T] : I\!R^d), y(\cdot) \in \mathcal{A}(w(\cdot), T)$

$$J_i(y(\cdot); w(\cdot), T) = y_i(T). \tag{4.3}$$

As each player tries to minimise his cost, the a priori bound (3.5) suggests that it may be fruitful to consider the restricted class

$$\begin{aligned} \mathcal{A}_h(w(\cdot), T) &= \{y(\cdot) \in \mathcal{A}(w(\cdot), T) : \\ &\quad 0 \leq y_i(t) \leq ((I - W)^{-1} h)_i(t), \ 0 \leq t \leq T, \ 1 \leq i \leq d\} \end{aligned} \tag{4.4}$$

where W is as in (A3), h given by (3.6). In addition to $Yw(\cdot)$, the y-part of the solution to $SP(w, b, R)$ under (A1) - (A3), another element of $\mathcal{A}_h(w(\cdot), T)$ is given by $y_i(t) = ((I - W)^{-1} h)_i(t), 0 \leq t \leq T, 1 \leq i \leq d$; see Theorem 5.1 of [Ra 1].

A feasible control $y^*(\cdot) = (y_1^*(\cdot), \ldots, y_d^*(\cdot)) \in \mathcal{A}_h(w(\cdot), T)$ is said to be a *Nash equilibrium* in $\mathcal{A}_h(w(\cdot), T)$ if for each $i = 1, 2, \ldots, d$,

$$J_i(y^*(\cdot); w(\cdot), T) \leq J_i((y_i(\cdot), y_{-i}^*(\cdot)); w(\cdot), T) \tag{4.5}$$

for any $y_i(\cdot) \in D([0,T] : I\!R)$ such that

$$(y_i(\cdot), y^*_{-i}(\cdot)) \equiv (y^*_1(\cdot), \ldots, y^*_{i-1}(\cdot), y_i(\cdot), y^*_{i+1}(\cdot), \ldots, y^*_d(\cdot)) \in \mathcal{A}_h(w(\cdot), T).$$

In order to understand the connection between the d-person game above and the Skorokhod problem in the orthant, it is convenient to consider the following control problem in a half space.

Let $H \equiv H_1 \equiv \{x \in I\!R^d : x_1 > 0\}$. Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1, h(\cdot)$ be as in Theorem 3.4. For $y_1(\cdot) \in D_{\uparrow}([0,T] : I\!R_+)$ we consider the state equation given by the system of d equations

$$\begin{aligned} z_1(t) &= w_1(t) + \int_0^t \ell_1(r, y_1(r-), z(r-))\ dr \\ &\quad + \sum_{j\neq 1} \int_0^t \Gamma_{1j}(r, y_1(r-), z(r-))\ dy_j(r) + y_1(t) \end{aligned} \qquad (4.6)$$

and for $i = 2, \ldots, d$

$$\begin{aligned} z_i(t) &= w_i(t) + \int_0^t \ell_i(r, y_1(r-), z(r-))\ dr \\ &\quad + \sum_{j\neq 1} \int_0^t \Gamma_{ij}(r, y_1(r-), z(r-))\ dy_j(r) \\ &\quad + \int_0^t \Gamma_{i1}(r, y_1(r-), z(r-))\ dy_1(r). \end{aligned} \qquad (4.7)$$

The function $y_1(\cdot)$ is regarded as the control; moreover we consider only those controls such that the solution $z(\cdot)$ of (4.6) - (4.7) lives in $\bar{H}_1$, that is, $z_1(\cdot) \geq 0$. We seek a feasible control such that $y_1(T)$ is minimal. In view of Theorem 3.4 (more precisely (3.10)) it is enough to focus on

$$\begin{aligned} \mathcal{V} \equiv \mathcal{V}_h^{(1)} &:= \{y_1(\cdot) \in D_{\uparrow}([0,T] : I\!R_+) : y_1(\cdot) \text{ is feasible} \\ &\quad \text{in the sense that } z_1(\cdot) \geq 0, \text{ and} \\ &\quad 0 \leq y_1(t) \leq ((I-W)^{-1}h)_1(t), 0 \leq t \leq T\}. \end{aligned} \qquad (4.8)$$

$T > 0$ is fixed for our discussion. Note that $\mathcal{V} \neq \phi$ by Theorem 3.4. We next define a suitable topology on $\mathcal{V}$.

Remark 4.2 It can be shown that $y_1(t) = ((I-W)^{-1}h)_1(t), 0 \leq t \leq T$ is an element of $\mathcal{V}$; the modifications needed in the proof of Theorem 5.1 of [Ra 1] are pretty obvious. Now, if $((I-W)^{-1}h)_1(T) = 0$ then by monotonicity $((I-W)^{-1}h)_1(t) = 0$ for all $0 \leq t \leq T$.

