p-STATISTICS

OR

SOME GENERALISATIONS IN ANALYSIS OF VARIANCE APPROPRIATE TO MULTIVARIATE PROBLEMS

By

S. N. ROY.

Statistical Laboratory

Introduction.

In an earlier paper[1] worked out by the author jointly with Mr. R. C. Bose the problem of discrimination between and classification of different multi-variate normal populations (with the same set of variances and covariances but with different sets of means) was sought to be tackled by defining the Studentized D^2-statistic and obtaining its sampling distribution. But the problem yet remained to be attacked when the populations differ in the sets of variances and covariances and when it is sought to discriminate between and classify them in the light of these sets. It is the object of the present paper to take a first step towards answering the following question. Can the two multi-variate normal populations from which two samples have been drawn be reasonably (*i.e.* on a given probability level) supposed to have the same set of variances and covariances or to have distinct sets? For the uni-variate case the question is completely answered by the sampling distribution of s_1/s_2 and Fisher's z.

In the earlier paper[1] by the author already quoted it was noted that if following Fisher one started from a linear compound with arbitrary coefficients, of the variates and so chose the compounding coefficients as to make a maximum the ratio between the square of the difference of the sample means (for the compound character) and the within variance of the samples for the same character, one could easily see after maximisation that this ratio is proportional to the studentised D^2-statistic referred to in that paper. In the present paper starting from a similar linear compound of the variates and so choosing the compounding coefficients as to make a maximum the ratio of the variance of the first sample (for the compound character) and the variance of the second sample for the same character one finds that the quest for the maximised value of the ratio spoken of above, leads one in the case of p-variate populations to as many as p studentised statistics instead of to merely one studentised statistic as in the previous case. Of these statistics it is found that one is a true maximum, another is a true minimum and the rest are what are known in analysis as stationary values. The invariance under linear transformations of these statistics is then shown. Next a geometrical interpretation is given of these p-statistics connecting them up with the p critical angles between two definite hyper-planes, one of

p dimensions and the other of $n'-1$ dimensions (where $n'-1$, is of course greater than p, n' being the size of one of the samples). Finally, partly with the help of the geometrical interpretation and partly with the help of earlier results derived by the author jointly with Mr. R. C. Bose in two previous papers[1, 3] the joint sampling distribution of the statistics is worked out for the case where the two populations in question are identical in the sets of variances and covariances. The statistic corresponding to the maximum value of the ratio is here called the maximum statistic and that for the minimum value the minimum statistic. Either the maximum or the minimum statistic or another specified statistic would serve according to circumstances to be set forth in the concluding section of this paper, to distinguish best between two samples drawn from a multi-vatiate normal population. The sampling distribution of either of these can be obtained by considering the joint distribution of the p-statistics and integrating out for all the others between suitable limits. Work has been carried up to this point in the present paper. New tables are needed to make the result available for practical purposes, *i.e.* for testing the hypothesis whether the two populations from which the two samples have been drawn are identical or distinct on any given probability level so far as the set of variances and covariances is concerned. The problem of analysis of variance is ofcourse closely connected with the above problem and both are really solved together.

In the next paper which is to shortly to appear in this Journal the incomplete probability integral for the distribution of the maximum, or the minimum or the other specified statistic would be given in a form which facilitates the construction of suitable tables. It is further proposed in that paper to work out the sampling distribution of these statistics for the non-zero case, *i.e.* when the two populations sampled are distinct in the sets of variances and covarianes; this, too, is proposed to be given in a form suitable for the construction of appropriate tables. As distinguished from the problem of discrimination this will, of course, solve, so far as the dispersion matrix is concerned, the problem of classification of multi-variate normal populations; the need for and significance of this problem was pointed out in the introduction to the earlier paper by the author already referred to.

§1. DERIVATION OF THE p-STATISTICS BY MAXIMISATION.

I shall take over now certain definitions and notations from the previous paper.

Consider two samples Σ and Σ' of sizes n and n' from two p-variate normal populations Π and Π' with the same set of variances and co-variances, α_{ij} ($i, j=1, 2, \ldots\ldots p$) where $\alpha_{ij}=\rho_{ij}.\ \sigma_i.\ \sigma_j$, σ_i and σ_j being the standard deviations for the i^{th} and j^{th} characters respectively, and ρ_{ij} the correlations between the i^{th} and the j^{th} character. The matrix $||\alpha_{ij}||$ will be said to be the common dispersion matrix for the two populations. Let a_{ij}, a'_{ij} ($i, j=1, 2, \ldots\ldots\ldots\ldots p$) denote the respective variances, and co-variances of the samples Σ and Σ' so that $||a_{ij}||$ and $||a'_{ij}||$ are their respective dispersion matrices. Let α_i, α'_i ($i=1, 2, \ldots\ldots\ldots p$) be the mean for the i^{th} character for the populations Π and Π' and let a_i and a'_i denote corresponding quantities for the samples Σ and Σ'. Let the sample readings for the sample Σ be x_{ik} ($i=1, 2, \ldots\ldots\ldots p$; $k=1, 2, \ldots\ldots\ldots n$) and those for the sample Σ' be $x'_{ik'}$ ($i=1, 2, \ldots\ldots p$; $k'=1, 2, \ldots\ldots n'$) the first suffix in both cases referring to the character and the second to the individual.

Then

$$\left.\begin{aligned} a_i &= \frac{1}{n}\sum_{k=1}^{n} x_{ik} \ (i=1,\ 2,\ldots\ldots\ldots\ldots\ldots\ldots p) \\ a_i' &= \frac{1}{n'}\sum_{k'=1}^{n'} x'_{ik'} \ (i=1,\ 2,\ldots\ldots\ldots\ldots\ldots\ldots p) \\ a_{ij} &= \sum_{k=1}^{n} (x_{ik}-a_i)\ (x_{jk}-a_j)/n \ \ (i,\ j=1,\ 2,\ldots\ldots\ldots p) \\ a'_{ij} &= \sum_{k'=1}^{n'} (x'_{ik'}-a'_i)\ (x'_{jk'}-a'_j)/n' \ (i,\ j=1,\ 2,\ldots\ldots\ldots p) \end{aligned}\right\} \quad \ldots \quad (1\cdot1)$$

Let the compounding co-efficients be λ_i $(i=1,\ 2,\ldots\ldots\ldots p)$; from the point of view of the compound character the sample Σ is now represented by the n readings $\sum_{i=1}^{p} \lambda_i\ x_{ik}$ $(k=1,\ 2,\ldots\ldots\ldots\ldots n)$ and the sample Σ' by the n' readings $\sum_{i=1}^{p} \lambda_i\ x'_{ik'}$ $(k'=1,\ 2,\ldots\ldots\ldots n')$; if we now denote by b and b' the variances of the samples Σ and Σ' for the compound character then it is easily seen from (1·1) that

$$\left.\begin{aligned} b &= \sum_{i,\ j=1}^{p} \lambda_i\ \lambda_j\ a_{ij} \\ &\text{and} \\ b' &= \sum_{i,\ j=1}^{p} \lambda_i\ \lambda_j\ a'_{ij} \end{aligned}\right\} \quad \ldots \quad (1\cdot2)$$

Let us set the ratio of $b/b'=K^2$ ... (1·3)

Then maximising this K with regard to the arbitrary co-efficients λ_i $(i,=1,\ 2,\ldots\ldots p)$ we are easily led on to the system of p equations.

$$\sum_{j=1}^{p} \lambda_j\ (a_{ij}-K^2\ a'_{ij})=0,\ (i=1,\ 2,\ldots\ldots\ldots\ldots\ldots p) \quad \ldots \quad (1\cdot4)$$

Eliminating λ_j $(j=1,\ 2,\ldots\ldots\ldots p)$ between these p equations we get the following p-fold determinantal equation in K^2 whose p roots give the stationary values for K

$$|\ a_{ij}-K^2\ a'_{ij}\ | \quad (1\cdot5)$$

It can be shown from algebra as well as from certain geometrical considerations to be given in a later section that of these p roots one corresponds to the maximum K, the other to the minimum K and the rest to stationary values of K

§2. Invariance of the p-Statistics.

