Identities Involving Reciprocals of Binomial Coefficients

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Abstract

In this paper, we deal with several combinatorial sums and some infinite series which involve the reciprocals of binomial coefficients. Many binomial identities as well as some polynomial identities are proved.

1 Introduction

As usual, the binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & n \geq m; \\ 0, & n < m. \end{cases}$$

where n and m are nonnegative integers.

In many subjects, such as combinatorial analysis, graph theory, and number theory, binomial coefficients often appear naturally and play an important role. However, it is well known that it is difficult to compute the values of combinatorial sums involving inverses of binomial coefficients. For some investigations in this respect, see [1–8]. It is the purpose of this paper to deal with several finite combinatorial sums and some infinite series involving the reciprocals of binomial coefficients. In [6], the first author used the identity

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

to observe that

$$\frac{1}{\binom{n}{r}}=\frac{r!(n-r)!}{n!}=\frac{\Gamma(r+1)\Gamma(n-r+1)}{\Gamma(n+1)}=(n+1)\int_0^1 t^r(1-t)^{n-r}dt.$$

Starting with this observation, it was proved in [6] that the following two identities hold:

$$\sum_{r=0}^{n}\frac{1}{\binom{n}{r}}=\frac{n+1}{2^n}\sum_{r=0}^{n}\frac{2^r}{r+1}=\frac{n+1}{2^n}\sum_{j\text{ odd}}\binom{n+1}{j}\frac{1}{j}.$$

In this paper, we exploit this method much further and prove, among other things, a polynomial identity and several interesting combinatorial identities where the binomial coefficients occur in the denominator.

2 A polynomial identity and applications

Theorem 2.1. *In the ring $\mathbb{Q}[T]$ of rational polynomials, the identity*

$$\begin{aligned}\sum_{r=m}^{n}\frac{T^r(1-T)^{n-r}}{\binom{n}{r}} &= (n+1)\sum_{r=m}^{n}\frac{T^{n+1}(1-T)^{n-r}}{r+1}\\ &\quad+(n+1)\sum_{r=0}^{n-m}\frac{T^{n-r}(1-T)^{n-m+1}}{(m+r+1)\binom{m+r}{r}}\end{aligned}\qquad(1)$$

holds for $m\le n$. An equivalent form is that for $\lambda\neq-1$,

$$\begin{aligned}\sum_{r=m}^{n}\frac{\lambda^r}{\binom{n}{r}} &= (n+1)\sum_{r=0}^{n-m}\frac{\lambda^{m+r}}{(\lambda+1)^{r+1}}\sum_{i=0}^{n-m-r}\binom{n-m-r}{i}\frac{(-1)^i}{m+1+i}\\ &\quad+(n+1)\frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\sum_{r=m}^{n}\frac{(\lambda+1)^{r+1}}{r+1}.\end{aligned}\qquad(2)$$

Proof. Let us consider, for a fixed real number λ, $I_{m,n}(\lambda) = \sum_{r=m}^{n} \frac{\lambda^r}{\binom{n}{r}}$. Then,

$$\begin{aligned} I_{m,n}(\lambda) &= \frac{1}{n!}\sum_{r=m}^{n} \lambda^r \Gamma(r+1)\Gamma(n-r+1) \\ &= (n+1)\sum_{r=m}^{n} \lambda^r \beta(r+1, n-r+1) \\ &= (n+1)\sum_{r=m}^{n} \lambda^r \int_0^1 t^r(1-t)^{n-r}dt \\ &= (n+1)\int_0^1 \sum_{r=m}^{n} (t\lambda)^r(1-t)^{n-r}dt \\ &= (n+1)\int_0^1 \frac{(t\lambda)^{n+1} - (t\lambda)^m(1-t)^{n-m+1}}{t\lambda - (1-t)}dt. \end{aligned}$$

