

On Product Systems arising from Sum Systems

B. V. RAJARAMA BHAT,

Indian Statistical Institute,
Bangalore-560 059, India.
bhat@isibang.ac.in

R. SRINIVASAN

Department of Mathematical Sciences,
University of Tokyo, Komaba, Tokyo, 153-8914, Japan.
vasanth@ms.u-tokyo.ac.jp

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Abstract

Boris Tsirelson constructed an uncountable family of type *III* product systems of Hilbert spaces through the theory of Gaussian spaces, measure type spaces and ‘slightly coloured noises’, using techniques from probability theory. Here we take a purely functional analytic approach and try to have a better understanding of Tsirelson’s construction and his examples.

We prove an extension of Shale’s theorem connecting symplectic group and Weyl representation. We show that the ‘Shale map’ respects compositions (This settles an old conjecture of K. R. Parthasarathy [8]). Using this we associate a product system to a sum system. This construction includes the exponential product system of Arveson, as a trivial case, and the type *III* examples of Tsirelson.

By associating a von Neumann algebra to every ‘elementary set’ in $[0, 1]$, in a much simpler and direct way, we arrive at the invariants of the product system introduced by Tsirelson, given in terms of the sum system. Then we introduce a notion of divisibility for a sum system, and prove that the examples of Tsirelson are divisible. It is shown that only type *I* and type *III* product systems arise out of divisible sum systems. Finally, we give a sufficient condition for a divisible sum system to give rise to a unitless (type *III*) product system.

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1 Introduction:

R. T. Powers [9] initiated a study of E_0 -semigroups, which are weakly continuous semigroups of unital $*$ -endomorphisms of some $\mathbf{B}(H)$, for a separable Hilbert space H . In this context Arveson [1] introduced the concept of product system of Hilbert spaces as an invariant for E_0 -semigroups. Up to cocycle conjugacy an E_0 -semigroup $\{\alpha_t\}$ is determined by the family of Hilbert spaces $\{H_t\}$, where

$$H_t = \{T \in \mathbf{B}(H) : \alpha_t(X)T = TX, \forall X \in \mathbf{B}(H)\}$$

with inner product $\langle T, S \rangle_{1_H} = S^*T$ (see [1]). Moreover the family $\{H_t\}$ forms a product system of Hilbert spaces (see Definition 1). Arveson also constructed an E_0 -semigroup from a given product system, thus proving that the product systems forms a complete invariant for the E_0 -semigroup (up to cocycle conjugacy).

Arveson classified product systems, according to the existence of units (see Definition 5), into three broad categories, such as type *I*, *II*, *III*. He also classified completely the type *I* product systems, up to isomorphism. We refer to [3] for general theory of E -semigroups and product systems and [12] for some recent developments.

The theory of product systems was lacking enough examples. For quite sometime there were essentially only one example each for type *II* and type *III* product systems (due to R. T. Powers (see [9]-[11])). Tsirelson produced an uncountable family of both type *II* and type *III* product systems (ref [14], [15]).

Tsirelson uses the theory of random sets arising from a Brownian motion to get type *II* product systems and the theory of FHS spaces, Gaussian spaces, measure type spaces and what he calls as ‘slightly coloured noises’ to get the examples of type *III* product systems. Tsirelson’s construction of type *III* product systems is complicated and involves lots of techniques from probability theory. Also it is not clear as how to work with the E_0 -semigroup associated with the product systems, and there is no information regarding other invariants of the product system, such as the automorphism group etc. Our work is inspired by the path breaking results of Tsirelson (which in turn borrow on some brilliant ideas of Vershik).

The basic idea of Tsirelson's construction of type *III* product systems is simple. Usual L^2 on sub-intervals on real line is a direct sum system in the sense that $L^2(0, s) \oplus L^2(s, s+t) = L^2(0, s+t)$ for positive s, t . Such a system on 'exponentiation' gives the type I or the Fock product system. Now if we replace a direct sum system by an 'almost' or 'quasi' direct sum system we get more exotic product systems. First job is to make precise as to what one means by quasi-direct sum and then one has to find a suitable procedure of exponentiation. Tsirelson does this by his notion of FHS equivalence, identifying the Hilbert spaces in the sum system with Gaussian type spaces, and then getting the product system, as the L^2 -space of the corresponding measure type spaces. We retain Tsirelson's notion of sum system though we don't use the language of probability theory. The essential difference in our approach is that we do the exponentiation using the theory of symmetric Fock spaces and a generalised version of Shale's theorem.

We first prove in Section 2, a generalisation of Shale's theorem. We also prove a functorial property in the Shale's theorem affirmatively settling a conjecture of K. R. Parthasarathy. Using this, after proving some lemmas, we associate product system with a sum system. We show that this gives the exponential product system, as a trivial case, and includes the examples of Tsirelson. We also prove some properties of sum systems, and provide an operator theoretic proof of some facts in Tsirelson's work.

In Section 3, given a product system we associate a von Neumann algebra to any elementary set (finite union of intervals) in the interval $[0, 1]$. We analyse these von Neumann algebras, and by simple application of double commutant theorem, strong-weak convergences, we arrive at the invariants for the product systems, given in terms of the original sum systems. This would prove the examples of Tsirelson are non-isomorphic to each other. This is in fact the difficult part of Tsirelson's work, and we give here a much more direct and simple proof of this fact.

In Section 4, we first define a notion of divisibility for a sum system, and study some basic properties of a divisible sum system. We prove that all examples of Tsirelson are divisible. We also show that only type *I* and *III* are possible under the divisibility assumption on the sum system. Finally, using some of the notions introduced by Tsirelson, we prove a sufficient condition for the product system arising from a divisible sum system to be of type *III*.

Almost after finishing this work we came to know about the new preprint of Tsirelson ([16]), where he has simplified many of the proofs in his earlier two preprints, for producing the uncountable family of type *II* and *III* product systems. We still believe that our method is more direct and simple, leading to new applications. We plan to consider some research ideas emerging from this approach in future.

We end this section by recalling some of basic definitions, which are initially defined by Arveson. For underlying measurability conditions we use a slightly modified, but essentially equivalent, definition, given by Volkmar Liebcher ([17]).

DEFINITION 1 *A product system of Hilbert spaces is an one parameter family of separable Hilbert spaces $\{H_t\}_{t \in (0, \infty)}$, together with unitary operators*

$$U_{s,t} : H_s \otimes H_t \mapsto H_{s+t} \text{ for } s, t \in (0, \infty),$$

satisfying the following two axioms of associativity and measurability.

(i) *(Associativity) For any $s_1, s_2, s_3 \in (0, \infty)$*

$$U_{s_1, s_2+s_3}(1_{H_{s_1}} \otimes U_{s_2, s_3}) = U_{s_1+s_2, s_3}(U_{s_1, s_2} \otimes 1_{H_{s_3}}).$$

(ii) *(Measurability) There exists a countable set H^0 of sections $R \ni t \rightarrow h_t \in H_t$ such that $t \mapsto \langle h_t, h'_t \rangle$ is measurable for any two $h, h' \in H^0$, and the set $\{h_t : h \in H^0\}$ is a total set in H_t , for each $t \in (0, \infty)$. Further it is also assumed that the map $(s, t) \mapsto \langle U_{s,t}(h_s \otimes h_t), h'_{s+t} \rangle$ is measurable for any two $h, h' \in H^0$.*

DEFINITION 2 *Two product systems $(H_t, U_{s,t})$ and $(H'_t, U'_{s,t})$ are said to be isomorphic if there exists a unitary operator $V_t : H_t \mapsto H'_t$, for each $t \in (0, \infty)$, satisfying the following two conditions.*

(i) $V_{s+t}U_{s,t} = U'_{s,t}(V_s \otimes V_t)$.

(ii) *The $t \in (0, \infty) \mapsto \langle V_t h_t, h' \rangle$ is measurable for any $h \in H^0, h' \in H'^0$.*

REMARK 3 *Volkmar Liebcher has proved in [17] that any two measurable structures give rise to isomorphic product systems, and as a consequence we get that two product systems are isomorphic if they are algebraically isomorphic. That is the condition (ii) in the above definition can be dropped.*

DEFINITION 4 For a product system $(H_t, U_{s,t})$, we define the opposite product system $(H_t^{op}, U_{s,t}^{op})$ by,

$$H_t^{op} = H_t, \quad U_{s,t}^{op} = U_{t,s}\tau_{s,t},$$

where $\tau_{s,t}$ is the flip operator on $H_s \otimes H_t$, $\tau_{s,t}(x \otimes y) = y \otimes x$.

A product system is said to be symmetric if it is isomorphic to its opposite product system, (i.e) it is anti-isomorphic to itself.

We next define the units, based on whose existence, the product systems are classified into three broad categories.

DEFINITION 5 A unit is a measurable section $\{u_t\}_{t \in (0, \infty)}$, ((i.e) $u_t \in H_t$, and the map $t \mapsto \langle u_t, h_t \rangle$ is measurable for any $h \in H^0$), satisfying

$$U_{s,t}(u_s \otimes u_t) = u_{s+t}, \quad \forall s, t \in (0, \infty), \quad \text{and } u_t \neq 0 \text{ for some } t \in (0, \infty).$$

We denote by \mathbb{U} the set of all units for a product system. We say a product system is of type *I*, if units exists for the product system and they generate the product system, (i.e.) for any fixed $t \in (0, \infty)$, the set

$$\{u_{t_1}^1 u_{t_2}^2 \cdots u_{t_n}^n : \sum_{i=1}^n t_i = t, u^i \in \mathbb{U}\},$$

is a total set in H_t , where the product is defined as the image of $u_{t_1}^1 \otimes u_{t_2}^2 \cdots \otimes u_{t_n}^n$ in H_t , under the canonical unitary given by the associativity axiom. It is of type *II* if units exists but they don't generate the product system. We say a product system to be of type *III* or unitless if there does not exist any unit for the product system. We are most concerned about this type *III* product systems in this paper.

2 The construction

In this section we construct a product system from a given sum system (see definition 16). We do this by proving a generalised version of Shale's theorem. Before that we fix our notation.

For a real Hilbert space G we denote by \overline{G} the complexification of G . (Throughout this paper we always denote a real Hilbert space by G , and if the Hilbert space is complex we denote it by H or \overline{G} or we specify it). We define, for a single Hilbert space G or for two Hilbert spaces G_1 and G_2 , $\mathbb{S}(G)$ and $\mathbb{S}(G_1, G_2)$ in the following way,

$$\mathbb{S}(G) = \{A \in \mathbb{B}(G) : A \text{ positive, invertible and } I - A \text{ is Hilbert-Schmidt}\},$$

$$\mathbb{S}(G_1, G_2) = \{A \in \mathbb{B}(G_1, G_2) : A \text{ invertible and } I - (A^*A)^{\frac{1}{2}} \text{ Hilbert-Schmidt}\}.$$

In the above definition, and elsewhere in this paper, by invertible we mean the inverse is also bounded. Note that $\mathbb{S}(G, G)$ is different from $\mathbb{S}(G)$.

Clearly $A \in \mathbb{S}(G_1, G_2)$ if and only if $A^{-1} \in \mathbb{S}(G_2, G_1)$, and $A \in \mathbb{S}(G_1, G_2)$ if and only if $(A^*A)^{\frac{1}{2}} \in \mathbb{S}(G_1)$.

If $A \in \mathbb{S}(G_1, G_2)$ and $B \in \mathbb{S}(G_2, G_3)$, we may conclude from the relation $I - A^*B^*BA = I - A^*A + A^*(I - B^*B)A$ that $I - A^*B^*BA$ is a Hilbert-Schmidt operator. The fact that $I - A^*A$ is a Hilbert-Schmidt operator is equivalent to saying that $I - (A^*A)^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, when A is invertible (see [6], and [15] Proposition 9.9, page 46), and the above verification now proves that $BA \in \mathbb{S}(G_1, G_3)$.