As in the previous paper we shall now show that if from the primary p statistical variates we obtain a new set of p variates by linear transformation with matrix $||\ \lambda_{ij}\ ||$ of rank p, then the p-statistics given by the p-roots of the equation (1·5) remain unchanged

If a_{ij} goes over to b_{ij}, it is easily seen that

$$b_{ij}=\lambda_{i1}\ \lambda_{j1}\ a_{11}+\ldots\ldots\ldots\lambda_{ip}\ \lambda_{jp}\ a_{pp}+(\lambda_{i1}\ \lambda_{j2}+\lambda_{i2}\ \lambda_{j1})a_{12}+\ldots\ldots\ldots(\lambda_{i,p-1}\ \lambda_{j,p}+\lambda_{i,p}\ \lambda_{j,p-1})a_{p-1,\ p} \quad (2\cdot1)$$

Thus

$$||\ b_{ij}\ ||=||\ \lambda_{ij}\ ||.\ ||\ a_{ij}\ ||.\ ||\ \lambda_{ji}\ || \quad \ldots \quad (2\cdot2)$$

where $||\lambda_{ji}||$ is the matrix obtained from $||\lambda_{ij}||$ by interchanging the rows and columns. Similarly if a'_{ij} goes over to b'_{ij} then we get a formula similar to (2·1). A similar relation holds for the corresponding determinants. Thus

$$\left.\begin{array}{l} |b_{ij}| = |a_{ij}| \cdot |\lambda_{ij}|^2 \\ \text{and} \\ |b'_{ij}| = |a'_{ij}| \cdot |\lambda_{ij}|^2 \end{array}\right\} \quad \ldots \quad (2\cdot3)$$

From (2·1) we easily see that if K be any arbitrary quantity, $||b_{ij}-K^2 b'_{ij}||$ transforms to $S_a |a_{ij}-K^2 a'_{ij}| \bar{S}_a$, where S_a denotes the transformation matrix $||\lambda_{ij}||$, and $\bar{S}_a$ the matrix obtained from it by interchanging the row and columns. Therefore $|b_{ij}-K^2 b'_{ij}|$ transforms to $|a_{ij}-K^2 a'_{ij}| \cdot |\lambda_{ij}|^2$; it should be noted that K^2 is the same in all these expressions. Hence the p roots of the equation $|b_{ij}-K b^2_{ij}|=0$ are both formally and numerically the same as the p roots of the equation $|a_{ij}-K^2 a'_{ij}|=0$. Hence the p-statistics are invariant under any linear transformation.

§3. Geometrical interpretation of the p-Statistics.

The sample Σ with the readings x_{ik} $(i=1, 2,\ldots\ldots p;\ k=1, 2,\ldots\ldots n)$ can be represented in the usual Fisherian space S_n of n dimensions by the p points with co-ordinates.

$$(x_{i1}, x_{i2},\ldots\ldots x_{in}),\ (i=1, 2,\ldots\ldots p) \quad \ldots \quad (3\cdot1)$$

or by p vectors $\mathbf{x}_i$ joining the points to the origin. We may take another space $S'_{n'}$ of n' dimensions absolutely orthogonal to the former space. and represent in it the sample Σ' by p other similar vectors $\mathbf{x}'_i$. Let

$$y_{ik}=x_{ik}-a_i,\ y'_{ik'}=x'_{ik'}-a'_i \quad \ldots \quad (3\cdot2)$$

where $i=1, 2,\ldots\ldots p$; $k=1, 2,\ldots\ldots n$; $k'=1, 2,\ldots\ldots n'$ and a_i and a'_i have been already defined in (1·1)

Let $\mathbf{y}_i, \mathbf{y}'_i$ denote the vectors, with components $(y_{i1}, y_{i2},\ldots\ldots y_{in})$ and $(y'_{i1}, y'_{i2},\ldots\ldots y'_{in'})$ lying in the spaces S_n and $S'_{n'}$ respectively. Then it is easily seen from (3·2) that the vectors $\mathbf{y}_i$ lie in a flat space S_{n-1} of $n-1$ dimensions, perpendicular to the equiangular line in S_n. Similar considerations apply to the vectors $\mathbf{y}'_i$. Denoting now the scalar product of $\mathbf{y}_i$ and $\mathbf{y}_j$ by $\mathbf{y}_i.\mathbf{y}_j$ we have clearly

$$a_{ij}=(\mathbf{y}_i.\mathbf{y}_j)/n,\ a'_{ij}=(\mathbf{y}'_i.\mathbf{y}'_j)/n' \quad \ldots \quad (3\cdot3)$$

Let us form the resultant of the vectors $\mathbf{y}_i$ and $\mathbf{y}'_i$ and call it $\mathbf{v}_i$ $(i=1, 2,\ldots\ldots p)$. The p vectors $\mathbf{v}_i$ form a flat of p-dimensions which makes with the flat $S'_{n'-1}$ containing the p vectors $\mathbf{y}'_i$, p critical angles which let us call φ_i $(i=1, 2,\ldots\ldots p)$. The significance and properties of these critical angles are, of course, well known to students of hyperspace geometry, we shall show that the p roots of the equation (1·5) are also equal to n'/n times the squares of the tangents of the p critical angles. That is to say,

$$K_i = \tan \varphi_i \sqrt{n'/n}\ (i=1, 2,\ldots\ldots p) \quad \ldots \quad (3\cdot4)$$

where K^2_i $(i=1, 2,\ldots\ldots p)$ are the p roots of the equation (1·5). This can be shown as follows.

Let us denote the flat space containing the vectors v_i by S_p. Then it is well known that if we take a line in S_p and any line in S_{n-1}, then a stationary value of the angle between these lines, obtained by taking appropriate positions of the two lines, is called a critical angle.