Putting $s = t\lambda - (1-t)$, one gets $I_{m,n}(\lambda) = (n+1)(I_1 + I_2)$ where

$$I_1 = \frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\int_{-1}^{\lambda} \frac{(s+1)^{n+1} - (s+1)^m}{s}ds$$

and

$$I_2 = \frac{\lambda^m}{(\lambda+1)^{n+2}}\int_{-1}^{\lambda} \frac{(s+1)^m}{s}(\lambda^{n-m+1} - (\lambda - s)^{n-m+1})ds.$$

Now, by writing

$$\frac{(s+1)^{n+1} - (s+1)^m}{s} = \sum_{r=m}^{n}(s+1)^r$$

and interchanging the order of the summation and the integration, one obtains

$$I_1 = \frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\sum_{r=m}^{n} \frac{(\lambda+1)^{r+1}}{r+1}.$$

Similarly,

$$I_2 = \sum_{r=0}^{n-m} \frac{\lambda^{m+r}}{(\lambda+1)^{r+1}} \sum_{i=0}^{n-m-r} (-1)^i \binom{n-m-r}{i}\frac{1}{m+1+i}.$$

Evidently, the above manipulations are valid when λ is any real number different from -1. Therefore, we have, for $\lambda \neq -1$,

$$\begin{aligned} \sum_{r=m}^{n} \frac{\lambda^r}{\binom{n}{r}} &= (n+1)\sum_{r=0}^{n-m} \frac{\lambda^{m+r}}{(\lambda+1)^{r+1}} \sum_{i=0}^{n-m-r} \binom{n-m-r}{i}\frac{(-1)^i}{m+1+i} \\ &\quad +(n+1)\frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\sum_{r=m}^{n} \frac{(\lambda+1)^{r+1}}{r+1}. \end{aligned}$$

This proves (2). The particular case $m = 0$ gives

$$\sum_{r=0}^{n} \frac{\lambda^r}{\binom{n}{r}} = (n+1) \sum_{r=0}^{n} \frac{\lambda^{n+1} + \lambda^{n-r}}{(r+1)(1+\lambda)^{n-r+1}}. \tag{3}$$

From (2), one can easily get a recursion formula for $I_{m,n}(\lambda)$. When $m = 0$, the recursive formula is

$$(1+\lambda^{-1})I_{0,n} = \lambda^n + \lambda^{-1} + \left(1 + \frac{1}{n}\right) I_{0,n-1}.$$

For $0 < \lambda < 1$, this formula proves also that $I_{0,n}(\lambda) \to 1$ as $n \to \infty$. Now, using (2) with λ replaced by $\frac{\theta}{1-\theta}$ for $\theta \neq 1$ (which can be done because $\frac{\theta}{1-\theta}$ takes all values except -1), we have

$$\begin{aligned}\sum_{r=m}^{n} \frac{\theta^r (1-\theta)^{n-r}}{\binom{n}{r}} &= (n+1) \sum_{r=0}^{n-m} \theta^{n-r} (1-\theta)^{n-m+1} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{r+i+1} \\ &\quad + (n+1) \sum_{r=m}^{n} \frac{\theta^{n+1} (1-\theta)^{n-r}}{r+1}. \end{aligned} \tag{4}$$

To show that this is a polynomial identity, let us denote by $P(T)$ and $Q(T)$ the polynomials over $\mathbb{Q}$ which correspond to the two sides of (4).

We have $P(\theta) = Q(\theta)$ for all $1 \neq \theta \in \mathbb{Q}$. This means that P and Q have to coincide in $\mathbb{Q}[T]$ and this is indeed a polynomial identity.