Also if $A \in \mathbb{S}(G_1, G_2)$, then the same fact implies that $A^*A \in \mathbb{S}(G_1)$. Now, as $A^{-1} \in \mathbb{S}(G_2, G_1)$ and $A^*A \in \mathbb{S}(G_1)$, we conclude that $(A^*)^{-1} \in \mathbb{S}(G_1, G_2)$.

Suppose $A \in \mathbb{S}(G_1, G_2)$, and $G'_1 \subset G_1$ be any subspace and $G'_2 = A(G'_1)$, then we want to check whether the restriction $A|_{G'_1} \in \mathbb{S}(G'_1, G'_2)$. As $I_{G'_1} - (A|_{G'_1})^*A|_{G'_1}$ is the compression of $I_G - A^*A$ to G'_1 it is a Hilbert-Schmidt operator. Then it is clear that $A|_{G'_1} \in \mathbb{S}(G'_1, G'_2)$.

We make some definitions and fix some notation.

DEFINITION 6 *We say two subspaces G_1 and G_2 , both contained in a real Hilbert space G , are quasi-orthogonal if there exists a map $A \in \mathbb{S}(G)$ such that $\langle Ax, Ay \rangle = 0$, $\forall x \in G_1, y \in G_2$.*

We use the notation $G_1 \uplus G_2 = G$ (respectively $\uplus_{i=1}^n G_i = G$), if G_1 is quasi-orthogonal to G_2 , and G is generated by G_1 and G_2 (respectively G_i 's are mutually quasi-orthogonal and $G = \vee_{i=1}^n G_i$). We also denote by

$$\mathbb{O}(\uplus_{i=1}^n G_i, G) = \{A \in \mathbb{S}(\uplus_{i=1}^n G_i, G); A(G_i) = G_i, \text{ for each } i = 1, 2, \dots, n\}.$$

LEMMA 7 Let $\{G_i\}_{i=1}^n$ be a family of real Hilbert spaces all contained in a one real Hilbert space G . Then the set $\mathcal{O}(\oplus_{i=1}^n G_i, G)$ is not empty if and only if $\uplus_{i=1}^n G_i = G$.

Proof: Suppose there exists $U \in \mathcal{O}(\oplus_{i=1}^n G_i, G)$ then define $A = ((U^{-1})^* U^{-1})^{\frac{1}{2}}$. Then for $x \in G_i, y \in G_j$ and $i \neq j$, $\langle Ax, Ay \rangle = \langle U^{-1}x, U^{-1}y \rangle = 0$. The invertibility of U and the condition $U(G_i) = G_i$ clearly imply that $G = \overline{\text{span}[\cup_{i=1}^n G_i]}$.

Now to prove the otherway, suppose there exists $A \in \mathcal{S}(G)$ such that $AG_i \perp AG_j$ if $i \neq j$, then as $G = \overline{\text{span}[\cup_{i=1}^n G_i]}$ we conclude that $\oplus_{i=1}^n AG_i = G$. Now define $U = A^{-1}(\oplus_{i=1}^n (A|_{G_i}))$, clearly $U \in \mathcal{O}(\oplus_{i=1}^n G_i, G)$. \square

REMARK 8 Note that we have also proved that $\mathcal{O}(\oplus_{i=1}^n G_i, G)$ is not empty if and only if the map $\oplus_{i=1}^n x_i \mapsto \sum_{i=1}^n x_i$ is in $\mathcal{S}(\oplus_{i=1}^n G_i, G)$.

The following lemma is proved in [15] using probability theory (Radon-Nikodym derivatives). We provide an operator theoretic proof here.

LEMMA 9 Let G_1, G_2, G_3 be real Hilbert spaces all contained in a real Hilbert space G . Let G_{12} (resp'tly. G_{23}) be the Hilbert space generated by G_1 and G_2 (resp'tly. by G_2 and G_3). Suppose that $G_1 \uplus G_{23} = G$ and $G_{12} \uplus G_3 = G$, then it also holds that $\uplus_{i=1}^3 G_i = G$.

Proof: Choose $A_1, A_2 \in \mathcal{S}(G)$ such that $A_1 G_1 \perp A_1 G_{23}$ and $A_2 G_{12} \perp A_2 G_3$. As we also have $G = \overline{\text{span}[G_{12}, G_3]}$, we conclude that $A_2 G_{12} \oplus A_2 G_3 = G$. Now define

$$A_0 = \left(((A_1 A_2^{-1}|_{A_2 G_{12}})^* A_1 A_2^{-1}|_{A_2 G_{12}})^{\frac{1}{2}} \oplus I|_{A_2 G_3} \right) A_2,$$

where $((A_1 A_2^{-1}|_{A_2 G_{12}})^* A_1 A_2^{-1}|_{A_2 G_{12}})^{\frac{1}{2}} \oplus I$ is defined on $A_2 G_{12} \oplus A_2 G_3$. Clearly $A_0 \in \mathcal{S}(G, G)$.

Now for $x \in G_1, y \in G_2$, we have

$$\begin{aligned} \langle A_0 x, A_0 y \rangle &= \langle (A_1 A_2^{-1}|_{A_2 G_{12}})^* A_1 A_2^{-1}|_{A_2 G_{12}}(A_2 x), A_2 y \rangle_{A_2 G_{12}} \\ &= \langle A_1 A_2^{-1}(A_2 x), A_1 A_2^{-1}(A_2 y) \rangle = \langle A_1 x, A_1 y \rangle = 0. \end{aligned}$$

Also if $x \in G_{12}$ and $y \in G_3$ $\langle A_0x, A_0y \rangle = \langle z, A_2y \rangle = 0$, where z is some element in A_2G_{12} . So A_0 satisfies $\langle A_0x, A_0y \rangle = 0$ whenever $x \in G_i, y \in G_j$, for $1 \leq i, j \leq 3$ and $1 \neq j$. If we define $A = (A_0^*A_0)^{\frac{1}{2}}$, then clearly $A \in \mathbf{S}(G)$ and it continues to satisfy $\langle Ax, Ay \rangle = 0$ whenever $x \in G_i, y \in G_j$, for $1 \leq i, j \leq 3$ and $1 \neq j$. \square

Let G_1, G_2 be two real Hilbert spaces and let $A \in \mathbf{S}(G_1, G_2)$, then define $S_A : \overline{G_1} \rightarrow \overline{G_2}$ by $S_A(u + iv) = Au + i(A^{-1})^*v$ for $u, v \in G_1$. Then S_A is a symplectic isomorphism between $\overline{G_1}$ and $\overline{G_2}$ (i.e. S_A is a real linear, bounded, invertible map with a bounded inverse satisfying $\text{Im}(\langle S_Ax, S_Ay \rangle) = \text{Im}\langle x, y \rangle$ for all $x, y \in \overline{G_1}$, see [8] page 162). Notice that, for a unitary operator $U \in \mathbf{B}(G_1, G_2)$ (which is clearly in $\mathbf{S}(G_1, G_2)$), S_U is a complex linear, unitary operator, and $S_U(x + iy) = Ux + iUy$.

We briefly recall the notions of the symmetric Fock space of a Hilbert space, exponential vectors and the Weyl operators. For a complex Hilbert space K , we know that the tensor product $\otimes_{i=1}^n K_i$, where $K_i = K$ for all $i = 1, 2, \dots, n$, admits an action of the symmetric group S_n , given by

$$\sigma(\otimes \xi_i) = \otimes \xi_{\sigma^{-1}(i)}.$$

The symmetric tensor product and symmetric Fock space corresponding to K are defined by

$$K^{\otimes s} = \{\xi \in K : \sigma(\xi) = \xi\}, \quad \Gamma_s(K) = \bigoplus_{i=0}^{\infty} K^{\otimes i},$$

where $K^{\otimes 0}$ is assumed to be \mathbb{C} . We call $1 \in \mathbb{C} \subset \Gamma_s(K)$, as the vacuum vector, and denote it by Φ . For any $x \in K$, we define,

$$e(x) = \bigoplus_{i=0}^{\infty} \frac{x^{\otimes i}}{\sqrt{i!}}.$$

It is a fact that the set $\{e(x) : x \in K\}$ is a linearly independent and total set in $\Gamma_s(K)$. The Weyl operator, corresponding to an element $x \in K$ is defined by,

$$W(x)(e(y)) = e^{-\frac{1}{2}\|x\|^2 - \langle y, x \rangle} e(y + x),$$

and $W(x)$ extends to an unitary operator on $\Gamma_s(K)$. Also, for a unitary operator U , between two Hilbert spaces K_1 and K_2 , $U \in \mathbf{B}(K_1, K_2)$, we

define another operator $Exp(U)$ between the corresponding symmetric Fock spaces, $Exp(U) \in \mathbf{B}(\Gamma_s(K_1), \Gamma_s(K_2))$, by,

$$Exp(U)(e(x)) = e(Ux).$$

Again, $Exp(U)$ extends to an unitary operator.

As $W(x)W(y) = e^{-Im\langle x,y \rangle} W_{x+y}$, the correspondence $x \mapsto W(x)$ provides a projective representation for the abelian group K . Notice that when $K = \overline{G}$, as $Im(\langle S_Ax, S_Ay \rangle) = Im\langle x, y \rangle$, the correspondence $x \mapsto W(S_Ax)$ also provides a projective representation. Shale's theorem answers the question as to when these two projective representations are equivalent. The following theorem is a generalisation of Shale's Theorem (see [8] page 169, Theorem 22.11), where now instead of maps from a real Hilbert space to itself we have maps from one real Hilbert space to another. More importantly we prove that the 'Shale map' $\Gamma(\cdot)$, of Shale's theorem respects composition - see (ii) of Theorem 10. This was left as an open problem in [8] page 170).

THEOREM 10 (i) *Let G_1, G_2 be real Hilbert spaces and $A \in \mathbf{S}(G_1, G_2)$, then there exists a unique unitary operator $\Gamma(A) : \Gamma_s(\overline{G_1}) \rightarrow \Gamma_s(\overline{G_2})$ such that*

$$\Gamma(A)W(u)\Gamma(A)^* = W(S_Au) \quad (2.1)$$

$$\langle \Gamma(A)\Phi_1, \Phi_2 \rangle \in \mathbb{R}^+ \quad (2.2)$$

where Φ_1 and Φ_2 are the vacuum vectors in $\Gamma_s(\overline{G_1})$ and $\Gamma_s(\overline{G_2})$ respectively.

(ii) *Suppose G_1, G_2, G_3 be three real Hilbert spaces, and $A \in \mathbf{S}(G_1, G_2)$, $B \in \mathbf{S}(G_2, G_3)$, then*

$$\Gamma(A^{-1}) = \Gamma(A)^* \quad (2.3)$$

$$\Gamma(BA) = \Gamma(B)\Gamma(A) \quad (2.4)$$

(iii) *If $\{T_n\} \subset \mathbf{S}(G, G)$, be any sequence of operators such that T_n converges strongly to $T \in \mathbf{S}(G, G)$ and $(T_n^*)^{-1}$ converges strongly to $(T^*)^{-1}$, then $\Gamma(T_n)$ converges weakly to $\Gamma(T)$.*

Proof: (i) Let $A_0 = (A^*A)^{\frac{1}{2}}$. As $I - A_0$ is a Hilbert-Schmidt operator on G_1 , there exists an orthonormal basis $\{e_i\} \subset G_1$ such that $A_0e_i = \lambda_i e_i$, with $\lambda_i > 0$ for each i and $\sum_i(\lambda_i - 1)^2 < \infty$.