Any vector in the flat S_p is given by $\sum_{i=1}^{p} \lambda_i v_i$, where λ_i's are arbitrary co-efficients. The projection of this on the flat S'_{n-1} is clearly equal to $\sum_{i=1}^{p} \lambda_i y'_i$ and that on the flat S_{n-1} which is absolutely orthogonal to S'_{n-1} is $\sum_{i=1}^{p} \lambda_i y_i$; therefore, if the angle between the vector $\sum_{i=1}^{p} \lambda_i v_i$ and the flat S'_{n-1} is ϕ, then it must be given by

$$\tan^2 \phi = \frac{\sum_{i,j=1}^{p} \lambda_i \lambda_j y_i . y_j}{\sum_{i,j=1}^{p} \lambda_i \lambda_j y'_i . y'_j} = \frac{n}{n'} . \frac{\sum_{i,j=1}^{p} \lambda_i \lambda_j a_{ij}}{\sum_{i,j=1}^{p} \lambda_i \lambda_j a'_{ij}} \quad \text{from (3·3)} \qquad \dots (3·5)$$

Differentiating this with the regard to λ_i's we easily see that the critical angles are given by the p roots of the p-fold determinantal equation.

$$\left| \; a_{ij} - \frac{n'}{n} \tan^2 \phi . a'_{ij} \; \right| \qquad \dots (3·6)$$

Thus it is seen that the p roots of the equation (3·6) in $\frac{n'}{n} \tan^2 \phi$ are the same as the p roots of the equation (1·5) in K^2 giving the p-statistics. Let us denote the vectors $\sum_{i=1}^{p} \lambda_i v_i$, $\sum_{i=1}^{p} \lambda_i y_i$, $\sum_{i=1}^{p} \lambda_i y'_i$ by L_i, M_i, N_i respectively. Then there are p such triads, each triad, of course, determining one ordinary plane; it is well known and can also be very easily proved that each such plane is perpendicular to all other planes, *i.e.*, L_i, M_i, N_i, are each perpendicular to each of L_j, M_j, N_j ($i \neq j$, $i, j = 1, 2, \dots\dots p$).

§4. The Distribution of the p-Statistics K_i ($i = 1, 2, \dots\dots p$) for the Case where the Two Populations Sampled have Identical Dispersion Matrices

In an earlier paper[3] published by the author jointly with Mr. R. C. Bose the joint sampling distribution of a_{ij}'s for a sample Σ drawn from a p-variate normal population Π was given in the form

$$\text{const.}\; e^{-\frac{n}{2} \sum_{i,j=1}^{p} \alpha^{ij} a_{ij}} \; |a_{ij}|^{(n-p-2)/2} \; \Pi da_{ij} \qquad \dots (4.1)$$

where $\Pi\, da_{ij}$ is the product $da_{11}\, da_{22} \dots\dots da_{pp}\, da_{12} \dots\dots da_{p-1,p}$ and α^{ij} is the minor of α_{ij} in the determinant $|\alpha_{ij}|$ divided by the determinant itself. By an entirely different method this distribution was also obtained by Wishart[4] in an earlier paper.

Similarly the joint distribution of a'_{ij}'s for the sample Σ' would be given by

$$\text{const. } e^{-\frac{n'}{2}\sum_{ij=1}^{p} a^{ij} a'_{ij}} \; |a'_{ij}|^{(n'-p-2)/2} \; \Pi\, da'_{ij} \qquad \dots \quad (4\cdot2)$$

Therefore the joint distribution of a_{ij} and a'_{ij} would be given by

$$\text{const. } e^{-\frac{1}{2}\sum_{ij=1}^{p} a^{ij}(n\, a_{ij}+n'\, a'_{ij})} \; |a_{ij}|^{(n-p-2)/2} \; |a'_{ij}|^{(n'-p-2)/2} \; \Pi\, da_{ij}\, \Pi\, da'_{ij} \qquad \dots \quad (4\cdot3)$$

In the flat S_p let us refer the vectors v_i to p orthogonal axes along the directions of the critical vectors L_i $(i=1, 2, \dots\dots p)$; let us call the projections of the vectors v_i along these directions, v_{ij} $(j=1, 2, \dots\dots\dots p)$; the vectors v_i are now replaced by the p^2 component vectors v_{ij} $(i, j=1, 2, \dots\dots\dots p)$; where the first suffix refers to the vector and the second suffix to its component along the axis in the direction of the critical vector L_j. The projection of v_i in the direction of L_j, when regarded as a vector shall be denoted by $\boldsymbol{v}_{ij}$ and its magnitude will be denoted simply by v_{ij}. It now follows from the considerations given at the end of Section 3 that

$$\begin{aligned} y'_i &= \text{projection of } v_i \text{ on } S'_{n'-1} \\ &= \text{vector sum of the projections of } v_{ij}\ (j=1, 2, \dots\dots\dots p) \text{ on } S'_{n'-1} \\ &= \sum_{j=1}^{p} v_{ij} \cos\varphi_j \text{ (the summation being of course a vector summation)} \end{aligned} \qquad \dots \quad (4\cdot4)$$

the different vectors $v_{ij}\cos\varphi_j$ being in the orthogonal directions N_j

$$\text{Similarly } y_i = \sum_{j=1}^{p} v_{ij} \sin\varphi_j \qquad \dots \quad (4\cdot5)$$

the summation being a vector summation and the different vectors $v_{ij} \sin\varphi_j$ $(j=1, 2, \dots\dots\dots p)$ being in the orthogonal directions M_j. Hence it is easily seen from (3·3) (4·4) and (4·5) that

$$\left.\begin{aligned} a_{ij} &= (y_i . y_j)/n = \sum_{k=1}^{p} v_{ik}\, v_{jk} \sin^2\varphi_k / n \\ a'_{ij} &= (y'_i . y'_j)/n' = \sum_{k=1}^{p} v_{jk}\, v_{ik} \cos^2\varphi_k / n' \end{aligned}\right\} \qquad \dots \quad (4\cdot6)$$

Therefore, $n\, a_{ij}+n'\, a'_{ij}$ which occurs in the density factor of the distribution (4·1) is easily seen to be given by

$$n\, a_{ij}+n'\, a'_{ij} = \sum_{k=1}^{p} v_{ik}\, v_{jk} \qquad \dots \quad (4\cdot61)$$

In the joint distribution (4·1) let us now transform from a_{ij} and a'_{ij} to the new variates v_{ij} $(i, j=1, 2 \ldots\ldots p)$ and ϕ_j $(j=1, 2, \ldots\ldots p)$. It should be noted that the total number of variates in the previous case is $\frac{p\,(p+1).2}{2}$ and in the latter case it is $p^2+p=p\,(p+1)$. They are, therefore, the same as they should be.

From (4·6) $| a_{ij} | = | \sum_{k=1}^{p} v_{ik}\, v_{jk} \sin^2 \phi_k/n |$

$$= (\sin \phi_1 \sin \phi_2 \ldots\ldots\ldots \sin \phi_p)^2 . \; | \sum_{k=1}^{p} v_{ik}\, v_{jk} | /n^p$$

$$= \prod_{i=1}^{p} \sin^2 \phi_i \, | v_{ij} |^2/n^p \qquad \ldots \quad (4{\cdot}62)$$

and

$$| a'_{ij} | = | \sum_{k=1}^{p} v_{ik}\, v_{jk} \cos^2 \phi_k/n' |$$

$$= (\cos \phi_1 \cos \phi_2 \ldots\ldots\ldots \cos \phi_p)^2 . \; | \sum_{k=1}^{p} v_{ik}\, v_{jk} | /n'^p$$

$$= \prod_{i=1}^{p} \cos^2 \phi_i \, | v_{ij} |^2 / n'^p \qquad \ldots \quad (4{\cdot}63)$$

Again from (4·6), the Jacobian

$$\frac{\partial\,(a_{ij},\; a'_{ij})}{\partial\,(v_{ij},\; \phi_i)} \; (i, j=1, 2, \ldots\ldots p;\; a_{ij}=a_{ji},\; a'_{ij}=a'_{ji}, \text{ but } v_{ij} \neq v_{ji})$$

is easily seen to be given by the determinant.