This is not yet the first identity (1) in the theorem. To obtain that, we make a few observations of independent interest. First, if we compare the coefficients of θ^m on both sides of the identity (4) above, we get

$$\frac{1}{\binom{n}{m}} = (n+1) \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \frac{1}{n-r+1}. \tag{5}$$

In terms of $k = n - m$, this is also equivalent to the identity

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{1}{m+r+1} = \frac{1}{(k+m+1)\binom{k+m}{m}}. \tag{6}$$

If we use this in the above polynomial identity (4), we get the identity (1) in Theorem 2.1. □

Corollary 2.2. *For arbitrary natural numbers m, n*

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{m+r+1} = \frac{1}{(n+m+1)\binom{n+m}{m}}.$$

Some consequences of Theorem 2.1 are :

Corollary 2.3.

$$\sum_{r=m}^{n} \frac{(-1)^{n-r}}{\binom{n}{r}} = (n+1)\sum_{r=0}^{n-m}(-1)^r \frac{\binom{n-m+1}{r}}{\binom{m+r}{r}(m+r+1)}. \tag{7}$$

$$\sum_{r=m}^{n} \frac{(-1)^{r}}{\binom{n}{r}} = \left((-1)^n + \frac{(-1)^m}{\binom{n+1}{m}}\right)\frac{n+1}{n+2}. \tag{8}$$

Consequently, $\sum_r \frac{1}{\binom{2n}{2r}} \to 2$ *and* $\sum_r \frac{1}{\binom{2n}{2r+1}} \to 0$ *as* $n \to \infty$.

$$\begin{aligned}\sum_{r=m}^{n} \frac{1}{\binom{n}{r}^2} &= \frac{(n+1)^2}{(m+1)(n-m+2)}\sum_{i=0}^{n-m}\frac{1}{\binom{m+i+1}{i}\binom{2n-m+2-i}{n-i}} \\ &\quad + \frac{(n+1)^2}{n+2}\sum_{i=m}^{n}\frac{1}{(i+1)\binom{2n+2-i}{n-i}}. \end{aligned} \tag{9}$$

$$\sum_{r\geq 0} \frac{(-1)^r}{\binom{n}{r}}\binom{n-r}{s-r} = (-1)^n(n+1)\sum_{i\geq 0}\frac{(-1)^i}{i+1}\binom{n+1}{s+i-n}. \tag{10}$$

$$\sum_{i=0}^{n} \frac{1}{i+1}\binom{n-i}{j} = \sum_{i\geq 0}\frac{(-1)^i}{i+1}\binom{n+1}{i+j+1}. \tag{11}$$

In particular, for a prime $p \geq 3$*, the numerator of* $1 + \frac{1}{2} + \cdots + \frac{1}{p-1}$ *is a multiple of* p.

$$\sum_{r\geq 0}\frac{1}{2r+1}\binom{n+1}{2r+1} = \sum_{i=0}^{n}\frac{2^i}{i+1}. \tag{12}$$

$$\sum_{r\geq 1}\frac{1}{2r}\binom{n+1}{2r} = \sum_{i=0}^{n}\frac{2^i-1}{i+1}. \tag{13}$$

Proof. We compare the coefficients of θ^n in (4) and get (7). If we further use the expression (5), we get the second identity (8). The special case $m = 0$ of (8) gives

$$\sum_{r=0}^{n}\frac{(-1)^r}{\binom{n}{r}} = (1+(-1)^n)\frac{n+1}{n+2}. \tag{14}$$

This shows that

$$\sum_r \frac{1}{\binom{2n}{2r}} \to 2$$

and

$$\sum_r \frac{1}{\binom{2n}{2r+1}} \to 0$$

as $n \to \infty$. The next identity (9) follows if we integrate (1) as a function of T from 0 to 1, and use the beta-gamma relation.

Equating the coefficients of T^s for $s \leq n$ in (1), one arrives at (10).

Further, for $m = 0$, since the right hand side of (1) has terms where T occurs with powers higher than n, one could compare the powers of T^{n+j+1} to get (11).