Let $f_i = \lambda_i^{-1}Ae_i$, then as

$$\langle Ae_i, Ae_j \rangle = \langle A^*Ae_i, e_j \rangle = \langle A_0e_i, A_0e_j \rangle = \langle \lambda_i e_i, \lambda_j e_j \rangle,$$

we conclude that $\{f_i\}$ is an orthonormal basis for G_2 . Also note that $Ae_i = \lambda_i f_i$ and $A^{*-1}e_i = \lambda_i^{-1}f_i$. Now identify $\overline{G_1}$ with $l^2(\{e_i\})$ ((i.e.) with $\oplus_{i=1}^{\infty} \mathbb{C}e_i$) and $\overline{G_2}$ with $l^2(\{f_i\})$ ((i.e.) with $\oplus_{i=1}^{\infty} \mathbb{C}f_i$). Also identify $\Gamma_s(\mathbb{C})$ with $L^2(\mathbb{R})$ by

$$e(z) \rightarrow (2\pi)^{-\frac{1}{4}} \exp(-\frac{1}{4}t^2 + zt - \frac{1}{2}z^2)$$

for $z \in \mathbb{C}$ (see [8] page 142, Proposition 20.9). Let U_i (resp. V_i) be the unitary operator between $\Gamma_s(\mathbb{C}e_i)$ (resp. $\Gamma_s(\mathbb{C}f_i)$) and $L^2(\mathbb{R})$. Then the following relations hold (and also with U_i replaced by V_i).

$$\begin{aligned} (U_i e(ze_i))(t) &= (2\pi)^{-\frac{1}{4}} \exp(-\frac{1}{4}t^2 + zt - \frac{1}{2}z^2), & (2.5) \\ (U_i W(xe_i)(U_i)^{-1}f)(t) &= f(t - 2x), \\ (U_i W(iye_i)(U_i)^{-1}f)(t) &= e^{ity}f(t), \end{aligned}$$

where $z \in \mathbb{C}$, $f \in L^2(\mathbb{R})$ and $z = x + iy$ (again see [8] page 142, Proposition 20.9).

For $\lambda > 0$ define L_λ on $L^2(\mathbb{R})$ by $L_\lambda(f)(x) = \lambda^{-\frac{1}{2}}f(\frac{x}{\lambda})$. L_λ is a unitary operator on $L^2(\mathbb{R})$. Also if we define $V_{\lambda_i} = V_i^{-1}L_{\lambda_i}U_i$, then clearly V_{λ_i} is a unitary operator between $\Gamma_s(\mathbb{C}e_i)$ and $\Gamma_s(\mathbb{C}f_i)$. Moreover a simple calculation, using the equations 2.5, shows that, for any $z = x + iy \in \mathbb{C}$, V_{λ_i} satisfies the following equations.

$$V_{\lambda_i}W(ze_i)V_{\lambda_i}^{-1} = W((\lambda_i x + \lambda_i^{-1}y)f_i) \quad (2.6)$$

$$\langle V_{\lambda_i}\Phi_1, \Phi_2 \rangle = \left(\frac{\lambda_i + \lambda_i^{-1}}{2}\right)^{-\frac{1}{2}}, \quad (2.7)$$

where Φ_1 and Φ_2 are the vacuum vector in $\Gamma_s(\mathbb{C}e_i)$ and $\Gamma_s(\mathbb{C}f_i)$ respectively.

Identify $\Gamma_s(\overline{G_1})$ (resp'tly. $\Gamma_s(\overline{G_2})$) with $\otimes_{i=1}^{\infty} \Gamma_s(\mathbb{C}e_i)$ (resp'tly. with $\otimes_{i=1}^{\infty} \Gamma_s(\mathbb{C}f_i)$), where the countable tensor product is with respect to the stabilising sequence of vacuum vectors. Define

$$\Gamma_n = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n} \otimes I_{[n+1]},$$

where $I_{[n+1]}$ is like an identity operator between $\otimes_{i=n+1}^{\infty} \Gamma_s(\mathbb{C}e_i)$ and $\otimes_{i=n+1}^{\infty} \Gamma_s(\mathbb{C}f_i)$,

$$(i.e.) \quad I_{[n+1]}(\otimes_{i=n+1}^l z_i e_i \otimes \Phi_1 \otimes \Phi_1 \cdots) = \otimes_{i=n+1}^l z_i f_i \otimes \Phi_2 \otimes \Phi_2 \cdots$$

For any $n > m > k$, we have

$$\begin{aligned} & \| (V_{\lambda_{k+1}} \phi_1 \otimes \cdots \otimes V_{\lambda_n} \phi_1) \otimes \Phi_2 \otimes \Phi_2 \otimes \cdots - (V_{\lambda_{k+1}} \phi_1 \otimes \cdots \otimes V_{\lambda_m} \phi_1) \otimes \Phi_2 \otimes \Phi_2 \otimes \cdots \|^2 \\ &= 2 \left(1 - \prod_{i=m+1}^n \left(\frac{\lambda_i + \lambda_i^{-1}}{2} \right)^{-\frac{1}{2}} \right), \end{aligned}$$

which converges to 0 as $n, m \rightarrow \infty$.

For $u \in \overline{G_1}$ define

$$\psi(u) = e^{-\frac{\|u\|^2}{2}} e(u).$$

Clearly $\|\psi(u)\| = 1$. Now we conclude, for any $u \in \oplus_{i=1}^k \Gamma_s(\mathbb{C}e_i)$, that

$$\lim_{n \rightarrow \infty} \Gamma_n(\psi(u)) = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k} \psi(u)) \otimes \bigotimes_{j=k+1}^{\infty} V_{\lambda_j} \Phi_1$$

exists. Define

$$\Gamma(A)(\psi(u)) = \lim_{n \rightarrow \infty} \Gamma_n(\psi(u)).$$

Furthermore, as $\|\Gamma(A)(\psi(u))\| = \|\psi(u)\| = 1$, $\Gamma(A)$ extends to an isometry between $\otimes_{i=1}^{\infty} \Gamma_s(\mathbb{C}e_i)$ and $\otimes_{i=1}^{\infty} \Gamma_s(\mathbb{C}f_i)$ ((i. e.). between $\Gamma_s(\overline{G_1})$ and $\Gamma_s(\overline{G_2})$).

Also by defining, for $u \in \otimes_{i=1}^k \Gamma_s(\mathbb{C}f_i)$,

$$\Gamma'(\psi(u)) = \lim_{n \rightarrow \infty} (V_{\lambda_1}^{-1} \otimes V_{\lambda_2}^{-1} \otimes \cdots \otimes V_{\lambda_k}^{-1} \psi(u)) \otimes \bigotimes_{j=k+1}^n V_{\lambda_j}^{-1} \Phi_1 \otimes \Phi_1 \otimes \Phi_1 \otimes \cdots,$$

and by using same arguments, we may conclude that Γ' extends to an isometry between $\Gamma_s(G_2)$ and $\Gamma_s(G_1)$, and that

$$\Gamma' = \Gamma(A)^* = \Gamma(A^{-1}).$$

Hence $\Gamma(A)$ is a unitary operator.

Clearly, as we may conclude from equations 2.6 and 2.7, the relations 2.1 and 2.2 are satisfied.

Now to prove the uniqueness, suppose there exists another unitary operator Γ' satisfying 2.1 and 2.2, then $\Gamma'\Gamma(A)^{-1}$ commutes with all Weyl operators $W(u)$. As the Weyl representation is irreducible, we conclude that $\Gamma' = c\Gamma(A)$, where c is a complex scalar of unit modulus. But the relation 2.2 implies that $\Gamma' = \Gamma(A)$.

(ii) Note that in the course of proving (i) we have also proved that $\Gamma(A)^{-1} = \Gamma(A^{-1})$.

Now, to prove 2.4, first notice, again by using the irreducibility of the Weyl representation, that $\Gamma(AB) = c\Gamma(A)\Gamma(B)$, for a complex number c of modulus 1. Also it is clear from the construction that when U is a unitary operator, $\Gamma(U) = \text{Exp}(S_U)$. This is clear because $\text{Exp}(S_U)$ satisfies both the relations 2.1 and 2.2 (note that all second quantised operators takes the vacuum vector to the vacuum vector). It is also easy to verify that the relation 2.4 is satisfied when either A or B is a unitary operator (Consider equation 2.2 and that the vacuum vector is fixed by $\text{Exp}(U)$). Hence, by using the above fact and polar decomposition, we may assume, without loss of generality, that $G_1 = G_2 = G_3$ and that $A, B \in \mathcal{S}(G)$.

We basically need to prove that $\langle \Gamma(A)\Gamma(B)\Phi, \Phi \rangle > 0$, where Φ is the vacuum vector in $\Gamma_s(\overline{G})$.

We apeal to Proposition 22.6 in [8] (page 166) for the validity of the relation $\Gamma(AB) = \Gamma(A)\Gamma(B)$, when G is finite dimensional. (The construction given in that proposition and the construction of Γ in Part (i) of this proposition are same as they both satisfy the relations 2.1 and 2.2.)

Let $\{A_n\}$ (resp'tly. $\{B_n\}$) $\subset \mathcal{S}(G)$ be a sequence of operators, such that $I - A_n$ (resp'tly. $I - B_n$) is a finite rank operator for each n , approximating $I - A$ (resp'tly. $I - B$), and that $\Gamma(A_n)$ (resp'tly. $\Gamma(B_n)$) converges strongly to $\Gamma(A)$ (resp'tly. $\Gamma(B)$). It is clear that such a sequence exists from the construction. Note that A_n (and similarly B_n also) is a direct sum of an invertible positive operator on the $\text{Range}(I - A_n)(= \text{Ker}^\perp(I - A_n))$ and the identity operator on $\text{Ker}(I - A_n)$, for each n . Let us define for each n , $G_n = \text{Span}[\text{Range}(I - A_n), \text{Range}(I - B_n)]$, then G_n s are finite dimensional

subspaces of G . The following relations

$$\begin{aligned} I - A_n B_n &= I - A_n - A_n(B_n - I) \\ &= I - B_n - (A_n - I)B_n \end{aligned}$$

imply that $I - A_n B_n$ is a finite rank operator, and that $\text{Range}(I - A_n B_n) \subset G_n$, $\text{Ker}^\perp(I - A_n B_n) \subset G_n$, for each n . Hence we have $A_n = P_n A_n P_n \oplus P_{[n]}$, $B_n = P_n B_n P_n \oplus P_{[n]}$, $A_n B_n = P_n A_n B_n P_n \oplus P_{[n]}$, where P_n and $P_{[n]}$ are the projections onto G_n and G_n^\perp respectively. Also it is clear from the construction of Γ that we also have $\Gamma(A_n) = \Gamma(P_n A_n P_n) \otimes I_{[n]}$, $\Gamma(B_n) = \Gamma(P_n B_n P_n) \otimes I_{[n]}$, $\Gamma(A_n B_n) = \Gamma(P_n A_n B_n P_n) \otimes I_{[n]}$, where $I_{[n]}$ is the identity operator on $\Gamma_s(\overline{G_n^\perp})$. Therefore, as G_n is finite dimensional, we may conclude that $\Gamma(A_n B_n) = \Gamma(A_n)\Gamma(B_n)$, and hence that $\langle \Gamma(A_n)\Gamma(B_n)\Phi, \Phi \rangle > 0$, for each n . The strong convergence of both $\Gamma(A_n)$ and $\Gamma(B_n)$ implies that $\Gamma(A_n)\Gamma(B_n)$ converges weakly to $\Gamma(A)\Gamma(B)$. Now it follows that $\langle \Gamma(A)\Gamma(B)\Phi, \Phi \rangle > 0$, and the proof of part (ii) of the proposition is complete.

(iii) Suppose let $\{T_n\} \subset S(G, G)$ converges strongly to $T \in S(G, G)$ and $(T_n^*)^{-1}$ converges strongly to $(T^*)^{-1}$. First we note that the bounded set $\{\overline{\Gamma(T_n)}\}$ (the closure is taken with respect to the weak topology) is compact with respect to the weak topology (Weak operator topology and weak* topology coincide on bounded sets). Also we know any compact T_2 space is metrizable, and hence the above set is sequentially compact. So we get a convergent subsequence $\Gamma(T_{n_k})$, say converging weakly to $V \in B(\Gamma_s(\overline{G}))$. To prove $\Gamma(T_n)$ converges weakly to $\Gamma(T)$, it is enough if we prove that $V = \Gamma(T)$. (This would mean that every subsequence of $\Gamma(T_n)$ has a further subsequence, which converges weakly to $\Gamma(T)$, which means $\Gamma(T_n)$ converges weakly to $\Gamma(T)$).

