ON THE INDEPENDENCE OF A SAMPLE CENTRAL MOMENT AND THE SAMPLE MEAN[1]

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1. Introduction. Let $X_1, X_2, \cdots, X_N$ be a random sample of size N (independently and identically distributed random variables) from a population with distribution function $F(x)$. It is known that the population can sometimes be characterized by the independence of a suitable statistic[2] $S = S(X_1, X_2, \cdots, X_N)$ and the sample mean $\bar{X} = \sum_{j=1}^{N} X_j/N$. If S is a polynomial statistic then the independence of S and $\bar{X}$ yields a differential equation for the characteristic function of $F(x)$. In order to determine $F(x)$ we must study this differential equation and find all its positive definite solutions. In the case of certain polynomial statistics, such as the k-statistics or quadratic polynomials, it is comparatively easy to obtain all positive definite solutions of this differential equation. In many cases however, this procedure is not feasible since it is often very difficult to decide whether a given function is positive definite. If we consider, for example, a normal population then any central sample moment $m_p = \sum_{j=1}^{N} (X_j - \bar{X})^p/N$ and the sample mean $\bar{X}$ are independent. But, when we investigate whether this property characterizes the normal population for $p > 3$, then it is practically impossible to determine all positive definite solutions of the corresponding differential equation.

In the present paper we prove the following theorem.

THEOREM. *Let $X_1, X_2, \cdots, X_N$ be a sample of size N from a certain population. Let p be a positive integer such that $(p - 1)!$ is not divisible by $N - 1$. The population is normal if and only if the sample central moment m_p of order p is distributed independently of the sample mean $\bar{X}$.*

REMARK. The condition that $(p - 1)!$ is not divisible by $N - 1$ is satisfied if $N > (p - 1)! + 1$.

For the proof of this statement we use a theorem which was recently derived by Linnik [1] and Zinger [2].

In Section 2 we derive two combinatorial lemmas which are essential for the proof of the theorem. In Section 3 we give some analytical results and deduce finally the theorem in Section 4.

2. Combinatorial lemmas. Let $x_0, x_1, \cdots, x_n$ be $n + 1$ real variables. Suppose that

$$(2.1) \qquad P = P(x_0, x_1, \cdots, x_n) = \sum{}^{*} A_{j_0 j_1 \cdots j_n} x_0^{j_0} x_1^{j_1} \cdots x_n^{j_n}$$

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[2] A statistic is a real, single valued and measurable function of the observations $X_1, X_2, \cdots, X_N$.

is a polynomial of degree p with real coefficients. Here and in the following the summation $\sum^*$ is extended over all non-negative integers $j_0, j_1, \cdots, j_n$ which satisfy the condition $j_0 + j_1 + \cdots + j_n = p$. If we replace each $x_k^{j_k}$ by $\nu^{(j_k)} = \nu(\nu - 1) \cdots (\nu - j_k + 1)$ in the polynomial P, we obtain a polynomial

$$\pi_p = \pi_p(\nu) = \sum^* A_{j_0 j_1 \cdots j_n} \nu^{(j_0)} \nu^{(j_1)} \cdots \nu^{(j_n)} \tag{2.2}$$

of degree p in the real variable ν. We write here $\nu^{(0)} = 1$. The polynomial $\pi_p(\nu)$ is called the adjoint polynomial of P.

In this section we study the adjoint polynomial when P has a special form.

LEMMA 2.1. *Let*

$$P = P(x_0, x_1, \cdots, x_n) = \sum_{k=0}^{n} (x_k - \bar{x})^p,$$

where $\bar{x} = \sum_{k=0}^{n} x_k/(n + 1)$. *The adjoint polynomial of P is then*

$$\pi_p(\nu) = \frac{(-1)^p p!}{(n+1)^{p-1}} \sum_{r=0}^{p} (-n)^r \binom{\nu}{r} \binom{n\nu}{p - r}.$$

PROOF. We note that

$$(x_k - \bar{x})^p = \frac{(-1)^p}{(n+1)^p} \sum^* (p; j_0, \cdots, j_n)(-n)^{j_k} x_0^{j_0} \cdots x_n^{j_n}. \tag{2.3}$$