It is then easily shown that the y-part of the solution to $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$ is given by $Y^{(1)}w(t) = 0, 0 \leq t \leq T$. Therefore in this case the optimal control is clearly given by $Y^{(1)}w(\cdot)$, the y-part of the solution to the Skorokhod problem in the half space. □

In view of the preceding remark we assume that $((I - W)^{-1}h)_1(T) > 0$. Extend $y_1(\cdot) \in \mathcal{V}$ to $I\!R$ by putting $y_1(r) = 0, r < 0$, and $y_1(r) = y_1(T), r \geq T$; identify it with the measure given by

$$\mu_{y_1}((a, b]) = \frac{1}{((I - W)^{-1}h)_1(T)} \; [y_1(b) - y_1(a)], \;\; a, b \in I\!R, \;\; a \leq b.$$

Note that $\{\mu_{y_1} : y_1(\cdot) \in \mathcal{V}\}$ is a family of subprobability measures on $I\!R$; these measures are supported on $[0, T]$. Any $r < 0$ or $r > T$ is a continuity point of $y_1(\cdot)$ (or equivalently μ_{y_1}). We say $\{y_1^{n)}(\cdot)\}$ *converges vaguely* to $y_1(\cdot)$, denoted $y_1^{(n)} \xrightarrow{v} y_1$, if $\mu_{y_1^{(n)}} \xrightarrow{v} \mu_{y_1}$; that is $y_1^{(n)} \xrightarrow{v} y_1$ if $[y_1^{(n)}(t) - y_1^{(n)}(s)] \to [y_1(t) - y_1(s)]$ for continuity points s, t of (extended) $y_1(\cdot)$. See [Ch] for a detailed discussion on vague convergence of subprobability measures. By Helly selection principle it follows that $\mathcal{V} \equiv \mathcal{V}_h^{(1)}$ is relatively compact in the above topology.

To prove that $\mathcal{V}$ is compact it is enough to show that it is closed. For this we make the following additional assumptions:

(B3): ℓ_1 does not depend on $z_l, l \neq 1$; that is, $\ell_1(t, \xi, z) = \ell_1(t, \xi, z_1)$.

(B4): Γ_{1j} is a function only of the time variable t; that is, $\Gamma_{1j}(t, \xi, z) = \Gamma_{1j}(t), j \neq 1$.

Now let $y_1^{(n)}(\cdot) \in \mathcal{V}$ and $y_1^{(n)} \xrightarrow{v} y_1$; we need to show that $y_1(\cdot) \in \mathcal{V}$. Let $z^{(n)}(\cdot), z(\cdot)$ denote the respective $I\!R^d$-valued solutions to the system of state equations (4.6), (4.7) corresponding to $y_1^{(n)}(\cdot), y_1(\cdot)$. It is fairly simple to check that $0 \leq y_1(t) \leq ((I - W)^{-1}h)_1(t)$ for all $0 \leq t \leq T$. It remains to show that $z_1(t) \geq 0, 0 \leq t \leq T$.

By (4.6) and the hypotheses (B1) - (B4),

$$
\begin{aligned}
|z_1^{(n)}(t) - z_1(t)| &\leq |y_1^{(n)}(t) - y_1(t)| + K \int_0^t |y_1^{(n)}(r) - y_1(r)| \, dr \\
&\quad + K \int_0^t |z_1^{(n)}(r) - z_1(r)| \, dr \\
&= \alpha_n(t) + \theta_n(t) + K \int_0^t |z_1^{(n)}(r) - z_1(r)| \, dr, \ \text{(say)} \ .
\end{aligned}
$$

Gronwall inequality now implies [BC],

$$
|z_1^{(n)}(t) - z_1(t)| \leq \alpha_n(t) + \theta_n(t) + K \ e^{Kt} \int_0^t (\alpha_n(r) + \theta_n(r)) \ e^{-Kr} dr.
$$

Note that $\theta_n(s) \to 0$ as $n \to \infty$ for all $s, \alpha_n(s) \to 0$ as $n \to \infty$ at points of continuity of $y_1(\cdot); \theta_n, \alpha_n$ are uniformly bounded. So the above implies that $z_1^{(n)}(t) \to z_1(t)$ for a.a.t. Hence $z_1(t) \geq 0$ for a.a.t; now right continuity of $z_1(\cdot)$ implies $z_1(t) \geq 0$ for all t. Thus $y_1(\cdot)$ is a feasible control, and we have proved

Theorem 4.3 *Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ satisfy (B1) - (B4) and (3.7). Let $\mathcal{V}_h^{(1)}$ be given by (4.8) and endowed with the topology of vague convergence of bounded measures. Then $\mathcal{V}_h^{(1)}$ is a nonempty compact space.* $\square$

Note that $y_1^{(n)} \xrightarrow{v} y_1 \Leftrightarrow \int f(r) \ dy_1^{(n)}(r) \to \int f(r) \ dy_1(r)$ for any continuous function f on $\mathbb{R}$ with compact support. Now let f_0 be a continuous function on $\mathbb{R}$ with compact support such that $f_0(\cdot) = 1$ on $[0, T]$; as μ_{y_1} is supported on $[0, T]$ for any $y_1(\cdot) \in \mathcal{V}$,

$$
y_1(T) = y_1(T) - y_1(0-) = \int f_0(r) \ dy_1(r).
$$

So the cost function given by $y_1(\cdot) \mapsto y_1(T)$ is a continuous function on the compact space $\mathcal{V}_h^{(1)}$. So there exists $y_1^{(0)}(\cdot) \in \mathcal{V}_h^{(1)}$ such that

$$
y_1^{(0)}(T) = \inf\{y_1(T) : y_1(\cdot) \in \mathcal{V}_h^{(1)}\}. \tag{4.9}
$$

We want to show that $Y^{(1)}w(T) = y_1^{(0)}(T)$, under suitable conditions.