First we conclude, from the strong continuity of the Weyl representation that $W(S_{T_n}x)$ converges strongly to $W(S_Tx)$ for all $x \in \overline{G}$. This basically means that $\Gamma(T_n)W(x)\Gamma(T_n)^*$ converges strongly to $\Gamma(T)W(x)\Gamma(T)^*$ for all $x \in \overline{G}$.

We have that

$$\langle \xi, \Gamma(T_{n_k})^* \Gamma(T_{n_k}) W(x) \Gamma(T_{n_k})^* \eta \rangle = \langle \Gamma(T_{n_k}) \xi, \Gamma(T_{n_k}) W(x) \Gamma(T_{n_k})^* \eta \rangle$$

converging to $\langle V, \Gamma(T)W(x)\Gamma(T)^*\eta \rangle$. We also have that $W(x)\Gamma(T_{n_k})^*$ con-

verges weakly to $W(x)V^*$, and so we conclude that

$$W(x)V^* = V^*\Gamma(T)W(x)\Gamma(T) \quad \forall x \in \overline{G},$$

which implies that $V^*\Gamma(T)$ commutes with all operators in $\mathbf{B}(\overline{G})$. We conclude that V is a scalar multiple of $\Gamma(T)$. But by the fact that V is the weaklimit of $\Gamma(T_{n_k})$, it follows that

$$\langle V\Phi, \Phi \rangle > 0.$$

Hence we conclude that $V = \Gamma(T)$, and the proof of the theorem is over. \square

REMARK 11 *The generalised version of Shale's theorem as presented here for two real Hilbert spaces (part (i)) can also be proved using the original Shale's theorem and polar decomposition. That is if $A = U(A^*A)^{\frac{1}{2}}$, then we can define,*

$$\Gamma(A) = \text{Exp}(S_U)\Gamma((A^*A)^{\frac{1}{2}}),$$

where $\Gamma((A^*A)^{\frac{1}{2}})$ is defined by original Shale's theorem. But we required the details of the construction of Γ in proving part (ii) of the Theorem.

REMARK 12 *It is clear from the construction (also a fact we have used in proof of (ii) in Theorem 10), that if $A \in \mathbf{S}(G_1, G_2)$ and $A' \in \mathbf{S}(G'_1, G'_2)$, then $A \oplus A' \in \mathbf{S}(G_1 \oplus G'_1, G_2 \oplus G'_2)$ and*

$$E_2(\Gamma(A) \otimes \Gamma(B))E_1^* = \Gamma(A \oplus B),$$

where E_i is the canonical unitary operator between $\Gamma_s(G_i \oplus G'_i)$ and $\Gamma_s(G_i) \otimes \Gamma_s(G'_i)$, $E_i(e(x \oplus y)) = e(x) \otimes e(y)$, for $i = 1, 2$.

Our aim is to get a product system out of what is called a 'sum system'. First we define a notion of sum system using a one parameter family of real Hilbert spaces. Later as a particular case we will define sum system as a two parameter family of Hilbert spaces, and consider only that definition throughout this paper. This definition is analogous to the definition of a product system, where the tensors are replaced by directsums, and unitaries by our special invertible operators, which are Hilbert-Schmidt perturbation of a unitary operators.

DEFINITION 13 A sum system is a one parameter family of real Hilbert spaces $\{G_t\}_{t \in (0, \infty)}$, together with operators $B_{s,t} \in \mathbf{S}(G_s \oplus G_t, G_{s+t})$ satisfying the following axioms of associativity and measurability.

(i) (Associativity) For any $s_1, s_2, s_3 \in (0, \infty)$

$$B_{s_1, s_2 + s_3}(1_{G_{s_1}} \oplus B_{s_2, s_3}) = B_{s_1 + s_2, s_3}(B_{s_1, s_2} \oplus 1_{G_{s_3}}).$$

(ii) (Measurability) There exists a countable set G^0 of sections $t \in \mathbb{R} \rightarrow x_t \in G_t$ such that $t \mapsto \langle x_t^n, x_t^m \rangle$ is measurable for any two $x^n, x^m \in G^0$, and the set $\{x_t^n : x^n \in G^0\}$ is a total set in G_t for all $t \in (0, \infty)$. Further it is also assumed that the following maps

$$t \in \mathbb{R} \mapsto B_{t, 1-t}(x_t^m \oplus 0) \in G_1, \quad t \mapsto B_{t, 1-t}(0 \oplus x_{1-t}^n)$$

are measurable for any fixed $n, m \in \mathbb{N}$.

Given a Sum system $(G_t, B_{s,t})$, define

$$H_t = \Gamma_s(\overline{G_t}), \quad U_{s,t} = \Gamma(B_{s,t}),$$

where the Hilbert spaces $\Gamma_s(\overline{G_s}) \otimes \Gamma_s(\overline{G_t})$ and $\Gamma_s(\overline{G_s \oplus G_t})$ are identified, using the canonical unitary operator taking $e(x) \otimes e(y)$ to $e(x \oplus y)$. Now we have produced a product system from a given sum system.

THEOREM 14 $(H_t, U_{s,t})$, defined as above, is a product system.

Proof: The associativity property follows from the associativity of the sum system, and from statement (ii) of theorem 10. So we basically have to prove the axiom of measurability.

To prove the measurability axiom, we use the group of unitary operators $\{\tau_t\}$ on H_1 , defined in [17]. Let π_t be the unitary map between $H_{1-t} \otimes H_t$ and $H_t \otimes H_{1-t}$ given by $\pi_t(x_{1-t} \otimes x_t) = x_t \otimes x_{1-t}$. Then define for each $t \in (0, 1)$, a unitary operator on H_1 , by $\tau_t = U_{t, 1-t} \pi_t U_{1-t, t}^*$, and we set $\tau_1 = 1_{H_1}$, and $\tau_{t+k} = \tau_t$ for any $k \in \mathbb{Z}$.

It is proved in [17] that $\{\tau_t\}_{t \in \mathbb{R}}$ forms an one parameter unitary group (see Proposition 2 in [17]). It is also proved in [17] that all measurable structures

on a given algebraic product system leads to isomorphic product systems, and an algebraic product system admits a measurable structure if and only if the unitary group $\{\tau_t\}$ is continuous (theorem 51 in [17]). Therefore we prove that $\{\tau_t\}$ is strongly continuous.

Define a group of operators, T_t on the real Hilbert space G_1 , by $T_t = B_{t,1-t}\sigma_t B_{1-t,t}^{-1}$, where $\sigma_t : G_{1-t} \oplus G_t \mapsto G_t \oplus G_{1-t}$, is the unitary operator, defined by $\sigma_t(x \oplus y) = y \oplus x$, for $t \in (0, 1)$. Also set $T_1 = 1_{G_1}$, and $T_{t+k} = T_t$ for any $k \in \mathbb{Z}$. The fact that T_t is a group can be checked in same way for τ_t . Now it is easy to check that $T_t \in \mathbf{S}(G_1, G_1)$, and using statement (ii) of theorem 10 it is also clear that $\tau_t = \Gamma(T_t)$. As adjoint of a strongly continuous semigroup is again a strongly continuous semigroup (see Theorem 4.3 of [7]), suppose if we prove that the group $\{T_t\}$ is strongly continuous, then the group $\{(T_t^*)^{-1}\}$ is also strongly continuous. Then this would imply the weak continuity, hence the strong continuity, of the unitary group $\{\tau_t\}$, by the statement (iii) in theorem 10. So we prove the strong continuity of $\{T_t\}$. This is equivalent to prove the strong measurability of T_t (see [5], part two, chapter X), and by the definition of T_t it is enough to prove the measurability for $t \in (0, 1)$.

Let us assume that the set of all measurable sections is indexed by \mathbb{N} . Define $y_t^k \in G_1$, for $k \in \mathbb{N}, t \in (0, 1)$ by

$$y_t^{2k-1} = B_{1-t,t}(x_{1-t}^k \oplus 0), \quad y_t^{2k} = B_{1-t,t}(0 \oplus x_t^k).$$

Then the invertibility of $B_{1-t,t}$ implies that the set $\{y_t^k\}_{k \in \mathbb{N}}$ is a linearly independent and total set in G_1 . Let ξ_k^t be the Gram-Schmidt orthogonalisation of y_k^t , i.e.

$$\xi_t^1 = \frac{y_t^k}{\|y_t^k\|}, \quad \xi_t^{k+1} = y_t^{k+1} - \sum_{i=1}^k \langle y_t^{k+1}, \xi_t^i \rangle \xi_t^i, \quad \xi_t^{k+1} = \frac{\xi_t^{k+1}}{\|\xi_t^{k+1}\|}, \quad \text{for } k \in \mathbb{N}.$$

The measurability axiom of the sum system says that map $t \mapsto y_t^k$ is measurable for $k \in \mathbb{N}$. It is an easy verification, using induction, to see that the map $t \mapsto \xi_t^k$ is also measurable.

We need to prove that the map $t \mapsto T_t x_1^n$ is measurable, for any fixed $n \in \mathbb{N}$. Now,

$$T_t(x_1^n) = \sum_k \langle x_1^n, \xi_t^k \rangle T_t \xi_t^k.$$

We basically need to prove that the map $t \mapsto T_t \xi_t^k$ is measurable. Notice that $T_t B_{1-t,t} = B_{t,1-t}$. Using the fact that $t \mapsto \xi_t^k$ is measurable and induction, we may conclude that the map $t \mapsto T_t \xi_t^k$ is measurable.

We have proved the measurability axiom for the product system, and that $(H_t, U_{s,t})$ forms a product system. \square

We call $(H_t, U_{s,t})$ as the exponential of the sum system $(G_t, B_{s,t})$ or as the product system arising out of this sum system. Next we define the notion of isomorphism for sum systems.

DEFINITION 15 *Two sum systems $(G_t, B_{s,t})$ and $(G'_t, B'_{s,t})$ are said to be isomorphic if there exists an operator $A_t \in \mathcal{S}(G_t, G'_t)$ for each $t \in (0, \infty)$, satisfying $A_{s+t} B_{s,t} = B'_{s,t} (A_s \oplus A_t)$.*

Clearly, by statement (ii) in theorem 10, if two sum systems are isomorphic, then the corresponding product systems are also isomorphic, where the isomorphism between the product systems are implemented by $\Gamma(A_t)$, $t \in (0, \infty)$. It is not clear as to whether the converse is true. Next we define a sum system given by a two parameter family of Hilbert spaces, and a semigroup of shift operators. Two parameter systems are more convenient. All our examples will be of this kind.

DEFINITION 16 *A two parameter sum system is a two parameter family of real Hilbert spaces $\{G_{(s,t)}\}$ for $0 < s < t < \infty$ all embedded into a single linear space $G_{(0,\infty)}^0$, satisfying $G_{(s,t)} \subset G_{(s',t')}$ if the interval (s, t) is contained in the interval (s', t') , together with a one parameter semigroup $\{S_t\}$, of linear maps on $G_{(0,\infty)}^0$ for $t \in (0, \infty)$ such that*

$$(i) \ S_s|_{G_{(0,t)}} \in \mathcal{S}(G_{(0,t)}, G_{(s,s+t)}).$$

$$(ii) \ G_{(0,s+t)} = G_{(0,s)} \uplus G_{(s,s+t)} \text{ for all } s, t.$$

(iii) *The semigroup $\{S_t\}$ is 'locally' strongly continuous, (i.e.) for any $x \in G_{(a,b)}$, $a, b \in (0, \infty)$, $S_t x$ converges to x , as $t \rightarrow 0$, where the convergence takes place in a bigger Hilbert space, $G_{(a,b+\epsilon)}$, for some $\epsilon > 0$.*

Notice that the condition (iii) in the above definition actually implies that S_t converges strongly to S_{t_0} , if $t \rightarrow t_0$, due to the semigroup property.

