A note on unitary representation problem with corrigenda to the articles Weak mixing and unitary representation problem, Bull. Sci. Math. 124 (2000) 517–523 and Identity excluding groups, Bull. Sci. Math. 126 (2002) 763–772

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Received 20 April 2004; accepted 23 April 2004

Abstract

Some of the assertions in the above mentioned articles turn out to be erroneous, as stated, on account of reliance on some earlier works. Here we describe modified (though more restrictive) hypothesis under which the analogues statements hold. We also uphold certain other results by providing a different proof.

MSC: 22D10

Keywords: Unitary representation; Probability measures; Spectral radius; Strong convergence

It was recently shown by A. Nevo that Theorem 2.11 and Corollary 2.11 of [4] are false. These results from [4] were employed in [6] and [7]. Specifically Lemma 2.1 of [6] and Lemma 2.2 of [7] are affected by this and through them certain other results in the paper. A. Nevo considered the unitary representation ρ of the circle group S^1 on the space of L_2-functions on S^1 with zero integral and a proper dense cyclic subgroup D of S^1 and proves that $\rho|_D$ weakly contains the trivial representation but obviously ρ has no non-

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trivial fixed vector. This example of Nevo can be constructed for higher-dimensional tori also using Diophantine approximation. C. Cuny has constructed a counterexample (which is an irreducible unitary representation ρ of the $\mathbb{Z}$-extension of S^1 and a dense subgroup D such that $\rho|_D$ weakly contains the trivial representation) which disproves Proposition 2.1 and Theorem 2.1 of [6].

The result in Theorem 2.1 of [6] is true under the additional assumption that μ is measurably strictly aperiodic as shown below. A probability measure μ on a locally compact group G is called *measurably adapted* if μ is not supported on a proper Borel subgroup and μ is called *measurably strictly aperiodic* if $\sum_{k\geqslant 1}\frac{1}{2^{k+1}}[\mu^k * \check{\mu}^k + \check{\mu}^k * \mu^k]$ is measurably adapted where $\check{\mu}(E) = \mu(E^{-1})$ for all Borel subsets E of G. One may easily see that any measurably adapted measure is adapted and any measurably strictly aperiodic measure is adapted and strictly aperiodic. It may also be easily seen that any adapted, strictly aperiodic and spread out measure (that is, for some $k \geqslant 1$, μ^k is not singular) is measurably strictly aperiodic. In fact, μ is measurably strictly aperiodic if μ is adapted, strictly aperiodic and $\sum_{k\geqslant 1}\frac{1}{2^{k+1}}[\mu^k * \check{\mu}^k + \check{\mu}^k * \mu^k]$ is spread out.

In [1] the strong convergence of T_μ^n is shown for spread out measures. Here we show that the spectral gap condition holds for T_μ for measurably strictly aperiodic measures on groups considered in Theorem 2.1 of [6] which implies in particular the strong convergence and rate of convergence of T_μ^n.

Lemma 1. *Let G be a locally compact σ-compact metrizable group and T be an unitary representation of G. Let μ be a measurably strictly aperiodic probability measure on G. If $\|T_\mu^n\| \not\to 0$, then T weakly contains the trivial representation.*

Proof. Suppose $\|T_\mu^n\| \not\to 0$. Then by Theorem 2.3 of [4], there exists a sequence (v_n) of unit vectors such that $\eta(H) = 1$ where $H = \{g \in G \mid \|T(g)v_n - v_n\| \to 0\}$ and $\eta = \sum_{k\geqslant 1}\frac{1}{2^{k+1}}[\mu^k * \check{\mu}^k + \check{\mu}^k * \mu^k]$. Since μ is measurably strictly aperiodic, $H = G$. Thus, T weakly contains the trivial representation. □

Theorem 1. *Let G be a locally compact σ-compact metrizable group and μ be a measurably strictly aperiodic probability measure on G. Suppose G has a compact normal subgroup K such that $K\backslash G$ is nilpotent. Then $\|T_\mu^n\| \to 0$ for any non-trivial irreducible unitary representation T of G. In particular, (T_μ^n) converges strongly for any unitary representation T of G.*

Proof. Suppose $\|T_\mu^n\| \not\to 0$ for some irreducible representation T of G. Then by Lemma 1, the trivial representation is weakly contained in T and hence $T(K)$ has non-trivial fixed vectors. Since K is normal, the $T(K)$-fixed vectors form a closed $T(G)$-invariant non-trivial subspace. Since T is irreducible, $T(K)$ is trivial. Thus, T may be regarded as a representation of $K\backslash G$. Since $K\backslash G$ is nilpotent, by Corollary 2.6 of [4], T is trivial. □

In view of the failure of Lemma 2.1 of [6], Theorem 2.2 of [6] is not proved yet. However, one can solve the unitary representation problem for measurably strictly aperiodic probabilities on IN-groups arguing as in Theorem 1 using the fact that any IN-group G has a compact normal subgroup K such that $K\backslash G$ is SIN and Theorem 3.3 of [4].