$$\text{const.} \begin{vmatrix}
\frac{\partial a'_{11}}{\partial v_{11}} & \cdots & \frac{\partial a'_{11}}{\partial v_{1p}} & \cdots & \frac{\partial a'_{11}}{\partial v_{p1}} & \cdots & \frac{\partial a'_{11}}{\partial v_{pp}} & \frac{\partial a'_{11}}{\partial \phi_1} & \cdots & \frac{\partial a'_{11}}{\partial \phi_p} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial a'_{1p}}{\partial v_{11}} & \cdots & \frac{\partial a'_{1p}}{\partial v_{1p}} & \cdots & \frac{\partial a'_{1p}}{\partial v_{p1}} & \cdots & \frac{\partial a'_{1p}}{\partial v_{pp}} & \frac{\partial a'_{1p}}{\partial \phi_1} & \cdots & \frac{\partial a'_{1p}}{\partial \phi_p} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial a'_{pp}}{\partial v_{11}} & \cdots & \frac{\partial a'_{pp}}{\partial v_{1p}} & \cdots & \frac{\partial a'_{pp}}{\partial v_{p1}} & \cdots & \frac{\partial a'_{pp}}{\partial v_{pp}} & \frac{\partial a'_{pp}}{\partial \phi_1} & \cdots & \frac{\partial a'_{pp}}{\partial \phi_p} \\
\frac{\partial a_{11}}{\partial v_{11}} & \cdots & \frac{\partial a_{11}}{\partial v_{1p}} & \cdots & \frac{\partial a_{11}}{\partial v_{p1}} & \cdots & \frac{\partial a_{11}}{\partial v_{pp}} & \frac{\partial a_{11}}{\partial \phi_1} & \cdots & \frac{\partial a_{11}}{\partial \phi_p} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial a_{1p}}{\partial v_{11}} & \cdots & \frac{\partial a_{1p}}{\partial v_{1p}} & \cdots & \frac{\partial a_{1p}}{\partial v_{p1}} & \cdots & \frac{\partial a_{1p}}{\partial v_{pp}} & \frac{\partial a_{1p}}{\partial \phi_1} & \cdots & \frac{\partial a_{1p}}{\partial \phi_p} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial a_{pp}}{\partial v_{11}} & \cdots & \frac{\partial a_{pp}}{\partial v_{1p}} & \cdots & \frac{\partial a_{pp}}{\partial v_{p1}} & \cdots & \frac{\partial a_{pp}}{\partial v_{pp}} & \frac{\partial a_{pp}}{\partial \phi_1} & \cdots & \frac{\partial a_{pp}}{\partial \phi_p}
\end{vmatrix}$$

$= J$ (Suppose) $\qquad \ldots \quad (4{\cdot}64)$

Therefore, by virtue of the relations (4·61), (4·62), (4·63) and (4·64) the joint distribution (4·3) now reduces to

$$\text{const. } e^{-\frac{1}{2} \sum_{i,j=1}^{p} a^{ij} \sum_{k=1}^{p} v_{ik}\, v_{jk}} \, | v_{ij} |^{n+n'-2p-4} \, \Big(\prod_{i=1}^{p} \sin \phi_i \Big)^{n-p-2} \Big(\prod_{i=1}^{p} \cos \phi_i \Big)^{n'-p-2} J . \prod_{i,j=1}^{p} dv_{ij} \prod_{i=1}^{p} d\phi_i \quad (4{\cdot}65)$$

where J is of course given by (4·64)

Let us for convenience put

$$\sin \varphi_i = s_i \text{ and } \cos \varphi_i = c_i \quad (i = 1, 2, \ldots\ldots\ldots p) \qquad \ldots \quad (4{\cdot}66)$$

Then (4·65) reduces to

$$\text{const. } e^{-\frac{1}{2}\sum\limits_{i,j=1}^{p} \alpha^{ij} \sum\limits_{k=1}^{p} v_{ik} v_{jk}} \; |v_{ij}|^{n+n'-2p-4} \left(\prod_{i=1}^{p} s_i\right)^{n-p-2} \left(\prod_{i=1}^{p} c_i\right)^{n'-p-2} \cdot \prod_{i,j=1}^{p} dv_{ij} \prod_{i=1}^{p} d\varphi_i \quad \ldots \quad (4{\cdot}67)$$

We have now to calculate J given by (4·64)

From the relations given by (4·6) the determinant J is easily seen to reduce to

$$\text{const.} \begin{vmatrix} 2v_{11}c_1^2 & \cdot & \cdot & 2v_{1p}c_p^2 & 0 & \cdot & 0 & \cdot & \cdot & 0 & -2v_{11}^2 c_1 s_1 & \cdot & \cdot & -2v_{1p}^2 c_p s_p \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ v_{p1}c_1^2 & \cdot & \cdot & v_{pp}c_p^2 & 0 & \cdot & v_{11}c_1^2 & \cdot & \cdot & v_{1p}c_p^2 & -2v_{11}v_{p1}c_1 s_1 & \cdot & \cdot & -2v_{1p}v_{pp}c_p s_p \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & 2v_{p1}c_1^2 & \cdot & \cdot & 2v_{pp}c_p^2 & -2v_{p1}^2 c_1 s_1 & \cdot & \cdot & -2v_{pp}^2 c_p s_p \\ 2v_{11}s_1^2 & \cdot & \cdot & 2v_{1p}s_p^2 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 2v_{11}^2 c_1 s_1 & \cdot & \cdot & 2v_{1p}^2 c_p s_p \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ v_{p1}s_1^2 & \cdot & \cdot & v_{pp}s_p^2 & 0 & \cdot & v_{11}s_1^2 & \cdot & \cdot & v_{1p}s_p^2 & 2v_{11}v_{p1}c_1 s_1 & \cdot & \cdot & 2v_{1p}v_{pp}c_p s_p \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & 2v_{p1}s_1^2 & \cdot & \cdot & 2v_{pp}s_p^2 & 2v_{p1}^2 c_1 s_1 & \cdot & \cdot & 2v_{pp}^2 c_p s_p \end{vmatrix}$$

This is a determinant of the order $p^2 + p$. Now adding to the first row the $\left\{\frac{p(p+1}{2} + 1\right\}$th row, to the second row the $\left\{\frac{p(p+1)}{2} + 2\right\}$th row and so on, and taking common $c_1 s_1$ out of the $(p^2+1)^{\text{th}}$ column, $c_2 s_2$ out of the $(p^2+2)^{\text{th}}$ column and so on this determinant J (by virtue of the relation $c_i^2 + s_i^2 = 1$) reduces to

$$\text{const.} \begin{vmatrix} v_{11} & \cdot & \cdot & v_{1p} & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ v_{p1} & \cdot & \cdot & v_{pp} & 0 & \cdot & v_{11} & \cdot & \cdot & v_{1p} & 0 & \cdot & \cdot & 0 \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & v_{p1} & \cdot & \cdot & v_{pp} & 0 & \cdot & \cdot & 0 \\ v_{11}^2 s_1^2 & \cdot & \cdot & v_{1p}s_p^2 & 0 & \cdot & 0 & \cdot & \cdot & 0 & v_{11}^2 & \cdot & \cdot & v_{1p}^2 \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ v_{p1}s_1^2 & \cdot & \cdot & v_{pp}s_p^2 & 0 & \cdot & v_{11}s_1^2 & \cdot & \cdot & v_{1p}s_p^2 & 2v_{11}v_{p1} & \cdot & \cdot & 2v_{1p}v_{pp} \\ \vdots & & & \vdots & & & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & v_{p1}s_1^2 & \cdot & \cdot & v_{pp}s_p^2 & v_{p1}^2 & \cdot & \cdot & v_{pp}^2 \end{vmatrix} \times \prod_{i=1}^{p} c_i s_i$$

This Jacobian determinant might be reduced in an elegant manner by the use of the advanced algebra of determinants and matrices. But in the present paper I purposely adopt more elementary and pedestrian methods to make the whole derivation simple and obvious, though rather lengthy. I reserve the more elegant and condensed treatment for a later paper where a purely geometrical method will also be given of deriving the distribution in view. With this remark I take to the more elementary method proposed.