Lemma 4.4 *Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ satisfy (B1) - (B4), (3.7). In addition, assume that ℓ_1 is a function of only the time variable. Then $Y^{(1)}w(T) = y_1^{(0)}(T)$. In fact, $(Y^{(1)}w)(t) \leq y_1(t), 0 \leq t \leq T$ for any feasible y_1.*

Proof: By our assumptions, $t \mapsto w_1(t) + \int\limits_0^t \ell_1(r)\ dr + \sum\limits_{j\neq 1} \int\limits_0^t \Gamma_{1j}(r)\ dy_j(r)$ is a known function of t. Clearly $(Y^{(1)}w)(\cdot), (Z^{(1)}w)_1(\cdot)$ solve the one dimensional Skorokhod problem for the above function. For any $y_1(\cdot) \in \mathcal{V}_h^{(1)}$, if $z(\cdot) = (z_1(\cdot), \ldots, z_d(\cdot))$ denotes the corresponding solution to the state equation (4.6), (4.7), then by our hypothesis.

$$z_1(s) - y_1(s) = w_1(s) + \int\limits_0^s \ell_1(r)\ dr + \sum_{j\neq 1} \int\limits_0^s \Gamma_{1j}(r)\ dy_j(r).$$

Hence it follows that

$$\begin{aligned} (Y^{(1)})w(t) &= \sup_{0\leq s\leq t} \max\left\{0, -\left[w_1(s) + \int\limits_0^s \ell_1(r)\ dr + \sum_{j\neq 1}\int\limits_0^s \Gamma_{1j}(r)\ dy_j(r)\right]\right\} \\ &\leq y_1(t) \end{aligned}$$

as $z_1(\cdot) \geq 0$. The lemma now follows. □

Theorem 4.5 *Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ satisfy (B1) - (B3), (3.7). In addition let the following hypotheses be satisfied:*

(B5)*: $\xi \leq \tilde{\xi} \Rightarrow \ell_1(r, \xi, z) \geq \ell_1(r, \tilde{\xi}, z)$*

(B6)*: ℓ_1 is differentiable w.r.t. z_1 and has a nonpositive derivative, viz.*

$$\ell_{13}(r, \xi, z_1) := \frac{\partial}{\partial z_1}\ell_1(r, \xi, z_1) \leq 0.$$

(Here we are implicitly assuming (B3) as well.)

(B7)*: Γ_{ij} depends only on the time variable $r, 1 \leq i, j \leq d$. (Clearly (B7) $\Rightarrow$ (B4); also recall $\Gamma_{ii} = 1$).*

Let $Y^{(1)}w(\cdot)$ denote the y-part of the solution to $SP(w, \ell, \Gamma; H_1, y_{-1}(\cdot))$. Then

$$(Y^{(1)}w)(t) \leq y_1(t), 0 \leq t \leq T \text{ for all } y_1(\cdot) \in \mathcal{V}_h^{(1)}. \qquad (4.10)$$

In particular $(Y^{(1)}w)(T) = y_1^{(0)}(T) =$ r.h.s. of (4.9).

Proof: Let $y_1^{(0)}(\cdot) \in \mathcal{V}_h^{(1)}$ be such that (4.9) holds. Let $z^{(0)}(\cdot) = (z_1^{(0)}(\cdot), \ldots, z_d^{(0)}(\cdot))$ denote the solution to the state equation (4.6), (4.7) corresponding to the feasible control $y_1^{(0)}(\cdot)$. Put $\tilde{z}^{(0)}(\cdot) = z^{(0)}(\cdot)$. Note that $\tilde{z}_1^{(0)}(\cdot) = z_1^{(0)}(\cdot) \geq 0$ as $y_1^{(0)}(\cdot)$ is feasible.

For $k = 1, 2, \ldots$ define $y_1^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot)$ inductively as follows. Set $\ell^{(k-1)}(r) = \ell(r, y_1^{(k-1)}(r-), \tilde{z}^{(k-1)}(r-)), \Gamma^{(k-1}$
$\Gamma(r, y_1^{(k-1)}(r-), \tilde{z}^{(k-1)}(r-)) = \Gamma(r)$. Note that once $y_1^{(k-1)}(\cdot), \tilde{z}^{(k-1)}(\cdot)$ are known, $\ell^{(k-1)}(\cdot), \Gamma^{(k-1)}(\cdot)$

are known functions of r. Let $y_1^{(k)}(\cdot), z^{(k)}(\cdot)$ be the solution to the Skorokhod problem $SP(w, \ell^{(k-1)}, \Gamma^{(k-1)}; H_1, y_{-1}(\cdot$
so that

$$
\begin{aligned}
z_1^{(k)}(t) &= w_1(t) + \int_0^t \ell_1^{(k-1)}(r)\ dr + \sum_{j\neq 1} \int_0^t \Gamma_{1j}(r)\ dy_j(r) + y_1^{(k)}(t) \\
&= w_1(t) + \int_0^t \ell_1(r, y_1^{(k-1)}(r-), \tilde{z}_1^{(k-1)}(r-))\ dr \\
&\quad + \sum_{j\neq 1} \int_0^t \Gamma_{1j}(r)\ dy_j(r) + y_1^{(k)}(t) \qquad (4.11)
\end{aligned}
$$

and for $i = 2, \ldots, d$

$$
\begin{aligned}
z_i^{(k)}(t) &= w_i(t) + \int_0^t \ell_i(r, y_1^{(k-1)}(r-), \tilde{z}^{(k-1)}(r-))\ dr \\
&\quad + \sum_{j\neq 1} \int_0^t \Gamma_{ij}(r)\ dy_j(r) + \int_0^t \Gamma_{i1}(r)\ dy_1^{(k)}(r). \qquad (4.12)
\end{aligned}
$$