We may assume that $G_{(0,\infty)}^0 = \cup_{t>0} G_{(0,t)}$, and define $G_{(0,\infty)} = \overline{G_{(0,\infty)}^0}$, as the Hilbert space completion. (The problem is we may not be able to extend the semigroup S_t to $G_{(0,\infty)}$.)

Let $(G_t, B_{s,t})$ be a (one parameter) sum system such that $B_{s,t}|_{G_s}$ is an isometry, for $s, t \in (0, \infty)$. Then $(G_t, B_{s,t})$ can be shown to be isomorphic to a sum system given by a two parameter family, in the following way. We can define $G_{(0,\infty)}$ as the inductive limit of the Hilbert spaces G_s . That is define

$$\tilde{G}_{(0,\infty)} = \bigcup_{t>0} G_t.$$

Define an equivalence relation on $\tilde{G}_{(0,\infty)}$ by the following, for $x \in G_s$ and $y \in G_t$ and $t > s$, $x \sim y$ if $B_{s,t-s}x = y$. The associativity axiom implies that (by taking $s_1 = s$, $s_2 = t - s$, $s_3 = t' - t$)

$$(B_{t,t'-t}B_{s,t-s})|_{G_s} = B_{s,t'-s}|_{G_s}.$$

Hence if $B_{s,t-s}x = y$ and $B_{t,t'-t}y = z$, then $B_{s,t'-s}x = z$. So we have an equivalence relation. Define

$$G_{(0,\infty)}^0 = \tilde{G}_{(0,\infty)} / \sim, \quad i_t : G_s \mapsto G_{(0,\infty)}^0, \quad i_t(x) = [x].$$

We can define

$$\lambda[x] = [\lambda x], \quad [x] + [y] = [x + y],$$

where the sum $x + y$ is taken by embedding x and y in a common bigger Hilbert space(which will be again consistent by the associativity axiom). If we define $\|[x]\| = \|x\|$ (which is well defined due to the isometric assumption on $B_{s,t}|_{G_s}$), then i_t is an embedding of G_t into $G_{(0,\infty)}^0$. Define

$$G_{(0,\infty)} = \overline{G_{(0,\infty)}^0}, \quad G_{(0,t)} = i_t(G_t), \quad \text{and clearly } G_{(0,\infty)} = \overline{\bigcup_{t>0} G_{(0,t)}}.$$

For $[x] \in G_{0,t} \subset G_{(0,\infty)}^0$ define

$$S_s([x]) = [B_{s,t}x], \quad \text{for } s, t \in (0, \infty).$$

It can be checked, again by using the associativity axiom, that the map S_s is well defined and that $\{S_t\}$ forms a semigroup also. Finally the the strong continuity of $\{S_t\}$ will follow from the measurability axiom.

REMARK 17 *It is not clear as to whether a general (one parameter) sum system is isomorphic to a sum system such that $B_{s,t}|_{G_s}$ is isometric for all $s \in (0, \infty)$.*

Given a two parameter sum system $(G_{(s,t)}, S_t)$ we get a one parameter sum system by defining,

$$G_t = G_{(0,t)}, \quad B_{s,t}(x_s \oplus y_t) = x_s + S_s y_t.$$

Then clearly the associativity axiom is satisfied by $(G_t, B_{s,t})$, due to the semigroup property of S_t . The measurability axiom may be proved as follows.

Let P_t denote the orthogonal projection from $G_{(0,\infty)}$ onto $G_{(0,t)}$, and let $\{x^n\}_{n \in \mathbb{N}}$ be any orthonormal basis for $G_{(0,\infty)}$. Define $x_t^n = P_t x^n$. Then clearly x_t^n is a countable total set in $G_{(0,t)}$, for each $t \in (0, \infty)$. Also clearly $P_t \uparrow I$ as $t \rightarrow \infty$, and $P_t \uparrow P_{t_0}$ as $t \uparrow t_0$ for any $t_0 \in (0, \infty)$. Hence the map $t \mapsto P_t x_n$ is measurable, and in particular the map

$$t \mapsto \langle x_t^n, x_t^m \rangle = \langle P_t x^n, x^m \rangle$$

is measurable for any $n, m \in \mathbb{N}$. Again clearly

$$t \mapsto B_{t,1-t}(x_t^n \oplus 0) = x_t^n$$

is measurable. So we only have to prove that the map

$$t \mapsto B_{t,1-t}(0 \oplus x_{1-t}^m) = S_t(x_{1-t}^m)$$

is measurable. Denote the above map by $f(t) = S_t(x_{1-t}^m)$. Now define for $k \in \mathbb{N}$, a function $f_k : (0, \infty) \mapsto G_{(0,1)}$ by

$$f_k(t) = S_t(x_{1-\frac{l}{k+1}}) \quad \text{if } t \in \left(\frac{l}{k}, \frac{l+1}{k}\right), \quad l = 0, 1, \dots, k-1.$$

Clearly the function f_k is measurable for each $k \in \mathbb{N}$ due to the strong continuity of S_t , and f_k converges to f pointwise, as $x_{t_n} \rightarrow x_t$ if $t_n \uparrow t$. Now the measurability of the function f proved.

REMARK 18 *In this construction of a one parameter sum system out of a two parameter sum system we have used the map: $B_{s,t}(x_s \oplus y_t) = A_{s,t}(x_s \oplus S_s y_t)$ where $A_{s,t}$ is the map $x \oplus y \mapsto x + y$. Instead of this $A_{s,t}$ we could have used any map $A_{s,t} \in \mathcal{O}(G_{(0,s)} \oplus G_{(s,s+t)}, G_{(0,s+t)})$ and this has no effect on the product system arising out of the sum system because of the following Lemma.*

LEMMA 19 Let $G_1, G_2 \subset G$ and $A \in \mathcal{O}(G_1 \oplus G_2, G)$. Define $H_i = \Gamma_s(\overline{G_i})$, $i = 1, 2$, $H = \Gamma_s(\overline{G})$, and a unitary operator V between $H_1 \otimes H_2$ and H , by

$$V = \Gamma(A) E (\Gamma(A^{-1}|_{G_1}) \otimes \Gamma(A^{-1}|_{G_2})), \quad (2.8)$$

where E is the canonical unitary operator between $\Gamma_s(G_1) \otimes \Gamma_s(G_2)$ and $\Gamma_s(G_1 \oplus G_2)$ and $\Gamma(A)$ provided by Shale's theorem. Then V does not depend on the particular choice of A .

Proof: Let $A_1, A_2 \in \mathcal{O}(G_1 \oplus G_2, G)$, then clearly $A_2^{-1}A_1 \in \mathcal{O}(G_1 \oplus G_2, G_1 \oplus G_2)$, which basically means that $A_2^{-1}A_1$ splits into direct sum of operators, i. e. $A_2^{-1}A_1 = (A_2^{-1}A_1)|_{G_1} \oplus (A_2^{-1}A_1)|_{G_2}$. Hence, we conclude that

$$\Gamma(A_2^{-1}A_1) = E\Gamma((A_2^{-1}A_1)|_{G_1}) \otimes \Gamma((A_2^{-1}A_1)|_{G_2})E^*,$$

where E is the canonical unitary operator between $\Gamma_s(G_1) \otimes \Gamma_s(G_2)$ and $\Gamma_s(G_1) \oplus \Gamma_s(G_2)$. By applying relations 2.3 and 2.4 we may conclude that

$$\Gamma(A_2)^*\Gamma(A_1) = E\Gamma(A_2^{-1}|_{G_1})\Gamma(A_1|_{G_1}) \otimes \Gamma(A_2^{-1}|_{G_2})\Gamma(A_1|_{G_2})E^*,$$

which would prove that V does not depend on the particular choice of A . \square

Here after we normally take only two parameter sum systems and we construct the one parameter sum system, and then the product system from it using the map $A_{s,t}(x_s \oplus y_t) = x_s + y_t$.

To begin with we present two sets of examples for sum systems. First one was given by Arveson producing the type *I* exponential product system, when the sum system comes from usual L^2 on intervals. The other one is the example of Tsirelson, producing type *III* product system where the L^2 spaces are completed with respect to a different inner product coming from carefully chosen positive definite kernels. In the next Section we will see that under some simplifying assumptions only type *I* and type *III* arise as product systems of sum systems. In particular it seems to be impossible to produce type *II* product systems from a sum system.

EXAMPLE 20 Let $G_{(a,b)} = L^2((a, b), K) = \{f : (a, b) \rightarrow K : \int \|f\|^2 < \infty\}$, where K is a separable Hilbert space, and S_t be the usual shift $S_t(f)(s) = f(s - t)$. Then exponential of this sum system is the exponential or Fock product system of Arveson, given in [1]. These are completely classified by the dimension of K .

EXAMPLE 21 In [15], Tsirelson defines a scalar product on $L^2(a, b)$, given by

$$\langle f, g \rangle = \int \int f(s)g(t)B(s-t)dsdt, \quad (2.9)$$

where $B \in L^1(\mathbb{R})$ is continuous and positive definite. Let $G_{(a,b)}$ be the completion of $L^2(a, b)$ with respect to this inner product and let S_t be the usual shift $S_t(f)(s) = f(s-t)$ extended. Then $\{S_t\}$ is a strongly continuous semigroup of isometries. It is also assumed that B satisfies the following property

$$\exists \epsilon > 0 \text{ such that } \forall t \in (0, \epsilon) \quad B(t) = \frac{1}{t \ln^\alpha(\frac{1}{t})} \quad (2.10)$$

and the function B is positive, decreasing and convex. With this assumption it is proved that the map $x \oplus y \rightarrow x + y$ is in $\mathcal{S}(G_{(0,s)} \oplus G_{(s,s+t)}, G_{(0,s+t)})$ (see proposition 9.9 in page 48, [15]). So $(G_{(a,b)}, S_t)$ forms a sum system. We will prove in the next two sections, that the corresponding product systems (for different α in the condition 2.10) are unital and non-isomorphic.

Before ending this section, we prove some facts regarding sum systems. First we prove that single points does not matter in a sum system, in the following sense. Given a sum system, we can naturally associate a real Hilbert space to any given interval. It does not matter whether the end points of the interval are included or not. This basically follows from the our assumption that the shift semigroup is strongly continuous.

We prove two easy lemmas before that. The first one is about the uniform boundedness of the shift semigroup over any finite interval, which is a well known fact for any strongly continuous semigroup on Banach spaces.

LEMMA 22 For any $a, b, s_1, s_2 \in (0, \infty)$,

$$\sup_{t \in (s_1, s_2)} \|S_t|_{G_{(a,b)}}\| < \infty.$$

Proof: We use the uniform boundedness principle.(We may consider the family $\{S_t\}_{t \in (s_1, s_2)}$, as operators between the two Banach spaces, $G_{(a,b)}$ and $G_{(a+s_1, b+s_2)}$.) So we only need to prove that for any $x \in G_{(a,b)}$,

$$\sup_{t \in (s_1, s_2)} \|S_t|_{G_{(a,b)}}x\| < \infty.$$

Suppose there exist a sequence $\{t_n\} \subset (s_1, s_2)$ converging to $t \in (s_1, s_2)$, and $\|S_{t_n}x\| \geq n$, for each $n \in \mathbf{N}$. But then $S_{t_n}x$ can not converge to $S_t x$, which contradicts the strong continuity assumption of $\{S_t\}$. \square

From here onwards we denote the restriction of the shift semigroup as just S_t , unless there is any confusion.