To illustrate the method by which this determinant may be reduced let us take for simplicity the case of three variates. The method is, of course, perfectly general and it would be obvious from what follows that we could immediately generalise from 3 to p variates. In the case of 3 variates let us set down for convenience

$$\left.\begin{array}{lll} v_{11}=x_1, & v_{12}=x_2, & v_{13}=x_3 \\ v_{21}=y_1, & v_{22}=y_2, & v_{23}=y_3 \\ v_{31}=z_1, & v_{32}=z_2, & v_{33}=z_3 \end{array}\right\} \quad \ldots \quad (4{\cdot}68)$$

Rearrange the rows as if the Jacobian determinant had been laid out by differentiating (a'_{ij}, a_{ij}) row after row in the following order: a'_{11}, a'_{22}, a'_{33}, a'_{12}, a'_{13}, a'_{23}, a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}. Then the determinant J is easily seen to be given in the present case by

$$J = \text{const.} \prod_{i=1}^{3} c_i s_i$$

$$\times \begin{vmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1 & z_2 & z_3 & 0 & 0 & 0 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 & x_1 & x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 & z_2 & z_3 & y_1 & y_2 & y_3 & 0 & 0 & 0 \\ x_1s_1^2 & x_2s_2^2 & x_3s_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & x_1^2 & x_2^2 & x_3^2 \\ 0 & 0 & 0 & y_1s_1^2 & y_2s_2^2 & y_3s_3^2 & 0 & 0 & 0 & y_1^2 & y_2^2 & y_3^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1s_1^2 & z_2s_2^2 & z_3s_3^2 & z_1^2 & z_2^2 & z_3^2 \\ y_1s_1^2 & y_2s_2^2 & y_3s_3^2 & x_1s_1^2 & x_2s_2^2 & x_3s_3^2 & 0 & 0 & 0 & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ z_1s_1^2 & z_2s_2^2 & z_3s_3^2 & 0 & 0 & 0 & x_1s_1^2 & x_2s_2^2 & x_3s_3^2 & 2x_1z_1 & 2x_2y_2 & 2x_3z_3 \\ 0 & 0 & 0 & z_1s_1^2 & z_2s_2^2 & z_3s_3^2 & y_1s_1^2 & y_2s_2^2 & y_3s_3^2 & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix} \quad (4{\cdot}69)$$

The reduction of this determinant will be shown in some detail as it will immediately make obvious the reduction in the general case of p-variates. Denote the columns by (1), (2), (3), (4) etc. now multiply (2) by x_1 and subtract from it x_2. (1); multiply (3) by x_1 and subtract from it x_3. (1); treat the columns (4), (5) (6) similarly, taking the multipliers to be (y_1, y_2, y_3); then the columns (7), (8), (9) taking as multipliers (z_1, z_2, z_3) To keep the determinant unaltered in value each multiplication will have to be balanced by a corresponding division. In all that follows such divisions will always be understood to have been made. The determinant (leaving out the factor $\prod_{i=1} c_i s_i$) now reduces to

$$\text{const.} \; \frac{1}{x_1 y_1 z_1}$$

$$\times \begin{vmatrix} (y_2x_1) & (y_3x_1) & (x_2y_1) & (x_3y_1) & 0 & 0 & 0 & 0 & 0 \\ (z_2x_1) & (z_3x_1) & 0 & 0 & (x_2z_1) & (x_3z_1) & 0 & 0 & 0 \\ 0 & 0 & (z_2y_1) & (z_3y_1) & (y_2z_1) & (y_3z_1) & 0 & 0 & 0 \\ x_1x_2s_{21}^2 & x_1x_3s_{31}^2 & 0 & 0 & 0 & 0 & x_1^2 & x_2^2 & x_3^2 \\ 0 & 0 & y_1y_2s^2_{21} & y_1y_3s^2_{31} & 0 & 0 & y_1^2 & y_2^2 & y_3^2 \\ 0 & 0 & 0 & 0 & z_1z_2s^2_{21} & z_1z_3s^2_{31} & z_1^2 & z_2^2 & z_3^2 \\ (y_2x_1s_2^2) & (y_3x_1s_3^2) & (x_2y_1s^2_2) & (x_3y_1s^2_3) & 0 & 0 & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ (z_2x_1s_2^2) & (z_3x_1s_3^2) & 0 & 0 & (x_2z_1s^2_2) & (x_3z_1s^2_3) & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ 0 & 0 & (z_2y_1s^2_2) & (z_3y_1s^2_3) & (y_2z_1s^2_2) & (y_3z_1s^2_3) & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix}$$

where*

$s_{21} = s_2^2 - s_1^2$, $s_{31}^2 = s_3^2 - s_1^2$;
$(y_2\, x_1)$ stands for $y_2\, x_1 - x_2\, y_1$ and similar meanings attach to $(y_3\, x_1)$ etc.
$(y_1\, x_1\, s^2_2)$ stands for $y_2\, x_1\, s^2_2 - x_2\, y_1\, s^2_1$ and similar meanings attach to $(y_3\, x_1\, s^2_3)$ etc. ... (4·695)

Adding to column (1), columns (3) and (5), and to column (2) columns (4) and (6), and taking out the factor $(s_2^2 - s_1^2)$ from the first column and $(s_3^2 - s_1^2)$ from the second column, the determinant reduces to

$$\text{const.}\ \frac{s^2_{21}\ s^2_{31}}{x_1\ y_1\ z_1}$$

$$\times \begin{vmatrix} 0 & 0 & (x_2y_1) & (x_3y_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (x_2z_1) & (x_3z_1) & 0 & 0 & 0 \\ 0 & 0 & (z_2y_1) & (z_3y_1) & (y_2z_1) & (y_3z_1) & 0 & 0 & 0 \\ x_1x_2 & x_1x_3 & 0 & 0 & 0 & 0 & x^2_1 & x^2_2 & x^2_3 \\ y_1y_2 & y_1y_3 & y_1y_2s_{21}^2 & y_1y_3s^2_{31} & 0 & 0 & y^2_1 & y^2_2 & y^2_3 \\ z_1z_2 & z_1z_3 & 0 & 0 & z_1z_2s^2_{21} & z_1z_3s^2_{31} & z^2_1 & z^2_2 & z^2_3 \\ [x_1y_2] & [x_1y_3] & (x_2y_1s^2_2) & (x_3y_1s^2_3) & 0 & 0 & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ [x_1z_2] & [x_1z_3] & 0 & 0 & (x_2z_1s^2_2) & (x_3z_1s^2_3) & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ [y_1z_2] & [y_1z_3] & (z_2y_1s^2_2) & (z_3y_1s^2_3) & (y_2z_1s^2_2) & (y_3z_1s^2_3) & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix}$$

where $[x_1\, y_2]$ stands for $x_1\, y_2 + y_1\, x_2$, and similar meanings attach to $[x_1\, z_3]$ etc. (4·696)