Note that $z_1^{(k)}(\cdot) \geq 0$. Next let $\tilde{z}^{(k)}(\cdot)$ be the solution to the state equations (4.6), (4.7) corresponding to $y_1^{(k)}(\cdot)$ so that

$$
\begin{aligned}
\tilde{z}_1^{(k)}(t) &= w_1(t) + \int_0^t \ell_1(r, y_1^{(k)}(r-), \tilde{z}_1^{(k)}(r-))\ dr \\
&\quad + \sum_{j\neq 1} \int_0^t \Gamma_{1j}(r)\ dy_j(r) + y_1^{(k)}(t) \qquad (4.13)
\end{aligned}
$$

and for $i = 2, \ldots, d$

$$
\begin{aligned}
\tilde{z}_i^{(k)}(t) &= w_i(t) + \int_0^t \ell_i(r, y_1^{(k)}(r-), \tilde{z}^{(k)}(r-))\ dr \\
&\quad + \sum_{j\neq 1} \int_0^t \Gamma_{ij}(r)\ dy_j(r) + \int_0^t \Gamma_{i1}(r)\ dy_1^{(k)}(r). \qquad (4.14)
\end{aligned}
$$

We claim that $\tilde{z}_1^{(k)}(t) \geq 0, 0 \leq t \leq T, k = 0, 1, 2, \ldots$ This is proved by induction on k. It is already done for $k = 0$. Assume it for $k \leq (n-1)$. As $\ell^{(n-1)}, \Gamma^{(n-1)}$ are functions only of r, by the preceding lemma, $y_1^{(n)}(t) \leq y_1(t), 0 \leq t \leq T$, for any $y_1(\cdot) \in D_\uparrow([0,T] : \mathbb{R}_+)$ such that for $0 \leq t \leq T$

$$
w_1(t) + \int_0^t \ell_1^{(n-1)}(r)\ dr + \sum_{j\neq 1} \int_0^t \Gamma_{1j}(r)\ dy_j(r) + y_1(t) \geq 0.
$$

But by induction hypothesis $\tilde{z}_1^{(n-1)}(\cdot) \geq 0$, and hence (4.13) for $k = n-1$ implies $y_1^{(n-1)}(\cdot)$ has the above property. So $y_1^{(n)}(t) \leq y_1^{(n-1)}(t)$ for all t. Consequently (4.11) with $k = n$, and (4.13) with $k = n-1$ give $z_1^{(n)}(t) \leq \tilde{z}_1^{(n-1)}(t), 0 \leq t \leq T$.

Next (4.11), (4.13) with $k = n$, (B3) and the mean value theorem give

$$\begin{aligned} \tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) &= \int_0^t \alpha^{(n)}(r)\, dr \\ &\quad + \int_0^t \ell_{13}(r, y_1^{(n-1)}(r), \theta^{(n)}(r))\, [\tilde{z}_1^{(n)}(r) - z_1^{(n)}(r)]\, dr \end{aligned} \tag{4.15}$$

where $\theta^{(n)}(r)$ is a point between $\tilde{z}_1^{(n)}(r)$ and $\tilde{z}_1^{(n-1)}(r)$, and

$$\begin{aligned} \alpha^{(n)}(r) &= [\ell_1(r, y_1^{(n)}(r), \tilde{z}_1^{(n)}(r)) - \ell_1(r, y_1^{(n-1)}(r), \tilde{z}_1^{(n)}(r)] \\ &\quad + [\ell_{13}(r, y_1^{(n-1)}(r), \theta^{(n)}(r))]\, [z_1^{(n)}(r) - \tilde{z}_1^{(n-1)}(r)]. \end{aligned}$$

In the above we have also used the fact that in dr integrals, $(r-)$ occurring in the integrands can be replaced by (r), as the discontinuities are countable. By the preceding paragraph and assumptions (B5), (B6) note that $\alpha^{(n)}(\cdot) \geq 0$. Therefore from (4.15) we get

$$\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) = \int_0^t \alpha^{(n)}(r)\, \exp\left(\int_r^t \ell_{13}(r', y_1^{(n-1)}(r'), \theta^{(n)}(r'))dr' \right)\, dr$$

and hence $\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) \geq 0$ for all t. As $z_1^{(n)}(\cdot) \geq 0$ we have $\tilde{z}_1^{(n)}(\cdot) \geq 0$, proving the claim.

It therefore follows that $y^{(n)}(\cdot) \in \mathcal{V}_h^{(1)}$ for all n. We have also proved $y_1^{(n)}(t) \leq y_1^{(n-1)}(t) \leq \cdots \leq y_1^{(0)}(t), 0 \leq t \leq T$ for all n. But by definition of $y_1^{(0)}(\cdot)$, note that $y_1^{(0)}(T) \leq y_1^{(n)}(T), \; \forall\, n$. Hence $y_1^{(n)}(T) = y_1^{(0)}(T)$ for all n.

The theorem will now follow once the following lemma is established. $\square$

Lemma 4.6 *Let $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ satisfy (B1), (B2), (3.7) and (B7). Let $y_1^{(0)}(\cdot) \in D_{\uparrow}([0,T] : \mathbb{R}_+), z^{(0)}(\cdot) \in D([0,T] : \mathbb{R}^d)$ be such that $0 \leq y_1^{(0)}(t) \leq ((I-W)^{-1}h)_1(t), z_1^{(0)}(t) \geq 0, 0 \leq t \leq T$, but otherwise arbitrary; (in particular they need not be as in the above proof). For $k = 1, 2, \ldots$ define $y_1^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot)$ inductively by analogues of (4.11) - (4.14). Then $y_1^{(n)}(\cdot) \to Y^{(1)}w$ in the total variation metric, $z^{(n)}(\cdot) \to Z^{(1)}w, \tilde{z}^{(n)}(\cdot) \to Z^{(1)}w$ both in the supremum metric.*