Here $(p; j_0, \cdots, j_n) = p!/(j_0! \cdots j_n!)$ is a multinomial coefficient. It follow from (2.3) that

$$P = \frac{(-1)^p}{(n+1)^p} \sum_{k=0}^{n} \sum^* (p; j_0 \cdots j_n)(-n)^{j_k} x_0^{j_0} \cdots x_n^{j_n}.$$

Therefore

$$\pi_p = \frac{(-1)^p p!}{(n+1)^{p-1}} \sum^* (-n)^{j_0} \binom{\nu}{j_0} \cdots \binom{\nu}{j_n}.$$

We write

$$c_p = \sum^* (-n)^{j_0} \binom{\nu}{j_0} \cdots \binom{\nu}{j_n}$$

so that

$$\pi_p = (-1)^p p! \, c_p/(n + 1)^{p-1}. \tag{2.4}$$

It is easy to verify that

$$\sum_{p=0}^{\infty} c_p x^p = (1 - nx)^\nu (1 + x)^{n\nu}. \tag{2.5}$$

Thus

$$c_p = \sum_{r=0}^{p} (-n)^r \binom{\nu}{r} \binom{n\nu}{p - r}. \tag{2.6}$$

Lemma 2.1 follows immediately from (2.4).

LEMMA 2.2. *Let*

$$P = P(x_0, x_1, \cdots, x_n) = \sum_{k=0}^{n} (x_k - \bar{x})^p,$$

where $\bar{x} = (n+1)^{-1}\sum_{k=0}^{n} x_k$. *If* $(p-1)!$ *is not divisible by* n, *the adjoint polynomial* $\pi_p(\nu)$ *of* P *has no non-zero integer roots.*

Suppose that for some integer ν, $\nu \neq 0$, we have $\pi_p(\nu) = 0$. Then, for that value of ν, $c_p = 0$ and, according to (2.6),

$$\binom{n\nu}{p} = n\binom{\nu}{1}\binom{n\nu}{p-1} - n^2\binom{\nu}{2}\binom{n\nu}{p-2} + \cdots + (-1)^{p-1}n^p\binom{\nu}{p}.$$

Thus, multiplying by $p!$ and cancelling the common factor $n\nu$, we find that

$$(n\nu - 1)(n\nu - 2)\cdots(n\nu - p + 1) \equiv 0 \pmod{n}$$

so that $(p-1)! \equiv 0 \pmod{n}$. Lemma 2.2 follows immediately.

3. Some analytical results. Let $P(x_1, x_2, \cdots, x_N)$ be a polynomial of degree $p \geq 1$. We say that it is an admissible polynomial if the coefficients of the terms $x_j^p (j = 1, 2, \cdots, N)$ are not zero.

We state the following lemma which is due to Zinger [2].

LEMMA 3.1. *Let* $X_1, X_2, \cdots, X_N$ *be a sample of size* N *from a certain population. Let* $P = P(X_1, X_2, \cdots, X_N)$ *be an admissible polynomial statistic and let* $\Lambda = \sum_{j=1}^{N} X_j$. *If* P *and* Λ *are independently distributed then the common characteristic function* $f(t)$ *of the* $X_1, X_2, \cdots, X_N$ *is an entire function of finite order.*

LEMMA 3.2 (*Theorem of Marcinkiewicz*). *Let* $P_m(t)$ *be a polynomial of degree* m *and suppose that* $f(t) = \exp[P_m(t)]$ *is a characteristic function. Then the degree* m *of* $P_m(t)$ *cannot exceed* 2.

LEMMA 3.3. *Suppose that the conditions of Lemma* 3.1 *are satisfied and that the characteristic function* $f(t)$ *has no zeros in the entire complex plane. Then the population is normal.*

The function $f(t)$ is an entire function of finite order m without zeros. According to Hadamard's factorization theorem, $f(t) = \exp[P_m(t)]$. The statement of Lemma 3.3 then follows from the theorem of Marcinkiewicz.