From row (7) subtract row (1) multiplied by s_1^2, from row (8) subtract row (2) multiplied by s_1^2, and from row (9) subtract row (3) multiplied by s_1^2. The determinant now becomes

$$\text{const.}\ \frac{s_{21}^2 s_{31}^2}{x_1\ y_1\ z_1}$$

$$\times \begin{vmatrix} 0 & 0 & (x_2y_1) & (x_3y_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (x_2z_1) & (x_3z_1) & 0 & 0 & 0 \\ 0 & 0 & (z_2y_1) & (z_3y_1) & (y_2z_1) & (y_3z_1) & 0 & 0 & 0 \\ x_1x_2 & x_1x_3 & 0 & 0 & 0 & 0 & x_1^2 & x_2^2 & x_3^2 \\ y_1y_2 & y_1y_3 & y_1y_2s_{21}^2 & y_1y_3s^2_{31} & 0 & 0 & y_1^2 & y_2^2 & y_3^2 \\ z_1z_2 & z_1z_3 & 0 & 0 & z_1z_2s_{21}^2 & z_1z_3s_{31}^2 & z_1^2 & z_2^2 & z_3^2 \\ [x_1y_2] & [x_1y_3] & x_2y_1s_{21}^2 & x_3y_1s_{31}^2 & 0 & 0 & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ [x_1z_2] & [x_1z_3] & 0 & 0 & x_2z_1s_{21}^2 & x_3z_1s_{31}^2 & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ [y_1z_2] & [y_1z_3] & z_2y_1s_{21}^2 & z_3y_1s_{31}^2 & y_2z_1s_{21}^2 & y_3z_1s_{31}^2 & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix}$$

* These abbreviations are introduced to keep within the available space.

Denote by $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ the co-factors of $(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$ in the determinant D given by

$$D = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Then in the 9th order determinant just preceding it multiply column (4) by $-\bar{z}_3$ and column (6) by $\bar{y}_3$ and subtract from them respectively column (3) multiplied by $\bar{z}_2$ and column (5) multiplied by $-\bar{y}_2$. The determinant D then becomes, by virtue of the relations $\bar{x}_2\bar{z}_3 - \bar{x}_3\bar{z}_2 = y_1 D$ and $\bar{x}_3\bar{y}_2 - \bar{x}_2\bar{y}_3 = z_1 D$,

$$\text{const.}\ \frac{s^2_{21}.\, s^2_{31}.\, D}{x_1}$$

$$\times \begin{vmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ x_1x_2 & x_1x_3 & 0 & 0 & x^2_1 & x^2_2 & x^2_3 \\ y_1y_2 & y_1y_3 & -y_3\bar{z}_3s^2_{31} - y_2\bar{z}_2s^2_{21} & 0 & y^2_1 & y^2_2 & y^2_3 \\ z_1z_2 & z_1z_3 & 0 & z_3\bar{y}_3s^2_{31} + z_2\bar{y}_2s^2_{21} & z^2_1 & z^2_2 & z^2_3 \\ [x_1y_2] & [x_1y_3] & -x_3\bar{z}_3s^2_{31} - x_2\bar{z}_2s^2_{21} & 0 & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ [x_1z_2] & [x_1z_3] & 0 & x_3\bar{y}_3s^2_{31} + x_2\bar{y}_2s^2_{21} & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ [y_1z_2] & [y_1z_3] & -z_3\bar{z}_3s^2_{31} - z_2\bar{z}_2s^2_{21} & y_3\bar{y}_3s^2_{31} + y_2\bar{y}_2s^2_{21} & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix}$$

Subtracting now column (4) from column (3) this is easily seen to reduce to

$$\text{const.}\ \frac{s^2_{21}.\, s^2_{31}.\, D}{x_1}$$

$$\times \begin{vmatrix} x_1x_2 & x_1x_3 & 0 & x^2_1 & x^2_2 & x^2_3 \\ y_1y_2 & y_1y_3 & -y_3\bar{z}_3s^2_{31} - y_2\bar{z}_2s^2_{21} & y^2_1 & y^2_2 & y^2_3 \\ z_1z_2 & z_1z_3 & -z_3\bar{y}_3s^2_{31} - z_2\bar{y}_2s^2_{21} & z^2_1 & z^2_2 & z^2_3 \\ [x_1y_2] & [x_1y_3] & -x_3\bar{z}_3s^2_{31} - x_2\bar{z}_2s^2_{21} & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ [x_1z_2] & [x_1z_3] & -x_3\bar{y}_3s^2_{31} - x_2\bar{y}_2s^2_{21} & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ [y_1z_2] & [y_1z_3] & -(z_3\bar{z}_3 + y_3\bar{y}_3)s_{31}^2 - (z_2\bar{z}_2 + y_2\bar{y}_2)s^2_{21} & 2y_1z_1 & 2y_2z_2 & 2y_3z_3 \end{vmatrix}$$

Subtracting from column (3), column (2) multiplied by $(s^2_3 - s^2_1)$, and adding to column (3), column (1) multiplied by $(s^2_2 - s^2_1)$ the determinant is easily seen to reduce to

$$\text{const.}\ s^2_{12}\, s^2_{13}\, s^2_{23}.\, D$$

$$\times \begin{vmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x^2_1 & x^2_2 & x^2_3 \\ y_1y_2 & y_1y_3 & y_2y_3 & y^2_1 & y^2_2 & y^2_3 \\ z_1z_2 & z_1z_3 & z_2z_3 & z^2_1 & z^2_2 & z^2_3 \\ [x_1y_2] & [x_1y_3] & [x_2y_3] & 2x_1y_1 & 2x_2y_2 & 2x_3y_3 \\ [x_1z_2] & [x_1z_3] & [x_2z_3] & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \\ [y_1z_2] & [y_1z_3] & [y_2z_3] & 2x_1z_1 & 2x_2z_2 & 2x_3z_3 \end{vmatrix}$$

Multiply columns (2), (3), (4) and (5) respectively by x_3, x_1, x_3, x_1 and subtract from them respectively column (1) multiplied by x_3, x_2, x_1, x_2; also multiply column (6) by x_1 and subtract from it column (2) multiplied by x_3. The above expression then reduces (after substitution from 4·695 and 4·696) to

$$\text{const.} - \frac{(s^2_1 - s^2_2)\,(s^2_1 - s^2_3)\,(s^2_2 - s^2_3)\;\text{D}.}{x^2_1 . x_3}$$

$$\times \begin{vmatrix} y_1\bar{z}_1 & -y_2\bar{z}_2 & y_1\bar{z}_3 & y_2\bar{z}_3 & -y_3\bar{z}_2 \\ -z_1\bar{y}_1 & z_2\bar{y}_2 & -z_1\bar{y}_3 & -z_2\bar{y}_3 & z_3\bar{y}_2 \\ x_1\bar{z}_1 & -x_2\bar{z}_2 & x_1\bar{z}_3 & x_2\bar{z}_3 & -x_3\bar{z}_2 \\ -x_1\bar{y}_1 & x_2\bar{y}_2 & -x_1\bar{y}_3 & -x_2\bar{y}_3 & x_3\bar{y}_2 \\ z_1\bar{z}_1 - y_1\bar{y}_1 & -z_2\bar{z}_2 + y_2\bar{y}_2 & z_1\bar{z}_3 + y_1\bar{y}_3 & z_2\bar{z}_3 - y_1\bar{y}_3 & -z_3\bar{z}_2 + y_3\bar{y}_2 \end{vmatrix}$$