Proof: For $k = 1, 2, \ldots$ note that $y_1^{(k)}(\cdot), z^{(k)}(\cdot)$ solve the Skorokhod problem $SP(w, \ell^{(k-1)}, \Gamma^{(k-1)}; H_1, y_{-1}(\cdot))$ where

$$\ell^{(k-1)}(r) = \ell(r, y_1^{(k-1)}(r-), \tilde{z}^{(k-1)}(r-)), \Gamma^{(k-1)}(r) = \Gamma(r)$$

(which are 'known' functions of r), and $\tilde{z}^{(k)}(\cdot)$ solves the state equation (4.6), (4.7) for the control $y_1^{(k)}(\cdot)$; note that $\tilde{z}^{(k)}(\cdot)$ need not be $\bar{H}_1$-valued. As $y_1^{(k+1)}(\cdot), y_1^{(k)}(\cdot)$ are maximal functions for one dimensional Skorokhod problem, (and as $y_1^{(k+1)}(0) = y_1^{(k)}(0)$), by Shashiashvili's estimate [Sh 2], we get, denoting $\varphi^{(k+1)}(r) = \text{Var } (y_1^{(k+1)} - y_1^{(k)} : [0,r])$,

$$\varphi^{(k+1)}(t) \leq K \int_0^t \varphi^{(k)}(s)\ ds + K \int_0^t \left(\sup_{0 \leq r \leq s} |\tilde{z}^{(k)}(r) - \tilde{z}^{(k-1)}(r)| \right)\ ds. \tag{4.16}$$

By the state equations for $\tilde{z}^{(k+1)}(\cdot), \tilde{z}^{(k)}(\cdot)$,

$$\sup_{0 \leq s \leq t} |\tilde{z}^{(k+1)}(s) - \tilde{z}^{(k)}(s)| \leq K\ \varphi^{(k+1)}(t) + K \int_0^t \left(\sup_{0 \leq r \leq s} |\tilde{z}^{(k+1)}(r) - \tilde{z}^{(k)}(r)| \right) ds$$

and hence by Gronwall inequality

$$\sup_{0 \leq s \leq t} |\tilde{z}^{(k+1)}(s) - \tilde{z}^{(k)}(s)| \leq K_1 e^{K_1 t} \varphi^{(k+1)}(t) \tag{4.17}$$

for some constant K_1. Put

$$g^{(n)}(t) = \varphi^{(n)}(t) + \sup_{0 \leq s \leq t} |\tilde{z}^{(n)}(s) - \tilde{z}^{(n-1)}(s)|, 0 \leq t \leq T, n = 1, 2, \ldots$$

Then (4.16), (4.17) and iteration gives

$$\left. \begin{array}{rcl} g^{(n)}(t) & \leq & C \int\limits_0^t g^{(n-1)}(s)\ ds \leq \frac{C_1 C^{(n-1)} t^{n-1}}{(n-1)!} \\ & \to & 0, \text{ as } n \to \infty. \end{array} \right\} \tag{4.18}$$

So $\{y_1^{(n)}(\cdot)\}$ is a Cauchy sequence in $D_\uparrow[0,T]$ under total variation metric, and $\{\tilde{z}^{(n)}(\cdot)\}$ is Cauchy under supremum metric; these converge in the respective metrics to say, $\tilde{y}_1(\cdot), \tilde{z}(\cdot)$.

By analogues of (4.11) - (4.14) for $z^{(n)}(\cdot), \tilde{z}^{(n)}(\cdot)$ it now follows that for $0 \leq t \leq T$,

$$\sup_{0 \leq s \leq t} |z^{(n)}(s) - \tilde{z}^{(n)}(s)| \leq K\ g^{(n)}(t) \to 0.$$

Therefore $\lim z^{(n)}(\cdot) = \lim \tilde{z}^{(n)}(\cdot) = \tilde{z}(\cdot)$ in the supremum metric. Consequently it is clear that $\tilde{z}_1(\cdot) \geq 0$ (as $z_1^{(n)}(\cdot) \geq 0$ for all n) and

$$\begin{aligned} \tilde{z}_1(t) &= w_1(t) + \int_0^t \ell_1(r, \tilde{y}_1(r), \tilde{z}(r))\ dr \\ &\quad + \sum_{j \neq 1} \int_0^t \Gamma_{1j}(r)\ dy_j(r) + \tilde{y}_1(t), \\ \tilde{z}_i(t) &= w_i(t) + \int_0^t \ell_i(r, \tilde{y}_1(r), \tilde{z}(r))\ dr \\ &\quad + \sum_{j \neq 1} \int_0^t \Gamma_{ij}(r)\ dy_j(r) + \int_0^t \Gamma_{i1}(r)\ d\tilde{y}_1(r). \end{aligned}$$

In view of well posedness of Skorokhod problem in the half space, to complete the proof it is enough to prove that

$$\tilde{y}_1(t) = \int_0^t 1_{\{0\}}(\tilde{z}_1(s))\, d\tilde{y}_1(s), \tag{4.19}$$

that is, $\tilde{y}_1(\cdot)$ can increase only when $\tilde{z}_1(\cdot) = 0$. Note that (4.19) would follow once it is established that $\int_0^t \tilde{z}_1(s)\, d\tilde{y}_1(s) = 0$ for all t. Since $y_1^{(k)}(\cdot) \to \tilde{y}_1(\cdot)$ in total variation and $z_1^{(k)}(\cdot) \to \tilde{z}_1(\cdot)$ uniformly, it is easily seen that

$$\int_0^t \tilde{z}_1(s)\, d\tilde{y}_1(s) = \lim_{k\to\infty} \int_0^t z_1^{(k)}(s)\, dy_1^{(k)}(s) = 0$$

as $y_1^{(k)}(\cdot), z^{(k)}(\cdot)$ solve Skorokhod problem for each k. □

We now go back to the d-person game in the orthant $\bar{G}$ with state space constraints.