Applying (11) with $n = p-1$ and $j = 0$ yields the assertion on numerators. When $j = 0$, the identity (11) becomes:

$$\sum_{i=0}^{n} \frac{1}{i+1} = \sum_{r \geq 0} \frac{1}{2r+1} \binom{n+1}{2r+1} - \sum_{r \geq 1} \frac{1}{2r} \binom{n+1}{2r}. \tag{15}$$

Combining this with the following identities from [6]

$$\sum_{r=0}^{n} \frac{1}{\binom{n}{r}} = \frac{n+1}{2^n} \sum_{r=0}^{n} \frac{2^r}{r+1} = \frac{n+1}{2^n} \sum_{j \text{ odd}} \binom{n+1}{j} \frac{1}{j}, \tag{16}$$

we get the last two identities (12) and (13). □

3 Combinatorial binomial identities

In this section, we use the basic method of Theorem 2.1 to prove various combinatorial identities where binomial coefficients occur in the denominator. We start with the following identity which was discovered by D. H. Lehmer [1]; his proof is different.

Theorem 3.1. *If* $|x| < 1$, *then*

$$\sum_{m \geq 1} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x}{\sqrt{1-x^2}} \sin^{-1} x.$$

Proof. For $|x| < 1$,

$$\begin{aligned} \sum_{m \geq 1} \frac{(2x)^{2m}}{m\binom{2m}{m}} &= \sum_{m \geq 1} (2x)^{2m} \beta(m+1, m) \\ &= \sum_{m \geq 1} (2x)^{2m} \int_0^1 t^m (1-t)^{m-1} dt = \int_0^1 \frac{4x^2 t}{1 - 4x^2 t(1-t)} dt. \end{aligned}$$

after interchanging the sum and the integral and evaluating the sum. This is further equal to

$$\int_{-x}^{x} \frac{s}{s^2 + (1-x^2)} ds + x \int_{-x}^{x} \frac{1}{s^2 + (1-x^2)} ds.$$

on changing variables as $s = x(2t-1)$. Thus, we get

$$\sum_{m \geq 1} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x}{\sqrt{1-x^2}} \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{2x}{\sqrt{1-x^2}} \sin^{-1} x.$$

□

Corollary 3.2. *We have*

$$\sum_{m\geq 1}\frac{1}{m\binom{2m}{m}}=\frac{\pi\sqrt{3}}{9}.$$

$$\sum_{m\geq 1}\frac{1}{\binom{2m}{m}}=\frac{1}{3}+\frac{2\pi}{9\sqrt{3}}.$$

$$\sum_{m\geq 1}\frac{1}{m^2\binom{2m}{m}}=\frac{\pi^2}{18}.$$

Proof. The first equality is the special case $x = 1/2$ of Lehmer's identity. The second is obtained by differentiating Lehmer's identity and then putting $x = 1/2$. To get the third identity, one integrates and then puts $x = 1/2$. □

Theorem 3.3. *If n and p are positive integers with $p > 1$, then*

$$\sum_{k=0}^{pn}(-1)^k\binom{pn}{k}\binom{2pn}{2k}^{-1}=\frac{(2pn+1)(1+(-1)^{pn})}{2(pn+1)}. \tag{17}$$

$$\begin{aligned}&\sum_{k=0}^{pn}(-1)^k k\binom{pn}{k}\binom{2pn}{2k}^{-1}\\ = &-\frac{pn(2pn+1)}{8}\left(\frac{1-(-1)^{pn}}{pn+2}-\frac{2(1+(-1)^{pn})}{pn+1}+\frac{1-(-1)^{pn}}{pn}\right).\end{aligned} \tag{18}$$

$$\begin{aligned}&\sum_{k=0}^{pn}(-1)^k k^2\binom{pn}{k}\binom{2pn}{2k}^{-1}\\ = &-\frac{(2pn+1)pn}{8}\left(\frac{1-(-1)^{pn}}{pn+2}+\frac{2(1+(-1)^{pn})}{pn+1}+\frac{1-(-1)^{pn}}{pn}\right)\\ &+\frac{(2pn+1)pn(pn-1)}{32}\left(\frac{1+(-1)^{pn}}{pn-1}-\frac{4(1-(-1)^{pn})}{pn}\right.\\ &\left.+\frac{6(1+(-1)^{pn})}{pn+1}-\frac{4(1-(-1)^{pn})}{pn+2}+\frac{1+(-1)^{pn}}{pn+3}\right).\end{aligned} \tag{19}$$

Proof. We give the proofs of (17) and (18). The proof of (19) is similar and is omitted here.