Multiply column (2) by $y_1 z_1$ and subtract from it column (1) multiplied by $-y_2\bar{z}_2$, multiply columns (3), (4), (5) respectively by $\bar{z}_1$, $-\bar{z}_1$, y_1 and subtract from them respectively column (1) multiplied by $\bar{z}_3$, column (2) multiplied by $\bar{z}_3$ and column (2) multiplied by y_3.

Remembering[2] the property (2) that a minor of the order m formed out of the inverse constituents is equal to the complementary of the corresponding minor of the original determinant D multiplied by the $(m-1)$th power of D, the above expression is easily seen to reduce to

$$\text{const.}\ \frac{(s^2_1 - s^2_2)\,(s^2_1 - s^2_3)\,(s^2_2 - s^2_3).\ \text{D}^2}{y_1.x_1.y_3.\ \bar{z}'_1\bar{z}^3_3}$$

$$\times \begin{vmatrix} y_1\bar{z}_1 & 0 & 0 & 0 & 0 \\ -z_1\bar{y}_1 & y_1z_2\bar{y}_2\bar{z}_1 - z_1y_2\bar{z}_2\bar{y}_1 & z_1 & z_3 & \bar{y}_2\bar{x}_1 \\ x_1\bar{z}_1 & +\bar{z}_3\bar{z}_1\bar{z}_3 & 0 & 0 & +\bar{z}_2\bar{z}_1 \\ -x_1\bar{y}_1 & y_1x_2\bar{z}_1\bar{y}_2 - x_1y_2\bar{y}_1\bar{z}_2 & x_1 & x_3 & -\bar{y}_2\bar{z}_1 \\ -z_1\bar{z}_1 - y_1\bar{y}_1 & \bar{x}_3\bar{z}_1\bar{z}_2 - y_1y_2x_3.\text{D} & y_1 & y_3 & -\bar{z}_2\bar{x}_1 \end{vmatrix}$$

i.e. to

$$\text{const.}\ \frac{(s^2_1 - s^2_2)\,(s^2_1 - s^2_3)\,(s^2_2 - s^2_3).\ \text{D}^2}{x_1y_2\bar{z}_1\bar{z}_3}$$

$$\times \begin{vmatrix} \bar{x}_1\bar{y}_3\bar{z}_1 - x_3z_1y_1\text{D} & z_1 & z_3 & \bar{y}_2\bar{x}_1 \\ -\bar{z}_1\bar{z}_2\bar{z}_3 & 0 & 0 & -\bar{z}_2\bar{z}_1 \\ -\bar{z}_3\bar{z}_1\bar{y}_2 - x_1y_2x_3\text{D} & x_1 & x_3 & -\bar{y}_2\bar{z}_1 \\ \bar{x}_3\bar{z}_1\bar{z}_3 - y_1y_2x_3\text{D} & y_1 & y_3 & \bar{z}_2\bar{x}_1 \end{vmatrix}$$

i.e. to

$$\text{const.}\ \frac{(s^2_1-s^2_2)(s^2_1-s^2_3)(s^2_2-s^2_3)\,D^3}{x_1 y_2 z_3}$$

$$\times\begin{vmatrix} \bar{x}_3\bar{y}_2 & z_1 & z_2 & \bar{y}_2\bar{x}_1 \\ \bar{z}_3\bar{z}_3 & 0 & 0 & \bar{z}_3\bar{z}_1 \\ -\bar{y}_3\bar{z}_3 & x_1 & x_2 & -\bar{y}_2\bar{z}_1 \\ -\bar{x}_3\bar{z}_2 & y_1 & y_2 & -\bar{z}_3\bar{x}_1 \end{vmatrix}$$

Multiplying column (4) by $\bar{z}_3$ and subtracting from it column (1) multiplied by $\bar{z}_1$ this easily becomes

$$\text{const.}\ \frac{(s_1^2-s^2_2)(s^2_1-s^2_3)(s^2_2-s^2_3)D^4}{x_1}$$

$$\times\begin{vmatrix} z_1 & z_2 & \bar{y}_3 \\ x_1 & x_2 & 0 \\ y_1 & y_2 & -\bar{z}_3 \end{vmatrix}$$

Or, const. $\dfrac{(s^2_1-s^2_2)\ (s^2_1-s^2_3)\ (s^2_2-s^2_3)}{x_1}\ D^4.\,(\bar{y}_3\bar{z}_3-\bar{z}_3\bar{y}_3)$

Or, const. $(s^2_1-s^2_2)(s^2_1-s^2_3)(s^2_2-s^2_3)\,D^3$

(absorbing everywhere the negatives into the const)

Therefore, from (4·69) the determinant J would be given by

$$J=\text{const.}\ \prod_{i=1}^{3} c_i s_i\,(s^2_1-s^2_2)\,(s^2_1-s^2_3)\,(s^2_2-s^2_3).\ D^3 \qquad \ldots\ (4{\cdot}7)$$

From what has been given above it would be obvious that this method of evaluating J and finally reducing it to the form (4·7) is perfectly general and in the case of p-variates, J would be evidently given by

$$J=\text{const.}\ \prod_{i=1}^{p} c_i s_i\,(s^2_1-s^2_2)\ldots(s^2_1-s^2_p)\,(s^2_2-s^2_3)\ldots(s^2_2-s^2_p)\ldots(s^2_{p-1}-s^2_p)\ |\,v_{ij}\,|^{p+2} \quad \ldots\ (4{\cdot}71)$$

From the relations (4·61), (4·62), (4·63) and (4·71), the joint distribution (4·3) transforms to

$$\text{const.}\ e^{-\frac{1}{2}\sum\limits_{i,j=1}^{p}\alpha^{ij}\sum\limits_{k=1}^{p} v_{ik}v_{jk}}\ |\,v_{ij}\,|^{n-p-2}\ |\,v_{ij}\,|^{n'-p-2}\ \prod_{i=1}^{p} s_i^{n'-p-2}\,c_i^{n'-p-2}$$

$$\times\,(s^2_1-s^2_2)\ldots\ldots(s^2_1-s^2_p)\,(s^2_2-s^2_3)\ldots\ldots(s^2_2-s^2_p)\ldots\ldots(s^2_{p-1}-s^2_p).\ |\,v_{ij}\,|^{p+2}\prod_{i=1}^{p} c_i s_i$$

$$\times\ \prod_{i,j=1}^{p} dv_{ij}\ \prod_{i=1}^{p} d\varphi_i \qquad \ldots\ (4{\cdot}72)$$