LEMMA 23 *For any $x \in G_{(0,1)}$, $S_t T_t x$ converges to x and $S_1 x$, as t tends to 0 and 1 respectively, where T_t is the semigroup which is already defined by*

$$T_t(x) = x'_{1-t} + S_{1-t}x_t, \quad \text{if } x = x_t + S_t x'_{1-t}, \quad \text{for } x_t \in G_{(0,t)}, \quad x'_{1-t} \in G_{(0,1-t)}.$$

Proof: We have,

$$\|S_t T_t(x) - x\| \leq \|S_t\| \|T_t x - x\| + \|S_t x - x\|.$$

Similarly we also have,

$$\|S_t T_t(x) - S_1 x\| \leq \|S_t\| \|T_t x - x\| + \|S_t x - S_1 x\|.$$

\square

PROPOSITION 24 *For $t \in (0, 1)$, let $x \in G_{(0,1)}$ be such that $x = x_t + S_t x'_{1-t}$ for $x_t \in G_{(0,t)}$, $x'_{1-t} \in G_{(0,1-t)}$. Then x_t and $S_t x'_{1-t}$ converges to 0, as t tends to 0 and 1 respectively.*

Proof: We have

$$x = x_t + S_t(x'_{1-t}) = x_t + S_t T_t(x) - S_1 x_t.$$

Hence by the above lemma $(I - S_1)x_t$ converges to 0 as $t \rightarrow 0$. Similarly we also have

$$S_t T_t(x) = S_t(x'_{1-t} + S_{1-t}x_t) = (S_t x'_{1-t} + S_1 x - S_1(S_t x'_{1-t})).$$

Again the above lemma implies that $(I - S_1)S_t x'_{1-t}$ converges to 0.

The map $(I - S_1) : G_{(0,1)} \mapsto G_{(0,2)}$, is clearly injective, and hence a bijection between $G_{(0,1)}$ and its range. Notice that the proof of the Proposition is over if we prove that the inverse is bounded. To prove that first notice that the map between $G_{(0,1)} \mapsto G_{(0,1)} \oplus G_{(0,1)}$ given by $x \mapsto x \oplus -x$, is a bijection between $G_{(0,1)}$ and its range, with a bounded inverse. The remaining part of the proof follows from the property (ii) in the definition of a sum system. \square

COROLLARY 25 *Let $(G_{(a,b)}, S_t)$ be a sum system, then*

$$G_{t+} = \bigcap_{s>0} G_{(t,t+s)} = \{0\}, \quad G_{t-} = \bigcap_{s>0} G_{(s,t)} = \{0\}.$$

Proof: As each S_t in the shift semigroup is a bijective map, it is enough if we prove that

$$G_{0+} = \{0\} = G_{1-}.$$

Suppose $x \in G_{0+}$, then the decomposition in the above proposition becomes $x_t = x$ and $x'_{1-t} = 0$ for any $t \in (0, 1)$. Hence $x_t = 0$. In an exactly similar way, from the other part of the above proposition, we may conclude that $G_{1-} = \{0\}$. \square

3 Invariants

In this section we get an invariant for any product system constructed out of a sum system. The invariant we get is same as the one got by Tsirelson in [15], but we prove it in our setup. Also the proof turns out to be more direct and simple.

Let $(H_t, U_{s,t})$ be any product system. Associate for any closed interval $[s, t] \subset [0, 1]$, a von Neumann algebra defined by

$$\mathcal{A}_{[s,t]} = U_{s,t,1} (1_{H_s} \otimes \mathbf{B}(H_{t-s}) \otimes 1_{H_{1-t}}) U_{s,t,1}^*,$$

where $U_{s,t,1}$ is the canonical unitary operator between the Hilbert spaces $H_s \otimes H_{t-s} \otimes H_{1-t}$ and H_1 , determined uniquely by the associativity of the product system. We define an elementary set to be a subset of $[0, 1]$, which is disjoint union of finite number of closed intervals. We denote by $\mathcal{F}^e = \mathcal{F}_{[0,1]}^e$ the collection of all elementary sets in $[0, 1]$. For an elementary set $E = \sqcup_{i=1}^n [s_i, t_i]$, define the associated von Neumann algebra to be the von Neumann algebra generated by all the von Neumann algebras associated with the individual intervals, i.e.

$$\mathcal{A}_E = \bigvee_{i=1}^n \mathcal{A}_{[s_i, t_i]}.$$

We define the concept of \liminf for a sequence of von Neumann algebras as follows.

DEFINITION 26 For a sequence of von Neumann algebras \mathcal{A}_n we define $\liminf \mathcal{A}_n$ as the von Neumann algebra generated by limits of all subsequences $\{T_{n_k}\}$, of any sequence $\{T_n\}$ such that $T_n \in \mathcal{A}_n$, where the limit is taken in the weak operator topology.

Clearly the set of all sequences of elementary sets E_n such that $\liminf \mathcal{A}_{E_n} = \mathbb{C}$, is an invariant of the product system under isomorphisms. From this observation we get the invariants for the product system, given in terms of the sum system, by Tsirelson.

When the product system arises from a sum system, we define G_E for $E \in \mathcal{F}^e$, to be the Hilbert space generated by all Hilbert spaces corresponding to the individual intervals. We will talk about G_E and \mathcal{A}_E , only when E is an elementary set, so it does not matter whether the intervals are closed or not, due to Corollary 25 in the previous section.

In order to get the invariants for the product systems, arising from a sum system, we also make some definitions of \liminf and \limsup of subspaces of a Hilbert space. We will be making use of these concepts in the next section also. The definitions are same as in [15].

DEFINITION 27 Let G be a real Hilbert space, and $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of subspaces of G , then $\liminf G_n = \{x \in G : x = \lim x_n, x_n \in G_n\}$. Also we define the $\limsup G_n$ to be the closed subspace generated by weak limits of all subsequences of x_n , such that $x_n \in G_n$, , i. e.

$$\limsup G_n = \overline{\text{span}}\{x : w - \lim x_{k_n} = x, x_{k_n} \in G_{k_n}, \{k_n\} \subset \mathbb{N}\}.$$

LEMMA 28 Let G be a real Hilbert space. For any sequence of subspaces G_n , $\limsup G_n = (\liminf G_n^\perp)^\perp$.

Proof: First we will prove the inclusion $\limsup G_n \subset (\liminf G_n^\perp)^\perp$. That is we need to prove that

$$\liminf G_n^\perp \subset (\limsup G_n)^\perp.$$

Let $y \in \liminf G_n^\perp$, that is there exists a sequence $\{y_n\}$ such that $y_n \in G_n^\perp$ and y_n converges to y . Also let $x \in G$, be the weak limit of of some sequence

x_{k_n} , where $x_{k_n} \in G_{k_n}$. Then it is easy to verify that $\langle x, y \rangle = \lim \langle x_{k_n}, y_{k_n} \rangle = 0$. This proves the required inclusion.

To prove the other inclusion, it is enough if we prove that

$$(\limsup G_n)^\perp \subset \liminf G_n^\perp.$$

Let $y \in (\limsup G_n)^\perp$, and let $y_n = P_n y \in G_n^\perp$, where P_n is the orthogonal projection onto G_n^\perp . It is enough to prove that y_n converges to y , that is $y - y_n \in G_n$ converges to 0. Note that

$$\|y - y_n\|^2 = \langle y - y_n, y - y_n \rangle = \langle y, y - y_n \rangle.$$

Hence it is enough to prove that every subsequence of $y - y_n$ has a further weakly convergent subsequence which converges weakly to 0. As $y - y_n$ is bounded, every subsequence has a weakly convergent subsequence. Now suppose $\{y - y_{k_n}\}$ be a convergent subsequence of $y - y_n$ converging to x , then by definition $x \in \limsup G_n$, and hence by our assumption $\langle y, x \rangle = 0$. Now $\|y - y_{k_n}\|^2 = \langle y, y - y_{k_n} \rangle$ converges to $\langle y, x \rangle = 0$. The proof of the lemma is over. \square

In our setup (i.e. when the product system is constructed from a sum system), for a set $E \in \mathcal{F}^e$, and $E = \bigsqcup_{i=1}^n [s_i, t_i]$, we have

$$\mathcal{A}_E = \Gamma(A_E) (\otimes_{i=1}^n \mathbf{B}(\Gamma_s(\overline{G(s_i, t_i)}))) \otimes_{i=0}^n \mathbf{1}_{\Gamma_s(\overline{G(t_i, s_{i+1})})} \Gamma(A_E)^{-1},$$

where we assume $t_0 = 0$ and $s_{n+1} = 1$, and $A_E \in \mathbf{S}(\oplus_{i=1}^n G(s_i, t_i) \oplus_{i=0}^n G(t_i, s_{i+1}), G(0, 1))$ is the map taking $\oplus x_{s_i, t_i} \oplus x_{t_i, s_{i+1}}$ to $\sum x_{s_i, t_i} + \sum x_{t_i, s_{i+1}}$.

Noting that the von Neumann algebra $\otimes_{i=1}^n \mathbf{B}(\Gamma_s(\overline{G(s_i, t_i)}))$ is generated by the set of Weyl operators $\{W(x + iy) : x, y \in \oplus_{i=1}^n G(s_i, t_i)\}$, it is easily seen that \mathcal{A}_E is generated by the set of Weyl operators

$$\{W(A_E x + i(A_E^*)^{-1} y) : x, y \in \oplus_{i=1}^n G(s_i, t_i)\}.$$

Then, as A_E (resp. $(A_E^*)^{-1}$) is a bijection between $\oplus_{i=1}^n G(s_i, t_i)$ and $G_E \subset G(0, 1)$ (resp. $G_{E^c}^\perp \subset G(0, 1)$), we conclude that

$$\mathcal{A}_E = \text{VNalg}\{W(x + iy) : x \in G_E, y \in G_{E^c}^\perp\},$$

for $E \in \mathcal{F}^e$.

Also, using the fact that $W(x)$ (resp. $W(ix)$) commutes with $W(iy)$ (resp. with $W(y)$) when x and y are orthogonal vectors, and by looking at the generators, it is easy to check that, for $E \in \mathcal{F}^c$, we have

$$\mathcal{A}'_E = \mathcal{A}_{E^c} = \text{VNalg}\{W(x+iy) : x \in G_{E^c}, y \in G_E^\perp\} \quad (*)$$

We prove a lemma which will be used in the main theorem of this section.

LEMMA 29 *Let $\{F_n\}$ be any sequence of elementary sets, then*

$$(i) \text{VNalg}\{W(x+iy) : x \in \limsup G_{F_n}, y \in \limsup G_{F_n}^\perp\} \subset \liminf \mathcal{A}_{F_n}.$$

$$(ii) \text{VNalg}\{W(x+iy) : x \in \liminf G_{F_n^c}, y \in \liminf G_{F_n}^\perp\} \subset (\liminf \mathcal{A}_{F_n})'.$$

Proof: (i) Let $x \in \limsup G_{F_n}$, that is $x = w - \lim x_{n_k}$, for some subsequence x_{n_k} such that $x_{n_k} \in G_{F_{n_k}}$. Then $e^{\|x_{n_k}\|^2} W(x_{n_k}) \in \mathcal{A}_{F_{n_k}}$, and it is an easy verification to check that $\langle e^{\|x_{n_k}\|^2} W(x_{n_k})e(y), e(z) \rangle$ converges to $\langle e^{\|x\|^2} W(x)e(y), e(z) \rangle$, for all $y, z \in G_{(0,1)}$. Hence we conclude that $W(x) \in \liminf \mathcal{A}_{F_n}$. Using the same argument we may conclude that $W(iy) \in \liminf \mathcal{A}_{F_n}$, for $y \in \limsup G_{F_n}^\perp$.