Before proceeding further we introduce a special class of polynomials. Let

$$P(x_1, x_2, \cdots, x_N) = \sum_{j_1+\cdots+j_N \leq p} A_{j_1,\cdots,j_N} x_1^{j_1} \cdots x_N^{j_N}$$

be a polynomial of degree p. It can be written as the sum

$$P(x_1, x_2, \cdots, x_N) = P_0(x_1, x_2, \cdots, x_N) + P_1(x_1, x_2, \cdots, x_N),$$

where

$$P_0(x_1, x_2, \cdots, x_N) = \sum_{j_1+\cdots+j_N = p} A_{j_1,\cdots,j_N} x_1^{j_1} \cdots x_N^{j_N}$$

is a homogeneous polynomial of degree p, while $P_1(x_1, x_2, \cdots, x_N)$ is a polynomial of degree less than p. We say that the polynomial $P(x_1, x_2, \cdots, x_N)$ is non-singular if the following two conditions are satisfied:

(i) $P(x_1, x_2, \cdots, x_N)$ contains the pth power of at least one variable.

(ii) The adjoint polynomial $\pi_p(\nu)$ of $P_0(x_1, x_2, \cdots, x_N)$ does not have a positive integer root.

For the proof of the theorem we need the following lemma.

LEMMA 3.4. *Let $X_1, X_2, \cdots, X_N$ be a sample of size N from a certain population. Let $P = P(X_1, X_2, \cdots, X_N)$ be a non-singular admissible polynomial statistic and let $\Lambda = \sum_{j=1}^N X_j$. If P and Λ are independently distributed, then the population is normal.*

Lemma 3.4 is due to Linnik [1]. In his paper Linnik made the additional assumption that the population distribution function has moments up to a certain order. In view of Lemma 3.1 (due to Zinger) this assumption is superfluous. Since Linnik's article [1] is not easily accessible while Zinger [2] only states (a somewhat generalized version) of Lemma 3.4 without proof, we give here its derivation.

Since P and Λ are independent, we conclude from Lemma 3.1 that the common characteristic function $f(z)$ of the random variables $X_1, X_2, \cdots, X_N$ is an entire function of finite order. The relation

$$\mathcal{E}(Pe^{iz\Lambda}) = \mathcal{E}(P)\mathcal{E}(e^{iz\Lambda}) \tag{3.1}$$

holds for all complex $z(z = t + iv;\ t, v \text{ real})$. First we show that the function $f(z)$ has no zeros in the entire complex plane. We write

$$f^{(j)} = f^{(j)}(z) = (d^j/dz^j)f(z) = i^j\mathcal{E}(X^je^{izX})$$

and note that $f^{(0)}(z) = f(z) = f$. We see from (3.1) that

$$\sum_{j_1+\cdots+j_N\leq p} A_{j_1\cdots j_N}f^{(j_1)}\cdots f^{(j_N)} = C[f(z)]^N, \tag{3.2}$$

where $C = i^p\mathcal{E}(P)$.

We give an indirect proof and assume therefore that the function $f(z)$ has zeros. Let the point $z = z_0$ be one of the zeros of $f(z)$ which are nearest to the origin and denote the order of the zero z_0 by ν(ν a positive integer). We show that this assumption leads to a contradiction.

Since $f(z)$ does not vanish in the circle $|z| < |z_0|$, we may divide (3.2) by $[f(z)]^N$ and obtain

$$R_0 + R_1 = C, \tag{3.3}$$

where

$$R_0 = \sum_{j_1+\cdots+j_N=p} A_{j_1\cdots j_N}\frac{f^{(j_1)}\cdots f^{(j_N)}}{f^N} \tag{3.4}$$

and

$$R_1 = \sum_{j_1+\cdots+j_N<p} A_{j_1\cdots j_N}\frac{f^{(j_1)}\cdots f^{(j_N)}}{f^N}$$