Theorem 4.7 *Let $b : [0,\infty) \times I\!R^d \times I\!R^d \to I\!R^d, R : [0,\infty) \times I\!R^d \times I\!R^d \to I\!M_d(I\!R)$ be as in Section 3. In addition to (A1) - (A3) let the following hypotheses hold:*
(A4): *R_{ij} are functions only of the time variable for $1 \le i, j \le d$, that is, $R(t, y, z) = R(t)$.*

(A5): *For $1 \le i \le d, b_i$ are independent of $z_l, l \ne i$, that is, $b_i(t, y, z) = b_i(t, y, z_i)$.*

(A6): *For fixed $1 \le i \le d, y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d) \in I\!R^{d-1}, t \ge 0, z \in I\!R^d$*

$$b_i(t, (\xi, y_{-i}), z) \ge b_i(t, (\tilde{\xi}, y_{-i}), z)$$

whenever $\xi \le \tilde{\xi}$, (here $(\xi, y_{-i}) = (y_1, \ldots, y_{i+1}, \xi, y_{i+1}, \ldots, y_d)$).

(A7): *For fixed i, the function $z_i \mapsto b_i(t, y, z_i)$ is differentiable and*

$$\frac{\partial}{\partial z_i}\, b_i(t, y, z_i) \le 0.$$

For $w \in D([0,T] : I\!R^d)$ let Yw, Zw be the solution to the Skorokhod problem $SP(w, b, R; G)$ in the orthant. Then Yw is a Nash equilibrium in $\mathcal{A}_h(w(\cdot), t)$ for any $0 \le t \le T$ for the cost functions given by (4.3), viz. $J_i(y(\cdot); w(\cdot), t) = y_i(t)$.

Proof: Fix $1 \le i \le d$. Set $y_{-i}(\cdot) = (Yw)_{-i}(\cdot),\ \ell(r, \xi, z) = b(r, (\xi, (Yw)_{-i}(r)), z),\ \Gamma(r) = R(r)$. Then (B1) - (B3), (3.7), (B5) - (B7) are satisfied with 1, H_1 replaced by i, H_i respectively, where $H_i = \{x \in I\!R^d : x_i > 0\}$. By uniqueness of solution to the Skorokhod problem in the half space $\bar{H}_i$, and as $\bar{G} \subset \bar{H}_i$, it follows that $(Yw)_i(\cdot), Zw$ solve $SP(w, \ell, \Gamma; H_i, (Yw)_{-i}(\cdot))$. So by the preceding theorem, $(Yw)_i(t) \le y_i(t)$ for any $y_i(\cdot) \in \mathcal{V}_h^{(i)}$, where $\mathcal{V}_h^{(i)}$ is defined analogous to (4.8). It is now clear that $(Yw)_i(t) \leq y_i(t), 0 \leq t \leq T$ for any $y_i(\cdot)$ such that $(y_i(\cdot), (Yw)_{-i}(\cdot)) \in \mathcal{A}_h(w(\cdot), t)$, because any feasible solution in the orthant is a feasible solution in the half space $\bar{H}_i$. This completes the proof. □

In the converse direction we have the following result

Theorem 4.8 *In addition to (A1) - (A3), assume the following hypotheses:*
(A8)*: For* $1 \leq i, j \leq d, b_i, R_{ij}$ *are independent of the* z*-variables.*
(A9)*: For* $1 \leq i, k \leq d$,

$$b_i(t, (y_k, y_{-k})) \geq b_i(t, (\tilde{y}_k, y_{-k}))$$

whenever $y_k \leq \tilde{y}_k$.
(A10)*: For* $1 \leq i, j, k \leq d, i \neq j$

$$0 \geq R_{ij}(t, (y_k, y_{-k})) \geq R_{ij}(t, (\tilde{y}_k, y_{-k}))$$

whenever $y_k \leq \tilde{y}_k$.

If $y^*(\cdot) = (y_1^*(\cdot), \ldots, y_d^*(\cdot))$ *is a Nash equilibrium in* $\mathcal{A}_h(w(\cdot), t)$ *for each* $0 \leq t \leq T$ *for the cost functions given by (4.3), then* $y^*(\cdot) = Yw(\cdot)$.

Proof: The proof of Theorem 5.11 of [Ra 1] can be applied here. □

Note: Existence of a Nash equilibrium is not asserted in the above result.

The following result is now immediate from the preceding theorems.

Corollary 4.9 *Assume the combined hypotheses of Theorems 4.7 and 4.8, that is, (A1) - (A3), (A4), (A8), (A9) hold and* $R_{ij}(t) \leq 0, i \neq j$. *Then* Yw *is the unique Nash equilibrium in* $\mathcal{A}_h(w(\cdot), t)$ *for each* t. □

Remark 4.10 In addition to the hypotheses of the above corollary, assume that $t \mapsto R_{ij}(t)$ is nondecreasing. Then it is shown in Theorem 5.3 of [Ra 1] that $(Yw)_i(t) \leq y_i(t), t \geq 0, 1 \leq i \leq d$, for *any* feasible control $y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot))$; that is, Yw can not be improved upon in any company; see [Re], [CM] for earlier results; see also Theorem 2.6 of [Ra 2]. In the terminology of [Ra 2], Yw is the unique utopian equilibrium in this case. □

Note: Example 4.14 in [Ra 2], in the context of absolutely continuous $w(\cdot)$, shows that Nash equilibrium (even serving for all t) need not be unique. □

In Theorem 4.7 note that the conditions on R_{ij} are more stringent than those on b_i. One reason for this is that we do not know of a suitable Gronwall inequality when $dy_j(\cdot)$ measures have atoms (that is, when $y_j(\cdot)$'s have discontinuities). When $y_j(\cdot)$ are continuous, conditions on R_{ij} or equivalently Γ_{ij} can be weakend. This is quite useful because, when $w(\cdot)$ come from sample paths of Brownian motion (for example, in the network model in Section 2), the solution to Skorokhod problem is continuous; so $(Yw)_i(\cdot)$ are continuous.