In view of the failure of Lemma 2.2 of [7], in Theorems 3.1 and 4.1 and Corollaries 3.1, 4.1 and 4.2 of [7] the stated conclusion that G is identity excluding does not follow. However the argument shows, in each of the respective results that the group considered is weak identity excluding. We say that a locally compact group is *weak identity excluding* if no non-trivial irreducible unitary representation weakly contains the trivial representation. Thus, we have

Theorem 2. *The following groups are weak identity excluding.*

(1) *Split compact extensions of totally disconnected nilpotent groups*;

(2) *Compactly generated groups of polynomial growth whose connected component of identity is compact*;

(3) *Suppose G is a p-adic algebraic group with unipotent radical U. Let $U_0 = U$ and $U_i = [U, U_{i-1}]$ for $i > 0$. For any $i \geqslant 0$ and for any G-invariant subspace W of $U_{i+1}\backslash U_i$, define $\phi_{i,W} : G \to GL(W)$ by $\phi_{i,W}(g)U_{i+1}x = U_{i+1}gxg^{-1}$ for all $g \in G$ and all $U_{i+1}x \in W \subset U_{i+1}\backslash U_i$. Suppose for each $i \geqslant 0$ and W as above, either $\phi_{i,W}(G)$ is non-amenable or W is of type $R_{\phi_{i,W}(G)}$. In particular, if the solvable radical of G is of type R or G is the semidirect product of $GL(n, \mathbb{Q}_p)$ and $\mathbb{Q}_p^n$.*

The condition weak identity excluding though weaker is of significance in certain contexts. The next two theorems deal with motivation and relevance of the notion of weak identity excluding groups.

Theorem 3. *Let G be a locally compact σ-compact metrizable group. If G is a weak identity excluding group, then for any non-trivial irreducible unitary representation T and for any measurably strictly aperiodic probability measure μ, the spectral radius of T_μ is strictly less than one and hence $\|T_\mu^n\| \to 0$.*

Proof. Let T be an irreducible unitary representation of G. Suppose for some measurably strictly aperiodic probability measure μ, $\|T_\mu^n\| \not\to 0$. Then by Lemma 1, T weakly contains the trivial representation. Since G is weak identity excluding, T is trivial. □

Lemma 2. *Let G be a locally compact σ-compact metrizable group and μ be a measurably adapted probability measure on G. Suppose for an irreducible unitary representation T of G, $\|T_\mu^n\| \not\to 0$. Then T weakly contains an one-dimensional unitary representation of G.*

Proof. By Theorem 2.2 of [3], there exist a sequence of unit vectors (v_n), a measurable subgroup D of G and a character χ of D such that $\mu(D) = 1$ and $\|T(g)v_n - \chi(g)v_n\| \to 0$ for all $g \in D$. Since μ is measurably adapted, $D = G$. By Theorem 2.2(iii) of [3], $\mu(a \ker \chi) = 1$ for some $a \in G$. This implies that $\bigcup a^n \ker \chi$ is a Borel subgroup supporting μ. Since μ is measurably adapted, $G = \bigcup a^n \ker \chi$. Thus, $\ker \chi$ is open and hence χ is a continuous character of $D = G$. This proves the result. □

Theorem 4. *Let G be a locally compact σ-compact metrizable group. If G is a weak identity excluding group, then for any measurably adapted probability measure μ on G and for any irreducible unitary representation T of G of dimension strictly greater than one, the spectral radius of T_μ is strictly less than one.*

Proof. Let T be an irreducible unitary representation of G and μ be a measurably adapted probability measure on G. Suppose spectral radius of T_μ is one. Then by Lemma 2, T weakly contains the one-dimensional representation defined by a continuous character χ on G. This implies that the representation $\pi = \chi^* T$ weakly contains the trivial representation where χ^* is the conjugate of χ. Since G is weak identity excluding, π is the trivial representation and hence T is one-dimensional. □

The following is an easy consequence of Theorem 4 and Theorem 3.1 of [3] and answers a question considered in [3] for weak identity excluding groups: the following is of particular interest because the asymptotic behavior of the following is not true for some measures on certain discrete groups (see Example 3.4 of [3]).

Theorem 5. *Let G be a locally compact σ-compact metrizable group. Suppose G is a weak identity excluding group and μ is a measurably adapted probability measure on G. Let N_μ be the smallest closed normal subgroup a coset of which contains the support of μ. Let T be any unitary representation and E_μ be the orthogonal projection onto the subspace of fixed vectors of $T(N_\mu)$. Then $T_\mu^n - T(a)^n E_\mu \to 0$ strongly for any $a \in G$ with $\mu(aN_\mu) = 1$.*

C. Cuny's example shows that the result in Corollary 3.1 of [7] is not true in its full generality but we uphold the result in Corollary 3.1 of [7] for p-adic Lie groups in the following.