We have

$$\begin{aligned}
&\sum_{k=0}^{pn}(-1)^k\binom{pn}{k}\binom{2pn}{2k}^{-1}\\
=&\ \sum_{k=0}^{pn}(-1)^k\binom{pn}{k}(2pn+1)\int_0^1 t^{2k}(1-t)^{2pn-2k}dt\\
=&\ (2pn+1)\int_0^1\left((1-t)^{2pn}\sum_{k=0}^{pn}\binom{pn}{k}\left(-\frac{t^2}{(1-t)^2}\right)^k\right)dt\\
=&\ (2pn+1)\int_0^1(1-t)^{2pn}\left(1-\frac{t^2}{(1-t)^2}\right)^{pn}dt\\
=&\ (2pn+1)\int_0^1(1-2t)^{pn}dt=\frac{(2pn+1)(1+(-1)^{pn})}{2(pn+1)}.
\end{aligned}$$

Similarly, once again we obtain

$$\begin{aligned}
&\sum_{k=0}^{pn}(-1)^k k\binom{pn}{k}\binom{2pn}{2k}^{-1}\\
=&\ (2pn+1)\int_0^1\left((1-t)^{2pn}\sum_{k=0}^{pn}k\binom{pn}{k}\left(-\frac{t^2}{(1-t)^2}\right)^k\right)dt\\
=&\ -pn(2pn+1)\int_0^1\frac{t^2}{(1-t)^2}\left(1-\frac{t^2}{(1-t)^2}\right)^{pn-1}(1-t)^{2pn}dt\\
=&\ -pn(2pn+1)\int_0^1 t^2(1-2t)^{pn-1}dt.
\end{aligned}$$

Thus, equality (18) holds. □

We note that (17) becomes Theorem 1.2 of [7] when $p=2$.

Theorem 3.4. *If m, n, p and q are nonnegative integers with $p\geq q$, then*

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{m+n+p}{m+k+q}^{-1}\\
&=\frac{m+n+p+1}{m+n+p+2}\left(\binom{m+n+p+1}{m+q}^{-1}+(-1)^n\binom{m+n+p+1}{m+n+q+1}^{-1}\right).
\end{aligned}\qquad(20)$$

Proof. We have

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{m+n+p}{m+k+q}^{-1}\\
= \;& \sum_{k=0}^{n}(-1)^k(m+n+p+1)\int_0^1 t^{m+k+q}(1-t)^{n+p-k-q}dt\\
= \;& (m+n+p+1)\int_0^1\left(t^{m+q}(1-t)^{n+p-q}\sum_{k=0}^{n}\left(\frac{-t}{1-t}\right)^k\right)dt\\
= \;& (m+n+p+1)\Big(\int_0^1 t^{m+q}(1-t)^{n+p-q+1}dt\\
& +(-1)^n\int_0^1 t^{m+n+q+1}(1-t)^{p-q}dt\Big)\\
= \;& \frac{m+n+p+1}{m+n+p+2}\left(\binom{m+n+p+1}{m+q}^{-1}+(-1)^n\binom{m+n+p+1}{m+n+q+1}^{-1}\right).
\end{aligned}$$