(ii) Let $x \in \liminf G_{F_n^c}$, that is $x = \lim x_n$, where $x_n \in G_{F_n^c}$. Also let $a \in \liminf \mathcal{A}_{F_n}$, that is there exists a sequence $a_{n_k} \in \mathcal{A}_{F_{n_k}}$, such that a_{n_k} converges in the weak operator topology to a . We want to prove that $W(x)$ commutes with a . We have that $W(x_{n_k})$ (and its adjoint $(W(-x_{n_k}))$) converges strongly to $W(x)$ (respectively to its adjoint $W(-x)$), and that a_{n_k} (and its adjoint $a_{n_k}^*$) converges weakly to a (respectively to a^*). Using the observation (*) above, we note that that $W(x_{n_k})$ and a_{n_k} commutes with each other. For any $\xi, \eta \in H_1$,

$$\langle aW(x)\xi, \eta \rangle = \lim_k \langle W(x)\xi, a_{n_k}^* \eta \rangle,$$

and

$$\langle W(x)\xi, a_{n_k}^* \eta \rangle \leq \langle W(x_{n_k})\xi, a_{n_k}^* \eta \rangle + \|W(x)\xi - W(x_{n_k})\xi\| \|a_{n_k}^* \eta\|.$$

As $\|a_{n_k}^* \eta\|$ is bounded, we get

$$\langle aW(x)\xi, \eta \rangle = \lim_k \langle W(x_{n_k})\xi, a_{n_k}^* \eta \rangle = \langle a_{n_k} \xi, W(-x_{n_k})\eta \rangle.$$

Now using the same convergences of sequences and retracing the same arguments we may conclude that

$$\langle aW(x)\xi, \eta \rangle = \langle W(x)a\xi, \eta \rangle.$$

Hence $W(x) \in (\liminf \mathcal{A}_{F_n})'$. A similar calculation will imply that $W(iy) \in (\liminf \mathcal{A}_{F_n})'$, for any $y \in \liminf G_{F_n}^\perp$. The lemma is proved \square

The following theorem allows us to compare the invariants through the sum system.

THEOREM 30 *Let F_n be any given sequence of elementary sets, then the following two statements are equivalent.*

- (i) $\liminf \mathcal{A}_{F_n} = \mathbb{C}$.
- (ii) $\liminf G_{F_n^c} = G_{(0,1)}$, $\limsup G_{F_n} = \{0\}$.

Proof: We first prove (i) implies (ii). We conclude using lemma 28 and part (i) of lemma 29, that

$$VNalg\{W(x + iy) : x \in \limsup G_{F_n}, y \in (\liminf G_{F_n^c})^\perp\} \subset \liminf \mathcal{A}_{F_n},$$

and clearly (i) implies (ii).

Now we prove the other implication, (ii) implies (i). Again using lemma 28 and part (ii) of lemma 29 we have that

$$VNalg\{W(x + iy) : x \in \liminf G_{F_n^c}, y \in \liminf G_{F_n}^\perp\} \subset (\liminf \mathcal{A}_{F_n})'.$$

If we assume (ii) holds, then LHS in the above inclusion is $B(H_1)$ and the (*) implies that (i) is true. The proof of the theorem is over \square

REMARK 31 *The above theorem asserts that the collection of all sequence of elementary sets $\{E_n\}$ such that $\liminf G_{E_n} = G_{(0,1)}$, and $\limsup G_{E_n^c} = \{0\}$ is an invariant of the product systems. Tsirelson has produced sequence of elementary sets satisfying $\liminf G_{E_n} = G_{(0,1)}$, and $\limsup G_{E_n^c} = \{0\}$, for each α but it violates the condition $\limsup G_{E_n^c} = \{0\}$ for $\alpha' \neq \alpha$. This proves that the examples of Tsirelson are non-isomorphic for different values of α .*

4 Units in the product system

In this section we get a sufficient condition for the product system, arising from what is called as a divisible sum system, to be unitless. We prove a necessary condition for a unit to exist, and the sufficient condition for the product system to be unitless is to violate that. We first define the notion of divisibility for sum systems and prove that this property is satisfied by the examples of Tsirelson. All through this section, we assume that the restriction of the shift map $S_t|_{G_{(a,b)}}$ of the sum system, is a unitary map for all $t, a, b \in (0, \infty)$. (This would imply that the semigroup $\{S_t\}$ can be extended as a semigroup of isometries on $G_{(0,\infty)}$.) We denote by $A_{s,t}$ the map between $G_{(0,s)} \oplus G_{(s,s+t)} \rightarrow G_{(0,s+t)}$ defined by $x \oplus y \mapsto x + y$.

DEFINITION 32 *We call a family $\{x_t\}_{t \in (0,\infty)}$ such that $x_t \in G_{(0,t)}$, $\forall t \in (0, \infty)$, as a real additive unit for the sum system $(G_{(a,b)}, S_t)$, if*

(i) *The map $t \mapsto \langle x_t, x \rangle$ is a measurable map for any $x \in G_{(0,\infty)}$.*

(ii) *$x_s + S_s x_t = x_{s+t}$, $\forall s, t, \in (0, \infty)$, (i. e.) $A_{s,t}(x_s \oplus S_s x_t) = x_{s+t}$.*

Similarly we call a family $\{y_t\}_{t \in (0,\infty)}$ such that $y_t \in G_{(0,t)}$, $\forall t \in (0, \infty)$, as an imaginary additive unit, for the sum system $(G_{(a,b)}, S_t)$, if

(i) *The map $t \mapsto \langle y_t, y \rangle$ is a measurable map for any $y \in G_{(0,\infty)}$.*

(ii) *$\{y_t\}$ satisfies $(A_{s,t}^*)^{-1}(y_s \oplus S_s y_t) = y_{s+t}$, $\forall s, t, \in (0, \infty)$.*

We denote by *RAU* and *IAU*, the set of all real and imaginary additive units respectively. For any given real(resptly. imaginary) additive unit $\{x_t\}$ (resptly. $\{y_t\}$), we denote $x_{s,t} = S_s(x_{t-s}) \in G_{(s,t)}$ (resptly. $y_{s,t} = S_s(y_{t-s}) \in G_{(s,t)}$).

We also define for an imaginary additive unit $\{y_t\}$,

$$y'_{s,s_1,s_2} = (A^*)^{-1}(0 \oplus y_{s_1,s_2} \oplus 0), \text{ for any } (s_1, s_2) \subset (0, s),$$

where $A : G_{(0,s_1)} \oplus G_{(s_1,s_2)} \oplus G_{(s_2,s)} \mapsto G_{(0,s)}$, given by $x \oplus y \oplus z \mapsto x + y + z$. It is easy to check that $y'_{s,s_1,s_2} \in G_{(0,s_1) \cup (s_2,s)}^\perp$. To simplify notation we denote y'_{1,s_1,s_2} by y'_{s_1,s_2} , and $y'_{1,0,t}$ by just y'_t . Finally note that

$$x_s + x_{s,s+t} = x_{s+t}, \quad y'_s + y'_{s,s+t} = y'_{s+t}.$$

DEFINITION 33 A sum system $(G_{(a,b)}, S_t)$ is called as a divisible sum system if the additive units exists and generate the sum system, (i. e.)

$$G_{(0,s)} = \overline{\text{span}[x_{s_1, s_2} : s_1, s_2 \in (0, s), \{x_t\} \in \text{RAU}]}$$

and

$$G_{(0,s)} = \overline{\text{span}[y'_{s, s_1, s_2} : s_1, s_2 \in (0, s), \{y_t\} \in \text{IAU}]}$$

PROPOSITION 34 (i) The collection of all real (and also imaginary) additive units forms a real vector space, with usual addition and scalar multiplication,

$$\{x_t^1\} + \{x_t^2\} = \{x_t^1 + x_t^2\}; \quad \lambda\{x_t\} = \{\lambda x_t\}.$$

(ii) If $\{x_t\} \in \text{RAU}$ and $\{y_t\} \in \text{IAU}$, then

$$\langle x_t, y_t \rangle = \langle x_1, y_1 \rangle t \quad \forall t \in (0, \infty).$$

In general for any two intervals $(s_1, s_2), (t_1, t_2) \subset (0, \infty)$, it is true that

$$\langle x_{s_1, s_2}, y'_{t_1, t_2} \rangle = \langle x_1, y_1 \rangle \ell((s_1, s_2) \cap (t_1, t_2)), \quad (4.11)$$

where ℓ is the Lebesgue measure on \mathbb{R} .

(ii) If a single real additive unit (and also an imaginary additive unit) generates the sum system then the additive units are determined uniquely up to a scalar.

Proof: (i) Clear

(ii) Given any $\{x_t\} \in \text{RAU}$ and $\{y_t\} \in \text{IAU}$, consider the function $h_{x,y}(t) = \langle x_t, y_t \rangle$. First we notice that $h_{x,y}$ is a real valued measurable function. It may be proven as follows. We know that the map $t \mapsto \langle x_t, x \rangle$ (also $t \mapsto \langle y_t, x \rangle$) is measurable for any $x \in G_{(0,\infty)}$. Then $\|x\| = \sup_n \langle x_t, x_n \rangle$, for some countable set $\{x_n\}$, due to the separability of the Hilbert space. Hence we conclude that the function $t \mapsto \|x_t\|$ is measurable. Similarly we conclude that the function $t \mapsto \|y_t\|$ is also measurable. Now using the relation

$$\langle x_t, y_t \rangle = \frac{1}{4}(\|x_t + y_t\|^2 - \|x_t - y_t\|^2),$$

we can conclude that the function $h_{x,y}(t)$ is measurable.

We also notice that

$$h_{x,y}(s+t) = \langle A_{s,t}(x_s \oplus S_s(x_t)), (A_{s,t}^*)^{-1}(y_s \oplus S_s(y_t)) \rangle = h_{x,y}(s) + h_{x,y}(t).$$

Therefore we conclude that $h_{x,y}(t) = h_{x,y}(1)t$. Now it is an easy verification to see that for any two intervals $(s_1, s_2), (t_1, t_2) \subset (0, t)$ we have that

$$\langle x_{s_1, s_2}, y'_{t_1, t_2} \rangle = h_{x,y}(1) \ell((s_1, s_2) \cap (t_1, t_2)), \quad (4.12)$$

where ℓ is the Lebesgue measure on \mathbb{R} .

(iii) Clear from (ii). □

REMARK 35 (i) *If the product system is exponential, that is the sum system $(G_{(a,b)}, S_t)$ is $(L^2((a,b), K), S_t)$, with the standard shift S_t , then $x_t = y_t = \xi 1_{(0,t)}$, for any $\xi \in K$, exhausts all the real and imaginary additive units, and the sum system is divisible.*

(ii) *The dimension of the vector space of additive real (resp. imaginary) units may be defined as an index of the sum system, and it is clearly an invariant for the sum system. In the case when the sum system gives rise to a type I product system it is a complete invariant. But in general it is not, as all examples of Tsirelson are of index 1 and they are mutually non-isomorphic.*

We prove in the next proposition that all examples of Tsirelson are divisible.

PROPOSITION 36 *Let $(G_{(a,b)}, S_t)$ be a sum system, and suppose that $G_{(a,b)}$ is the completion of $L^2(a,b)$ with respect to some inner product, such that S_t the canonical shift becomes an isometry. Then*

(i) $x_t = 1_{0,t}$ is a real additive unit.

(ii) *The non-zero imaginary additive unit exists (which is unique up to a scalar, if it exists) if and only if the linear functional $f \mapsto \int f dt$ is continuous on the dense subspace $L^2(a,b) \subset G_{(a,b)}$.*

Proof: As we have assumed the map $A_{s,t} : G_{(0,s)} \oplus G_{(s,t)} \rightarrow G_{(0,s+t)}$ to be $x \oplus y \mapsto x + y$, it is clear that $x_t = 1_{0,t}$ is a real additive unit and also it generates the sum system.