Proposition 4.11 *Let* $\ell, \Gamma, w(\cdot), y_j(\cdot), j \neq 1$ *be as in Theorem 4.3 except that*
(a) $y_j(\cdot), j \neq 1$ *are continuous;*
(b) instead of (B4) assume
(B4)$'$*:* Γ_{1j} *does not depend on* $z_l, l \neq 1$*, for* $j \neq 1$*; that is,* $\Gamma_{1j}(t, \xi, z) = \Gamma_{1j}(t, \xi, z_1)$.

Then the conclusion of Theorem 4.3 still holds.

Proof: Proceeding as in the proof of Theorem 4.3 we get, using the same notation,

$$
\begin{aligned}
|z_1^{(n)}(t) - z_1(t)| \quad \leq \quad & |y_1^{(n)}(t) - y_1(t)| + K \int_0^t |y_1^{(n)}(r) - y_1(r)| \left[dr + \sum_{j \neq 1} dy_j(r) \right] \\
& + K \int_0^t |z_1^{(n)}(r) - z_1(r)| \left[dr + \sum_{j \neq 1} dy_j(r) \right].
\end{aligned}
$$

As $y_j(\cdot), j \neq 1$ are continuous, Gronwall inequality implies as before $z_1^{(n)}(t) \to z_1(t)$ a.a.t. Proof is completed as before using right continuity of $z_1(\cdot)$. $\square$

Note: In the above proposition, even if $w(\cdot), y_1^{(n)}(\cdot), n = 1, 2, \ldots$ are continuous, $y_1(\cdot)$ (and hence $z(\cdot)$) can possibly have discontuities; this is because vague/weak limit of continuous probability distributions can have atoms.

Lemma 4.12 *Let f be a bounded measurable function on $[0, T], \nu$ a bounded nonatomic signed measure, and μ a finite (nonnegative) nonatomic measure on $[0, T]$ such that*

$$
f(t) = \int_0^t d\mu(s) + \int_0^t f(s)\ d\nu(s), \ \forall\ 0 \leq t \leq T. \qquad (4.20)
$$

Then

$$
f(t) = \int_0^t \exp(\nu[s, t])\ d\mu(s), \ \ 0 \leq t \leq T \qquad (4.21)
$$

and hence f is nonnegative.

Proof: Since ν is nonatomic, by Fubini's theorem, the n-fold product $(\nu \times \cdots \times \nu)$ gives zero measure to any $(n-1)$ or lower dimensional hyperplane. Next note that $(\nu \times \cdots \times \nu)$ is invariant under permutation of coordinates. Therefore for any $s < t, n = 1, 2, \ldots$ it follows that

$$
\left[\int_s^t d\nu(r) \right]^n = n! \int_{s < s_1 < s_2 < \ldots < s_n < t} d\nu(s_1)\ d\nu(s_2) \ldots d\nu(s_n). \qquad (4.22)
$$

The lemma can now be proved by iterating (4.20), and repeatedly using Fubini's theorem and (4.22). $\square$

We are now in a position to give the analogue of Theorem 4.5.

Theorem 4.13 *Assume (B1) - (B3), (B4)$'$ and*
(B5)$'$: $\xi \leq \tilde{\xi} \Rightarrow \ell_1(r, \xi, z_1) \geq \ell_1(r, \tilde{\xi}, z_1), \Gamma_{1j}(r, \xi, z_1) \geq \Gamma_{1j}(r, \tilde{\xi}, z_1)$ *for* $r \geq 0, z_1 \in I\!R, j \neq 1$;

(B6)′: *ℓ_1, Γ_{1j} are dfferentiable w.r.t. z_1, have bounded derivatives and*

$$\begin{aligned}
\ell_{13}(r,\xi,z_1) &:= \frac{\partial}{\partial z_1}\ell_1(r,\xi,z_1) \leq 0, \\
\Gamma_{1j3}(r,\xi,z_1) &:= \frac{\partial}{\partial z_1}\Gamma_{1j}(r,\xi,z_1) \leq 0.
\end{aligned}$$

Let $w(\cdot)$ be continuous with $w_1(0) \geq 0$. For $j \neq 1$, let $y_j(\cdot)$ be continuous, $y_j(0) = 0, y_j(\cdot)$ nondecreasing and $y_j(\cdot)$ satisfies (3.7). Then $(Y^{(1)}w)(t) \leq y_1(t)$ for all $0 \leq t \leq T, y_1(\cdot) \in \mathcal{V}_h^{(1)}$.

Proof: Let $y_1^{(0)}(\cdot), z^{(0)}(\cdot), \tilde{z}^{(0)}(\cdot)$ be as in the proof of Theorem 4.5, denoting an optimal element in $\mathcal{V}_h^{(1)}$ and the corresponding solution to state equation. For $k = 1, 2, \ldots$ let $y_1^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot)$ be defined inductively by analogues of (4.11) - (4.14). Note that though $y_1^{(0)}(\cdot), z^{(0)}(\cdot), \tilde{z}^{(0)}(\cdot)$ may not be continuous, by continuity of $w(\cdot), y_j(\cdot), j \neq 1$ it follows that $y_1^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot), k \geq 1$ are all continuous.