We note that Theorem 1.5 of [7] is a special case of (20). □

Theorem 3.5. *If n and m are positive integers, then*

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k\binom{m+n}{m+k}^{-1} &= \frac{m+n+1}{m+n+3}\Bigg(\frac{(-1)^n(n+1)(m+n+3)}{m+n+2}-\\
&\qquad \binom{m+n+2}{m+1}^{-1}-(-1)^n\Bigg). \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k^2\binom{m+n}{m+k}^{-1} &= (m+n+1)\Bigg(\frac{(-1)^n(n+1)^2}{m+n+2}-\frac{(-1)^n(2n+3)}{m+n+3}\\
&\quad +\frac{2}{m+n+4}\left(\binom{m+n+3}{m+2}^{-1}+(-1)^n\right)\\
&\quad -\frac{1}{m+n+3}\binom{m+n+2}{m+1}^{-1}\Bigg). \qquad (22)
\end{aligned}$$

Proof. It is clear that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k\binom{m+n}{m+k}^{-1} &= \sum_{k=0}^{n}(-1)^k k(m+n+1)\int_0^1 t^{m+k}(1-t)^{n-k}dt\\
&= (m+n+1)\int_0^1\left(t^m(1-t)^n\sum_{k=0}^{n}k\left(\frac{-t}{1-t}\right)^k\right)dt\\
&= (m+n+1)\Big(\int_0^1(-1)^n(n+1)t^{m+n+1}dt\\
&\quad -\int_0^1 t^{m+1}(1-t)^{n+1}dt-\int_0^1(-1)^n t^{m+n+2}dt\Big).
\end{aligned}$$

Then equality (21) holds. Similarly, we can verify (22). □

Theorem 3.6. *For real* $|t|<1$ *the infinite series* $\sum\limits_{r\geq m}\frac{t^{n+r}}{\binom{n+r}{r}}$ *converges to*

$$n\sum_{i=1}^{n-1}\binom{n-1}{i}\frac{(t-1)^{n-1-i}}{i\binom{m+i}{i}}-n\sum_{i=1}^{m}\binom{m}{i}\frac{(t-1)^{n-1+i}}{i\binom{n-1+i}{i}}$$
$$+n(t-1)^{n-1}\sum_{i=m+1}^{n-1}\frac{1}{i}+n(t-1)^{n-1}\log\left(\frac{1}{1-t}\right). \tag{23}$$

Proof. The proof is completely similar to that of Theorem 2.1. The only point that has to be taken care of here is that we compute the sequence of partial sums $\sum\limits_{r=m}^{M}\frac{t^{n+r}}{\binom{n+r}{r}}$ by the above method and observe that it converges as $M\to\infty$, to

$$n\sum_{i=1}^{n-1}\binom{n-1}{i}\frac{(t-1)^{n-1-i}}{i\binom{m+i}{i}}-n\sum_{i=1}^{n}\binom{m}{i}\frac{(t-1)^{n-1+i}}{i\binom{n-1+i}{i}}$$
$$+n(t-1)^{n-1}\sum_{i=m+1}^{n-1}\frac{1}{i}+n(t-1)^{n-1}\log(\frac{1}{1-t}).$$

□

Corollary 3.7.

$$\sum_{r=m}^{\infty}\frac{1}{\binom{n+r}{r}}=\frac{n}{(n-1)\binom{m+n-1}{n-1}}. \tag{24}$$

$$\sum_{r=0}^{\infty}\frac{1}{(r+1)(r+2)\cdots(r+n)} = \frac{1}{(n-1)(n-1)!}. \tag{25}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{\binom{n+r}{r}} = 2^{n-1}n\left(\log 2-\sum_{i=1}^{n-1}\frac{1}{i}\right)-n\sum_{i=1}^{n-1}(-1)^i\binom{n-1}{i}\frac{2^{n-1-i}}{i}. \tag{26}$$

$$\sum_{r=0}^{\infty}\frac{1}{2^r\binom{n+r}{r}} = 2n(-1)^{n-1}\left(\log 2+\sum_{i=1}^{n-1}\frac{1+(-2)^i\binom{n-1}{i}}{i}\right). \tag{27}$$

Proof. It is easy to see that the right hand side in Theorem 3.11 has a finite limit at $t=\pm1$ and, hence the theorem is valid for $t=\pm1$ also. The specializations $t=1,-1,\frac{1}{2}$ give, respectively, (24), (26) and (27); (25) is just the case $m=0$ of (24). □