Or to

$$\text{const. } e^{-\frac{1}{2}\sum\limits_{i,j=1}^{p} \alpha^{ij} \sum\limits_{k=1}^{p} v_{ik} v_{jk}} \; | v_{ij} |^{n+n'-p-2} \prod_{i=1}^{p} c_i^{n'-p-1} \, s_i^{n-p-1}$$

$$\times (s^2_1 - s^2_2)......(s^2_1 - s^2_p)\; (s^2_2 - s^2_3)......(s^2_2 - s^2_p)......(s^2_{p-1} - s^2_p) \prod_{i,j=1}^{p} dv_{ij} \prod_{i=1}^{p} d\varphi_i \qquad ... \quad (4\cdot73)$$

Integrating out over v_{ij}'s we have the joint distribution of φ_i's in the form

$$\text{const. } \prod_{i=1}^{p} c_i^{n'-p-1} \, s_i^{n-p-1}$$

$$\times (s^2_1 - s^2_2)...(s^2_1 - s^2_p)\; (s^2_2 - s^2_3)...(s^2_2 - s^2_p)...(s^2_{p-1} - s^2_p) \prod_{i=1}^{p} d\varphi_i \qquad ... \quad (4\cdot74)$$

where of course c_i and s_i are given by (4·66)

Therefore from (3·4), (4·66) and (7 74) the joint distribution of K_i's (where K_i's are the roots of equation (1·5) and are accordingly our p-statistics) is obtained in the form

$$\text{const. } \prod_{i=1}^{p} \frac{K_i^{n-p-1}}{(1 + c^2 K^2_i)^{(n+n'-2)/2}}$$

$$\times (K^2_1 - K^2_2)...(K^2_1 - K^2_p)\; (K^2_2 - K^2_3)...(K_2^2 - K^2_p)...(K^2_{p-1} - K^2_p). \prod_{i=1}^{p} dK_i \qquad ... \quad (4\cdot75)$$

where $c^2 = n/n'$ ... (4·76)

It should be noted that if we put $p=1$ then the above can be shown to reduce as it should to

$$\text{const. } \frac{K^{n-2} \; dK}{(1 + c^2K^2)^{(n+n'-2)/2}} \qquad ... \quad (4\cdot77)$$

§5 CONCLUSION.

The hypothesis to be tested is that the two samples Σ and Σ' have come from two populations Π and Π' with the same set of variances and covariances. With that end in view we proceed as follows.

It has been noted at the end of section 3 that of these K_i's, one is a true maximum, one is a true minimum and the others are stationary. It is evident from (1·5), (4·3) and from the geometrical considerations in section 3 and section 4 that K_i's might each be taken to vary from 0 to ∞.

Let us assume that in any particular case K_1 is the maximum statistic, K_p, the minimum and K_r is the one nearest to unity. In any given case all these can of course be calculated to any desired degree of approximation by any suitable method, say, Horner's method.

To obtain the sampling distribution of K_1, the maximum statistic, we integrate out (4·75) successively over K_2, K_3......K_p each from 0 to K_1. This will be done in the second paper which is to appear shortly in the next issue of this journal. Suppose this distribution is

$$\text{const. } F\,(K_1)\, dK_1 \qquad ... \quad (5\cdot1)$$

To obtain the sampling distribution of K_p, we proceed otherwise ; changing the order of integration, *i.e.*, integrating out (4·75) successively over K_1, K_2, ... K_{p-1} each from K_p to ∞, the distribution of K_p, the minimum statistic would be obtained, say, in the form

$$\text{const. } f(K_p)\, d(K_p) \qquad \dots \quad (5{\cdot}2)$$

The sampling distribution of K_r, the statistic which or whose reciprocal is nearest to to unity can be similarly obtained by integrating out over the others between suitable limits. Suppose it is of the general form

$$\text{const. } \varphi(K_r)\, dK_r \qquad \dots \quad (5{\cdot}3)$$

The suitable limits involved in the derivation (5·3) as well as the general forms of $F(K_1)$, $f(K_p)$ and $\varphi(K_r)$ given in (5·1, (5·2) and (5·3) will be shown in detail in the next paper. Each of the functions $F(K_1)$, $f(K_p)$ *and* $\varphi(K_r)$ is the sum of a number of terms each of which involves the product of incomplete Γ-functions.

(*i*) To accept the hypothesis with safety (on any level of significance) we use either K_1 and (5·1), or K_p and (5·2) according as K_1 is greater or less than $1/K_p$. The underlying logic is that we now look at the samples from a point of view which puts them farthest apart. If even from such a point of view the difference looks insignificant we are entitled to assert with safety that the difference is in fact really insignificant (on a certain level of course). What is precisely implied is this. When we assert in such a situation that the difference is insignificant on the 5% level we are really putting a maximum limit to the margin of error involved in our assertion, which is here 5%. The error may be in fact less than this.

(*ii*) To reject the hypothesis with safety (on a given level of course) we have recourse to K_r and its sampling distribution given in (5·3). The underlying logic again is that we now look at the samples from a point of view which puts them closest to each other. If even from such a point of view the difference looks significant then it is in fact really significant. More precisely speaking, when we assert in such a situation that the difference is significant on the 5% level we are again putting a maximum limit to the error involved in our assertion, which is here 5%. Here also the error may in fact be much less than this.

For (5·1) ,(5·2) and (5·3) a transformation to a new set of variates has been found to be mathematically convenient, the new variates being defined by

$$K_i = e^{Z_i} \quad (i = 1, 2, \dots\dots\dots\dots p) \qquad \dots \quad (5{\cdot}4)$$

In practice these new variates and their sampling distributions should be preferred to the K_i's. The changes consequent upon this transformation will come out in the next paper already referred to.

Starting from the joint distribution of K_1, K_2, ... K_p given in (4·75) we can, however, tackle the problem of testing of hypothesis in an entirely different way, by considering is a space of p dimensions (such that a point in it has the co-ordinates K_1, K_2, ... K_p) the family of equifrequency surfaces $\mu = \text{const}$, where μ is given by the relation

$$\mu = \text{const.} \prod_{i=1}^{p} \frac{K_i^{n-p-1}}{(1 + c^2 K_i^2)^{(n+n'-2)/2}}$$

$$\times (K_1^2 - K_2^2)\dots\dots(K_1^2 - K_p^2)\,(K_2^2 - K_3^2)\dots\dots(K_2^2 - K_p^2)\dots\dots(K_{p-1}^2 - K_p^2) \qquad \dots \quad (5.5)$$

The expression on the right hand side of 5·5 is really the density factor in (4·75).

This latter method of testing of hypothesis will be dealt with in a later paper.

BIBLIOGRAPHY

1. BOSE, R. C. and ROY, S. N. : The Distribution of the Studentised D^2-Statistic. *Sankhyā* : Vol 4 (1), pp. 19-38.

2. BURNSIDE AND PANTON : Theory of equations, Vol. II, p. 42

3. MAHALANOBIS, P. C., BOSE, R. C. AND ROY, S. N. : Normalization of Statistical Variates and the use of Rectangular Co-ordinates in the Theory of Sampling Distributions. *Sankhyā* : Vol. 3 (1), 1937, p. 1-40.

4. WISHART, J. : The Generalised Product Moment Distribution in Samples drawn from a normal Population. *Biometrika* : Vol. XXA, 1928, pp. 32-40.

[*Paper received: 20th June, 1939*].