Let $\varphi = \varphi(z) = \ln f(z)$. It is then easily verified that

$$f^{(j)}/f = \varphi^{(j)} + \theta_j(\varphi', \varphi'', \cdots, \varphi^{(j-1)}) \qquad (j = 1, 2, \cdots) \tag{3.5}$$

where θ_j is a polynomial in $\varphi', \varphi'', \cdots, \varphi^{(j-1)}$. We also write $\varphi^{(0)} \equiv 1$ and $\theta_0 \equiv 0$. We substitute (3.5) into (3.3) and get, for $|z| < |z_0|$,

$$S_0 + S_1 = C, \tag{3.6}$$

where

$$S_0 = \sum_{j_1 + \cdots + j_N = \nu} A_{j_1 \cdots j_N}[\varphi^{(j_1)} + \theta_{j_1}] \cdots [\varphi^{(j_N)} + \theta_{j_N}] \tag{3.7}$$

and

$$S_1 = \sum_{j_1 + \cdots + j_N < \nu} A_{j_1 \cdots j_N}[\varphi^{(j_1)} + \theta_{j_1}] \cdots [\varphi^{(j_N)} + \theta_{j_N}].$$

Since

$$f(z) = (z - z_0)^\nu g(z), \tag{3.8}$$

where $g(z)$ is an entire function and $g(z_0) \neq 0$, it is easy to verify that

$$\varphi'(z) = \nu/(z - z_0) + h_1(z),$$

and, in general, that

$$\varphi^{(j)}(z) = [(-1)^{j-1}(j-1)!\nu]/[(z - z_0)^j] + h_j(z) \qquad (j = 1, 2, \cdots). \tag{3.9}$$

The functions $h_j(z)$ are regular at the point $z = z_0$. We substitute (3.9) into (3.6) and see that

$$\frac{\gamma_\nu}{(z - z_0)^\nu} + \frac{\gamma_{\nu-1}}{(z - z_0)^{\nu-1}} + \cdots + \frac{\gamma_1}{(z - z_0)} + H(z) = C, \tag{3.10}$$

where $H(z)$ is regular at the point $z = z_0$. We show next that $\gamma_\nu \neq 0$ and note that relation (3.10) leads therefore to a contradiction.

We remark that γ_ν depends only on ν and on the coefficients of the homogeneous polynomial $P_0(X_1, X_2, \cdots, X_N)$. We see that γ_ν is the coefficient of $(z - z_0)^{-\nu}$ in the expression which we obtain by substituting (3.9) into S_0 in (3.7). We get the same value for the coefficient of $(z - z_0)^{-\nu}$ if we substitute (3.8) into R_0 in (3.4). Since $f(0) = 1$ we see from (3.8) that $g(0) = C_1 \neq 0$. We note that γ_ν is also the coefficient of $(z - z_0)^{-\nu}$ in the expression obtained by substituting $\psi(z) = C_1(z - z_0)^\nu$ instead of $f(z)$ into the right-hand side of (3.4). We get

$$\psi^{(j)}(z) = C_1 \nu^{(j)}(z - z_0)^{\nu - j}, \qquad (j = 0, 1, 2, \cdots, \nu).$$

Therefore

$$\gamma_\nu = \sum_{j_1 + \cdots + j_N = \nu} A_{j_1 \cdots j_N} \nu^{(j_1)} \cdots \nu^{(j_N)}.$$

Thus $\gamma_\nu = \pi_p(\nu)$, the adjoint polynomial of P_0. Since P is a non-singular polynomial $\pi_p(\nu)$ does not vanish for any positive integer ν so that $\gamma_\nu \neq 0$. This leads to the desired contradiction in (3.10) so that $f(z)$ has no zeros.

The proof of Lemma 3.4 follows then from Lemma 3.3.

4. Proof of the Theorem. We show first that the condition is sufficient. It follows from Lemma 2.2 that the sample central moment m_p is a non-singular polynomial statistic if $(p - 1)!$ is not divisible by $N - 1$. Hence the theorem is an immediate consequence of Lemma 3.4. The necessity of the condition follows from the well-known fact that in a normal population any translation-invariant statistic is independent of the sample mean.

REFERENCES

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