Proposition 3.8.

$$\sum_{n\geq 2}\frac{(-1)^n}{(2n-2)(2n-1)2n}=\frac{\pi-3}{4}. \tag{28}$$

$$\sum_{n\geq 0}\frac{1}{(3n+1)(3n+2)(3n+3)}=\frac{\pi\sqrt{3}-3\log 3}{12}. \tag{29}$$

$$\sum_{n\geq 0}\frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)}=\frac{6\log 2-\pi}{24}. \tag{30}$$

Proof. Now,

$$\begin{aligned}
&\sum_{k\geq 2}(-1)^k\frac{(2k-3)!}{(2k)!}=\sum_{k\geq 2}(-1)^k\frac{\Gamma(2k-2)}{\Gamma(2k+1)}=\frac{1}{2}\sum_{k\geq 2}(-1)^k\frac{\Gamma(2k-2)\Gamma(3)}{\Gamma(2k+1)}\\
=\;&\frac{1}{2}\sum_{k\geq 2}(-1)^k\beta(2k-2,3)=\frac{1}{2}\sum_{k\geq 2}(-1)^k\int_0^1 t^{2k-3}(1-t)^2dt\\
=\;&\frac{1}{2}\int_0^1\frac{(1-t)^2}{t^3}\sum_{k\geq 2}(-t^2)^k dt=\frac{1}{2}\int_0^1\frac{t(1-t)^2}{1+t^2}dt.
\end{aligned}$$

This is an easy exercise to evaluate and turns out to be $\frac{\pi-3}{4}$.

The second sum is

$$\begin{aligned}
\sum_{n\geq 0}\frac{(3n)!}{(3n+3)!} &= \frac{1}{2}\sum_{n\geq 0}\frac{\Gamma(3n+1)\Gamma(3)}{\Gamma(3n+4)}=\frac{1}{2}\sum_{n\geq 0}\beta(3n+1,3)\\
&= \frac{1}{2}\int_0^1\sum_{n\geq 0}t^{3n}(1-t)^2dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t+t^2}dt.
\end{aligned}$$

One can easily compute this to be

$$\sum_{n\geq 0}\frac{1}{(3n+1)(3n+2)(3n+3)}=\frac{\pi\sqrt{3}-3\log 3}{12}.$$

The last one is similar. □

Proposition 3.9.

$$\begin{aligned}
&\sum\frac{1}{\binom{6n}{6r}}+\sum\frac{1}{\binom{6n}{6r+3}}-\sum\frac{1}{\binom{6n}{6r+1}}-\sum\frac{1}{\binom{6n}{6r+4}}\\
&=3+\frac{1}{3n}+(6n+1)\sum_{s<n}\left(\frac{1}{3s}+\frac{1}{6s+1}-\frac{1}{6s+2}-\frac{2}{6s+3}-\frac{1}{6s+4}+\frac{1}{6s+5}\right).
\end{aligned}$$

Proof. It is clear that the polynomial identity (1) of Theorem 2.1 holds for complex numbers z. Let $m=0$ and take z to be a root of $z(1-z)=1$; this means that z is a primitive 6-th root of unity. Let us write

$$L_i = \sum_{r \equiv i \bmod 3} \frac{1}{\binom{n}{r}} \quad \text{for} \quad i = 0, 1, 2$$

$$R_i = \sum_{\substack{1 \le r \le n+1 \\ r \equiv i \bmod 6}} \frac{1}{r} \quad \text{for} \quad r = 0, 1, 2, 3, 4, 5$$

We have then

$$L_0 - L_1 + z(L_1 - L_2) = (n+1)z^n(2R_0 + R_1 - R_2 - 2R_3 - R_4 + R_5). \tag{31}$$

The identity of the proposition is the special case where $n \equiv 0 \bmod 6$. □

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