To prove (ii), suppose a non-zero imaginary additive unit $\{y_t\}$ exists for the sum system, then the relation 4.12 (as $h(1) \neq 0$, and by choosing a real multiple of y_t if needed) can be written, using (i), as

$$\langle f, y_t \rangle = \int_0^t f dt,$$

for any simple function $f \in L^2(0, t) \subset G_{(0,t)}$. Now it follows that then the linear functional $f \mapsto \int_0^t f dt$ is continuous on the dense subspace $L^2((a, b)) \subset G_{(a,b)}$. Suppose if we assume that the linear functional $f \mapsto \int_0^t f dt$ is continuous on the dense subspace $L^2((a, b)) \subset G_{(a,b)}$, then we can choose $y_t \in G_{(0,t)}$ satisfying relation 4.12 with $h(1) = 1$. Now it is an easy verification to check that, we have for $s, t \in (0, \infty)$ and $s_1, s_2 \in (0, s + t)$ that

$$\langle x_{s_1, s_2}, (A^*)^{-1}(y_s \oplus S_s y_t) \rangle = \langle x_{s_1, s_2}, y_{s+t} \rangle.$$

As the set $\{x_{s,t} : s, t \in (0, s + t)\}$ is total in $G_{(0,s+t)}$, we conclude that $\{y_t\}$ is an imaginary additive unit. \square

COROLLARY 37 *All examples of Tsirelson (Example 21) are divisible.*

Proof: To prove that sum systems of Example 21) are divisible, we basically need to prove the existence of the imaginary additive unit, (i.e.) it is enough to prove that $f \mapsto \int_0^t f dt$ is continuous with respect to the scalar product (2.9) for $f \in L^2(0, t)$. That is we want a $g \in L^2(0, t)$ such that $g \star B = 1_{(0,t)}$, so that $\int_0^t f g \star B = \int_0^t f dt$. By taking Fourier transform we basically need a $\hat{g} \in \ell^2(\mathbb{Z})$, such that $\hat{g} \hat{B} = \hat{1}_{(0,t)}$, that is we need to verify $\frac{e^{int} - 1}{inB} \in \ell^2(\mathbb{Z})$. But we have that \hat{B} never vanishes and $\hat{B}(n) \sim \frac{C}{\ln^{\alpha-1}|n|}$ for $n \rightarrow \pm\infty$ (see [15], lemma 9.5, page 41), and the series $\sum_{n \in \mathbb{Z}} \frac{\ln^{2\alpha-2}|n|}{n^2}$ is convergent. \square

Now we prove that the product system arising from a divisible sum system is always symmetric.

PROPOSITION 38 *Suppose $(H_t, U_{s,t})$ be a product system constructed out of a divisible sum system $(G_{(a,b)}, S_t)$, then $(H_t, U_{s,t})$ is a symmetric product system.*

Proof: It is enough if we prove that the sum system is anti-isomorphic to itself. Let $\{\{x_t^i\} : i \in I\}$ be a spanning collection of real additive units for the sum system. Define $T_t : G_{(0,t)} \mapsto G_{(0,t)}$, by $T_t(x_{s_1, s_2}^i) = x_{t-s_2, t-s_1}^i$, for $(s_1, s_2) \subset (0, t)$, $i \in I$. Clearly

$$\|T_t(x_{s_1, s_2}^i)\| = \|x_{t-s_2, t-s_1}^i\| = \|S_{t-s_2}(x_{s_2-s_1}^i)\| = \|x_{s_1, s_2}^i\|,$$

as we have assumed that the shift map to be isometric. So T_t is an isometry on a total set, and it is also bijective on this total set. Hence the map T_t extends to a unitary operator on G_t . It is easy to check that this map provides the required anti-isomorphism. \square

Next we prove a theorem which asserts only type *I* and type *III* product systems can be constructed from a divisible sum system.

THEOREM 39 *Let $(H_t, U_{s,t})$ be a product system constructed out of a divisible sum system $(G_{(a,b)}, S_t)$. If $(H_t, U_{s,t})$ has a unit then it is a type *I* product system.*

Proof: We assume that a unit $u(t) \in H_t = \Gamma_s(\overline{G_{(0,t)}})$ exists for the product system, and prove that the product system is divisible.

Let $z_t \in \overline{G_{(0,t)}}$ be such that $z_t = c_1 x_t + i c_2 y_t$, where $\{x_t\} \in \text{RAU}$, $\{y_t\} \in \text{IAU}$ and c_1, c_2 are real scalars. Then clearly it holds that $S_{A_s, t}(z_s \oplus S_s z_t) = z_{s+t}$. So we have

$$U_{s,t}(W(z_s) \otimes W(z_t))U_{s,t}^* = \Gamma(A_{s,t})W(z_s \oplus S_s z_t)\Gamma(A_{s,t})^* = W(z_{s+t}).$$

This basically shows that the family of unitaries $W(z_t) \in \mathbf{B}(H_t)$, is an automorphism for the product system. As any automorphism of a product system preserves units, we conclude that the family of vectors $W(z_t)u_t \in H_t$ is also a unit for the product system (H_t) .

Fix a $t \in (0, \infty)$. The definition of divisibility asserts that the set of all vectors of the form $\sum_{j=1}^n c_j x_{s_{j-1}, s_j}^j + i c'_j y_{t, s_{j-1}, s_j}^j$, where c_j, c'_j varying over real

numbers, $s_0 = 0 < s_1 < s_2 \cdots < s_n = t$, and $\{x_t^j\}$ and $\{y_t^j\}$ varying over all real and imaginary units respectively, is dense in $\overline{G_{(0,t)}}$.

If we denote the unit $W(cx_t + ic'y_t)u_t$ by $\{v_{c,c'}(t)\}$, then the image of $\otimes_{j=1}^n v_{c_j,c'_j}(s_i - s_{i-1})$, under the canonical unitary of the product system is $W(\sum_{j=1}^n c_j x_{s_{j-1},s_j} + ic'_j y'_{t,s_{j-1},s_j})u_t$. So we conclude that the units generate the subspace

$$\overline{\text{span}}[W(x)u_t : x \in \overline{G_{(0,t)}}].$$

But this subspace is whole of $\Gamma_s(\overline{G_{(0,t)}})$, as the Weyl representation is irreducible. Hence the product system is divisible, i.e. of type *I*. \square

For any elementary set $E = \sqcup_{i=1}^n (s_i, s_{i+1}) \subset (0, 1)$, we define

$$x_E = \sum_{i=1}^n x_{s_i, s_{i+1}}, \quad y'_E = \sum_{i=1}^n y'_{s_i, s_{i+1}} \in G_{(0,1)} \quad \text{for } \{x_t\} \in \text{RAU}, \{y_t\} \in \text{IAU}.$$

The following theorem provides a necessary condition for the product system arising from divisible sum system to be of type *I*. By the next theorem the sufficient condition for the product system to be of type *III* is to violate this condition.

THEOREM 40 *Let $(G_{(a,b)}, S_t)$ be a divisible sum system, giving rise to a type *I* product system. Then for any sequence of elementary sets E_n satisfying $\liminf G_{E_n} = G_{(0,1)}$, it also holds that $\limsup G_{E_n^c} = \{0\}$*

Proof: Let $(H_t, U_{s,t})$ be the product system given by the sum system $(G_{(a,b)}, S_t)$. As it is of type *I* (see [1]), it is isomorphic to an exponential product system $(H'_t, U'_{s,t})$, given by the sum system $(G'_{(a,b)}, S'_t)$, where $G_{(a,b)} = L^2((a,b), K)$ for some separable Hilbert space K , and S'_t is the canonical shift. We denote by $(V_t)_{t \in (0, \infty)}$ a family of unitary maps implementing the isomorphism between the product systems H_t and H'_t .

First let us note that the condition, $\liminf G'_{E_n} = G_{(0,1)}$ forcing $\limsup G'_{E_n^c} = \{0\}$, is satisfied by the sum system $(G'_{(a,b)}, S'_t)$. This follows first by noticing that $G'_{E^c} = G'_E{}^\perp$ for any elementary set E , and then by using lemma 28.

Now we claim that the set $\{y'_B : B \in \mathcal{F}^e\} \subset G_{(0,1)}$, is bounded for any imaginary additive unit y . Suppose not, for each positive integer n , choose

an elementary set $B_n \subset (0, \frac{1}{n})$ such that $\|y'_{B_n}\| > n$. If this is not possible for some n , that is for each elementary set $B \in (0, \frac{1}{n})$, $\|y'_B\| \leq n$, then by shifting the y'_B s by the unitary operator $S_{\frac{k}{n}}$, $k = 1, 2, \dots, n$, and by using the triangle inequality we may conclude that $\|y'_E\| \leq n^2$, for any elementary set in $E \subset (0, 1)$. But this means that the set $\{y'_B : B \in \mathcal{F}_{[0,1]}^e\}$ is bounded. So we can indeed choose such $B_n \subset (0, \frac{1}{n})$ such that $\|y'_{B_n}\| > n$.

Now we know that $W(y_t)$ is an automorphism for the product system (H_t) , hence $V_t W(y_t) V_t^*$ is an automorphism of the product system (H'_t) . By the result in section 8 of [1] we can conclude that

$$V_t W(y_t) V_t^* = e^{i\lambda t} W(\xi 1_{(0,t)}) \text{Exp}(U^t), \text{ where } \lambda \in \mathbb{R}, \xi \in K, U \in \mathbf{U}(K),$$

and $U^t(\eta 1_A) = (U\eta)1_A$ for any $\eta \in K$ and $A \subset (0, t)$. It is easy to verify that

$$V_1 W(y'_{B_n}) V_1^* = e^{i\ell(E_n)} W(\xi 1_{B_n}) \text{Exp}(U^{E_n}),$$

where U^{E_n} is the unitary operator defined by $U^{E_n}(\eta 1_A) = U\eta 1_A$ if $A \subset E_n$, and $U^{E_n}(\eta 1_{A'}) = \eta 1_{A'}$ if $A' \subset E_n^c$, $\eta \in K$. Clearly the above sequence converges strongly to the identity operator. But the sequence $W(y'_{B_n})$ can not be a strongly convergent sequence, for the following reason. Suppose $W(y'_{B_n})$ converges strongly, then by applying on the vacuum vector, we first conclude that the sequence $e^{-\frac{\|y'_{B_n}\|^2}{2}} e(y'_{B_n})$ converges. But the projection of this sequence on to the k -th particle subspace converges to 0, for each $k \in \mathbf{N}$, as $e^{-\frac{\|y'_{B_n}\|^2}{2}} \left(\frac{\|y'_{B_n}\|^k}{\sqrt{n!}}\right)$ converges to 0, for each $k \in \mathbf{N}$. Hence the sequence $e^{-\frac{\|y'_{B_n}\|^2}{2}} e(y'_{B_n})$ should converge to 0, but this is not possible as $e^{-\frac{\|y'_{B_n}\|^2}{2}} \|e(y'_{B_n})\| = 1$ for all n . Hence we have proved our claim.

Now suppose E_n be any sequence of elementary sets satisfying $\liminf G_{E_n} = G_{(0,1)}$, we claim that $\ell(E_n^c)$ converges to 0, as n tends to ∞ . Suppose $\ell(E_n^c)$ does not converge to 0, let $y'_n = y'_{E_n^c} \in G_{E_n}^\perp$. Then $\|y'_n\|$ is a bounded sequence which does not converge weakly to 0, as

$$\langle y'_n, x_1 \rangle = \ell(E_n^c).$$

So we conclude that $\limsup G_{E_n}^\perp$ is not equal to $\{0\}$. But, by lemma 28, this contradicts our assumption that $\liminf G_{E_n} = G_{(0,1)}$. Hence we have proved our claim that $\ell(E_n^c) \rightarrow 0$.

As any $f \in L^2((0, 1), K)$ is the limit of $1_{E_n}f$, we have that

$$\liminf L^2(E_n, K) = L^2((0, 1), K).$$

But this also implies that $\limsup L^2(E_n^c, K) = \{0\}$. Now the theorem 30 implies that $\limsup G_{E_n^c} = \{0\}$ and the proof of the proposition is over. \square

REMARK 41 *Tsirelson in his examples produces a sequence of elementary sets $\{E_n\}$ such that $\liminf G_{E_n} = G_{(0,1)}$, but the condition that $\limsup G_{E_n^c} = \{0\}$ is violated. This once again proves that the examples of Tsirelson are of type III.*

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