Proceeding as in the proof of Theorem 4.5, we get

$$\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) = \int_0^t d\mu^{(n)}(r) + \int_0^t (\tilde{z}_1^{(n)}(r) - z_1^{(n)}(r))\, d\nu^{(n)}(r) \tag{4.23}$$

where

$$
\begin{aligned}
d\mu^{(n)}(r) &= [\ell_1(r, y_1^{(n)}(r), \tilde{z}_1^{(n)}(r)) - \ell_1(r, y_1^{(n-1)}(r), \tilde{z}_1^{(n)}(r))]\ dr \\
&\quad + \sum_{j\neq 1} [\Gamma_{1j}(r, y_1^{(n)}(r), \tilde{z}_1^{(n)}(r)) - \Gamma_{1j}(r, y_1^{(n-1)}(r), \tilde{z}_1^{(n)}(r))\ dy_j(r) \\
&\quad + [\ell_{13}(r, y_1^{(n-1)}(r), \theta^{(n)}(r))]\ [z_1^{(n)}(r) - \tilde{z}_1^{(n-1)}(r)]\ dr \\
&\quad + \sum_{j\neq 1} [\Gamma_{1j3}(r, y_1^{(n-1)}(r), \theta^{(n,j)}(r))]\ [z_1^{(n)}(r) - \tilde{z}_1^{(n-1)}(r)]\ dy_j(r) \\
d\nu^{(n)}(r) &= \ell_{13}(r, y_1^{(n-1)}(r), \theta^{(n)}(r))\ dr \\
&\quad + \sum_{j\neq 1} \Gamma_{1j3}(r, y_1^{(n-1)}(r), \theta^{(n,j)}(r))\ dy_j(r),
\end{aligned}
$$

$\theta^{(n)}(r), \theta^{(n,j)}(r)$ being points between $\tilde{z}_1^{(n-1)}(r)$ and $\tilde{z}_1^{(n)}(r)$. Note that (4.23) is the analogue of (4.15). As in the earlier proof, using induction and our hypotheses, it is seen that $d\mu^{(n)}(r)$ is a nonnegative measure; it is also clear that $\mu^{(n)}(\cdot)$ is a bounded nonatomic measure on $[0, T]$. Similarly $d\nu^{(n)}(r)$ is a bounded nonatomic signed measure; (in fact, $\nu^{(n)}(\cdot)$ is a nonpositive measure). Therefore the preceding lemma now implies that $\tilde{z}^{(n)}(t) - z_1^{(n)}(t) \geq 0$ for all t. Consequently it follows as before that $y_1^{(n)}(\cdot) \in \mathcal{V}_h^{(1)}$ for all n, and $y_1^{(n)}(t) \leq y_1^{(n-1)}(t) \leq \ldots \leq y_1^{(0)}(t), 0 \leq t \leq T$ and hence $y_1^{(n)}(T) = y_1^{(0)}(T)$ for all n.

To get the analogue of Lemma 4.6, let $g^{(n)}(\cdot)$ be as in the proof of that lemma. Mimicking the arguments there

$$
\begin{aligned}
g^{(n)}(t) &\leq C \int_0^t g^{(n-1)}(r) \left[dr + \sum_{j\neq 1} dy_j(r) \right] \\
&\leq C_1 \frac{C^{n-1} \left[t + \sum_{j\neq 1} y_j(t) \right]^{n-1}}{(n-1)!} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

In the above we have used Gronwall inequality for integrals involving dr and $dy_j(r), j \neq 1$. The other details being exactly the same as in the earlier case, the proof of the theorem is complete. □

The following analogue of Theorem 4.7 is now easy to obtain; (cf. Theorem 4.9 of [Ra 2]).

Theorem 4.14 *Let $w(\cdot)$ be continuous with $w(0) \in \bar{G}$. In addition to (A1) - (A3) assume*

(A4)′: *For $1 \leq i \leq d, b_i, R_{ij}$ are independent of $z_l, l \neq i$; that is, $b_i(t, y, z) = b_i(t, y, z_i)$, $R_{ij}(t, y, z) = R_{ij}(t, y, z_i)$ for all j.*

(A5)′: *For fixed $1 \leq i \leq d, y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d), t \geq 0, z \in I\!R^d$*

$$
\begin{aligned}
b_i(t, (\xi, y_{-i}), z) &\geq b_i(t, (\tilde{\xi}, y_{-i}), z), \\
R_{ij}(t, (\xi, y_{-i}), z) &\geq R_{ij}(t, (\tilde{\xi}, y_{-i}), z)
\end{aligned}
$$

whenever $\xi \geq \tilde{\xi}$.

(A6)′: *The functions* $z_i \mapsto b_i(t, y, z_i), z_i \mapsto R_{ij}(t, y, z_i)$ *are differentiable, have bounded derivatives and*

$$\frac{\partial}{\partial z_i}\, b_i(t, y, z_i) \leq 0,\ \frac{\partial}{\partial z_i}\, R_{ij}(t, y, z_i) \leq 0,\ 1 \leq i,\ j \leq d.$$

Let Yw, Zw *be the solution to* $SP(w, b, R; G)$ *in the orthant. Then the conclusion of Theorem 4.7 holds.* □

In view of Theorems 4.8 and 4.14, the analogue of Corollary 4.9, when $w(\cdot)$ is continuous, is now obvious.

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