Theorem 6. *Let G be a compactly generated p-adic Lie group of polynomial growth. Then G is identity excluding.*

Proof. By Theorem 2 of [5], there exists a compact normal subgroup K of G such that G/K is a real Lie group. Since G is totally disconnected, G/K is also totally disconnected and hence G/K is discrete. This implies that K is a compact open normal subgroup of G. Since K is a compact p-adic Lie group, for a given integer m, K contains only finitely many subgroups of index m. Thus, for any compact open subgroup L of K, $\bigcap \alpha(L)$ where the intersection runs over all continuous automorphisms α of K is a finite intersection. This implies that every compact open subgroup of K contains a characteristic compact open subgroup. Thus, $\mathrm{Aut}(K)$ is compact. In particular, every compact open subgroup of K contains a compact open normal subgroup of G. Now, for any open subgroup U of G, $K \cap U$ is a compact open subgroup of K and hence $K \cap U$ contains a compact open normal subgroup of G. Thus, G has a basis of compact open normal subgroups at identity.

Let T be any irreducible unitary representation of G on a Hilbert space V. Let $v \in V$ be such that $\|v\| = 1$. Then there exists a compact open normal subgroup K_1 of G such that $\|T(g)v - v\| < 1$ for all $g \in K_1$. Let ω_{K_1} be the normalized Haar measure on K_1. Then $\|T_{\omega_{K_1}}(v) - v\| < 1$. This implies that $T_{\omega_{K_1}}(v)$ is a nonzero vector in V. It can be easily seen that $T_{\omega_{K_1}}(v)$ is fixed by $T(K_1)$. Since K_1 is normal subgroup, space of all $T(K_1)$-fixed vectors form a closed $T(G)$-invariant non-trivial subspace. Since T is irreducible, $T(K_1)$ is trivial. Thus, T may be regarded as an irreducible representation of G/K_1 which is a finitely generated group of polynomial growth. By the Main Theorem of [2] and by Theorem 3.2 of [7], G is identity excluding. □

Remark 1. We would like to remark that using the ideas presented in the proof of Theorem 6, one can solve the unitary representation problem for IN-p-adic Lie groups. Thus, upholding Theorem 2.2 of [6] for IN-p-adic Lie groups.

The failure of Lemma 2.2 of [7], leaves a gap in the proof of Theorem 5.1 of [7] and some other minor mistakes are also there in the proof of Theorem 5.1 of [7] which could be corrected but we do not want to go into the details because we now give a slightly different proof of Theorem 5.1 of [7] and uphold the results in Theorems 5.1 and 5.2 of [7]. We in fact prove the result for split solvable algebraic groups over local fields of characteristic zero.

Theorem 7. *Let G be a solvable algebraic group over a local field of characteristic zero and let G^0 be the Zariski-connected component of identity in G. Let μ be an adapted and strictly aperiodic probability measure on G and T be an unitary representation of G. Suppose G is a split solvable algebraic group, that is, the maximal torus in G^0 is a split torus. Then (T_μ^n) converges strongly.*

Proof. We may assume that T is irreducible and that there exist a dense subgroup D of G and a sequence (v_n) of unit vectors such that $\|T(g)v_n - v_n\| \to 0$ for all $g \in D$. Let $G' = [G, G]$. Then G is solvable (algebraic group) implies G' is nilpotent. For $i \geqslant 1$, let $G'_i = [G', G'_{i-1}]$, where $G'_0 = G'$. Suppose T is trivial on G'. Then T may be regarded as a representation of $G' \backslash G$ which is an abelian group and we are done. So we may assume that $T(G')$ is non-trivial. Then there exists a $k \geqslant 0$ such that $T(G'_k)$ is non-trivial and $T(G'_{k+1})$ is trivial. Replacing G by $G'_{k+1} \backslash G$, we may assume that G'_k is in the center of G'. Let $N = G'_k$. Then the orbits of the dual action of G on the dual of N are locally closed in the local filed topology as G is an algebraic group. By Mackey's Theorem, there exists a character χ on N such that T is unitarily induced from an irreducible unitary representation ρ of the stabilizer G_χ of χ and $\rho(x) = \chi(x)$ for all $x \in N$. Since N is in the center of G', $G' \subset G_\chi$. This shows that G_χ is a normal subgroup of G. Let E be the Hilbert space on which ρ is defined. Let $L^2(G, G_\chi, \rho)$ be the space of all functions $f : G \to E$ such that $f(hx) = \rho(h) f(x)$ for all $x \in G$ and all $h \in G_\chi$ and $\int_{G_\chi \backslash G} \|f(x)\|^2 \, dm(G_\chi x) < \infty$ where m is the invariant measure on $G_\chi \backslash G$. Then T is defined on $L^2(G, G_\chi, \rho)$ by $T(g) f(x) = f(xg)$ for all $x, g \in G$ and all $f \in L^2(G, G_\chi, \rho)$. Now for any $f \in L^2(G, G_\chi, \rho)$, we define $f_1(G_\chi x) = \|f(x)\|$ for all $x \in G$. Then $f_1 \in L^2(G_\chi \backslash G)$. Let λ be the projection of μ onto $G_\chi \backslash G$. Now for $f \in L^2(G, G_\chi, \rho)$,

$$\|T_\mu f\|^2 = \left| \int \langle T_\mu f(x), T_\mu f(x) \rangle \, dm(G_\chi x) \right|$$

$$\leqslant \iiint \left|\langle f(xg_1), f(xg_2)\rangle\right| d\mu(g_1)\, d\mu(g_2)\, dm(G_\chi x)$$
$$\leqslant \iiint f_1(G_\chi x g_1) f_1(G_\chi x g_2)\, d\mu(g_2)\, d\mu(g_1)\, dm(G_\chi x)$$
$$\leqslant \left\| R_\lambda(f_1)\right\|^2,$$

where R is the regular representation of $G_\chi\backslash G$ on $L^2(G_\chi\backslash G)$. This implies that $\|T_\mu^n f\| \leqslant \|R_\lambda^n f_1\|$ for all $n \geqslant 1$ and all $f \in L^2(G, G_\chi, \rho)$.

Suppose $G_\chi\backslash G$ is compact. Since G_χ is an algebraic normal subgroup of G containing G', $G_\chi\backslash G$ is a compact commutative algebraic group. Thus, the Zariski-connected component of $G_\chi\backslash G$ is isomorphic to a Zariski-connected algebraic subgroup of any maximal torus in G^0. Since G^0 is split solvable, the Zariski-connected component of $G_\chi\backslash G$ is trivial and hence $G_\chi\backslash G$ is finite. Let $G_\chi\backslash G = \{G_\chi x_0, G_\chi x_1, \ldots, G_\chi x_l\}$ for some $x_0, x_1, \ldots, x_l$ in G with $x_0 = e$ and $x_j x_i^{-1} \notin G_\chi$ for all $i \neq j$. Then by assumption there exists a sequence $(f_n) \in L^2(G, G_\chi, \rho)$ such that $\|f_n\|^2 = 1$ and $\|T(g) f_n - f_n\| \to 0$ for all $g \in D$. Since D is dense in G, $D \cap G'$ is dense in G' and hence $D \cap G_i'$ is dense in G_i' for all $i \geqslant 1$. In particular, $D \cap N$ is dense in N. We may assume that m is the counting measure on $G_\chi\backslash G$. Since $\|f_n\|^2 = \sum_{i=0}^{l} \|f_n(x_i)\|^2 = 1$, there exists a i such that $\|f_n(x_i)\| \not\to 0$. Now, for $g \in D \cap N$, $\|T(g) f_n(x_i) - f_n(x_i)\|^2 \leqslant \|T(g) f_n - f_n\|^2 \to 0$ and hence for $g \in D \cap N$,

$$\left\|T(g) f_n(x_i) - f_n(x_i)\right\| = \left\|f_n(x_i g) - f_n(x_i)\right\| = \left\|f_n(x_i)\right\| \left|\chi(x_i g x_i^{-1}) - 1\right| \to 0.$$

This implies that $\chi(x_i g x_i^{-1}) = 1$ for all $g \in D \cap N$. Since $D \cap N$ is dense in N, χ is trivial. This is a contradiction to the assumption that $T(N)$ is non-trivial. So, $G_\chi\backslash G$ is a non-compact abelian group. This implies that $\|T_\mu^n f\| \leqslant \|R_\lambda^n f_1\| \to 0$ for all $f \in L^2(G, G_\chi, \rho)$. □

Remark 2. We would like to remark that Theorem 7 is true for certain solvable algebraic groups over any local field. In the positive characteristic case one has to provide an appropriate notion of split solvable algebraic groups.

We also take this opportunity to correct a few typos in [7].

In page 765, line 9 from below, '$\chi(g_i h g_i^{-1})$' should be changed to '$\chi(g_i^{-1} h g_i)$'.

In page 767, line 19 from top, '$\chi(g_i x g_i^{-1})$' should be changed to '$\chi(g_i^{-1} x g_i)$'.

In page 770, line 9 from top, '$A_s \setminus (e)$' should be changed to '$A_s \setminus (K)$ for some compact open subgroup K'.

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