ON THE PROBLEM OF CONFOUNDING IN THE GENERAL SYMMETRICAL FACTORIAL DESIGN

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Introduction

In all factorial designs in which the number of different sets of factors tested is large, or those in which their number is small but one or more of them involve a large number of variants, or finally, those which show both these characteristics simultaneously, the need for the reduction of the size of the block for the adequate elimination of fertility differences is an insistent consideration. The two techniques most extensively used for accomplishing this purpose are: (*i*) The Split-plot technique, and (*ii*) Confounding. In fact, the former is essentially a special case of the latter, split-plot arrangements being specialized types of designs involving the confounding of main effects, which are resorted to primarily for facilitating the performance of agricultural operations and only secondarily for effecting reduction of the block size. These will be found discussed in detail elsewhere by one of the authors[4] (K. Kishen). As, however, high-order interactions are as a rule negligible, sacrificing information on them only as far as possible by mixing them up indistinguishably with inter-block differences is the most elegant technique known to the experimenter for effecting the desired reduction in the block size and thereby improving the precision of the experiment.

No general solution of the problem of confounding in any symmetrical or unsymmetrical case is so far available. Asymmetrical designs like $p \times q \times r \times$ .... (all $p, q, r,$............ not equal) form a class apart and require special methods of attack: these will not be considered here. It may, however, be useful to recapitulate that the problem of confounding in the case of designs of the type $3^m \times 2^n$ (m, n being any positive integers) and all cases reducible to it has been completely solved by Yates As regards symmetrical arrangements, Nair has, in his recent communications[7, 8] developed a method of solving the problem in the general case $s \times s \times s \times s \times$ ......$(= s^m)$, where m is a positive integer and s a prime positive integer or a power of a prime, based on his theory of interchanges derivable from the associated Hyper-Graeco-Latin squares. He has demonstrated the working of his method in obtaining confounded arrangements for the s^m type of experiment in sub-blocks of s^2 plots, for (*i*) $s=3$, $m=3$ and 4, (*ii*) $s=4$, $m=3$, 4 and 5, and (*iii*) $s=5$, $m=3$ and 4.

For the case $s=2$, the problem has been solved in complete generality by Barnard[1], who has, by appealing to the principle of generalized interaction in the case of 2^m designs, enumerated the possible confounded arrangements reducing the size of the block from 2^m to 2^r plots $(r=1, 2, \ldots\ldots m-1)$, and has also given alongside the more useful balanced arrangements, confounding analogous sets of degrees of freedom in different replications. It is the purpose of the present paper to demonstrate, with the help of Galois fields and the associated finite hyperdimensional projective geometries, that the principle of generalized interaction is not a special feature of the 2^m factorial designs, but that it also holds in the general case s^m, where s is a prime or a power of a prime, and m a positive integer. This important principle and another allied concept flowing from it have been utilized to demonstrate a general method of forming sub-blocks of s^{m-k} plots $(k=1, 2, \ldots\ldots\ldots m-1)$ in the case of s^m factorial designs. The possible types of confounding for s^3 and s^4 designs in s^2- and s-plot blocks have been investigated in their entirety to serve as concrete illustrations, the consideration of other special cases having been reserved for a subsequent communication.

§1. Properties of Finite Geometries constructed from the Galois Fields

(1·1) The number of elements constituting a Galois field, *i.e.*, a field with a finite number of elements, is $s(=p^n)$ where p is a prime positive integer and n any positive integer. Conversely, given any number $s(=p^n)$, there always exists a Galois field with s elements in it and any two Galois fields with the same number of elements are structurally identical, so that it is possible to set up a correspondence between the elements of the two fields in such a way that the sum corresponds to the sum and the product to the product. The Galois field with s elements is symbolised by GF(s).

Let $\alpha_0=0, \alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_{s-1}$ be the elements of the G F(s), $s=p^n$, p being a prime positive integer and n any positive integer. There are different ways of indentifying $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_{s-1}$ with the $s-1$ non-zero elements of G F(s) when expressed in the standard form. In the case $n=1$, *i.e.*, when s is a prime number p, the identification we adopt is to set α_i equal to the residue class (i), modulo p. In the case when $n>1$, or s is a power of a prime higher than the first, our identification will be as follows:—

Let $f(x)$ be a specified minimum function, *i.e.*, an irreducible factor of the cyclotomic polynomial of the order p^n-1 of G F$_p(x)$. Then the elements of G F(s) can be represented uniquely by the residue classes modulo $f(x)$ of the polynomials 0, $x^0=1$, $x, x^2, \ldots\ldots\ldots x^{s-2} (s=p^n)$, the class with standard representative x being a primitive element of G F(s). Then the identification we adopt is to set $\alpha_0=0$ and $\alpha_i=x^{i-1}$. Since $x^{s-1}=1$, the rule for multiplication of the elements of the Galois field is as under:

$$\left.\begin{array}{l} \alpha_i\,\alpha_j=\alpha_0 \text{ if either } i=0 \text{ or } j=0, \\ \alpha_i\,\alpha_j=\alpha_t \text{ where } t\equiv(i+j-1) \bmod (s-1),\ 1\leqslant t\leqslant s-1, \\ \qquad \text{if } i\neq 0\ j\neq 0. \end{array}\right\} \qquad \ldots \quad (1)$$

A table of minimum functions which we shall use is given below :—

TABLE 1. MINIMUM FUNCTIONS FOR GALOIS FIELDS.

Field	Minimum function
$G\,F(2^2)$	x^2+x+1
$G\,F(2^3)$	x^3+x^2+1
$G\,F(2^4)$	x^4+x^3+1
$G\,F(3^2)$	x^2+x+2
$G\,F(3^3)$	x^3+2x+1
$G\,F(5^2)$	x^2+2x+3

The number p is said to be the characteristic of the field $G\,F(p^n)$. It may be observed that for fields of characteristic 2, addition is equivalent to subtraction. For further details about Galois fields, reference may be made to the recent papers by R. C. Bose[2,3].

(1·2) *The Finite Projective Geometry $P\,G(m, s)$.*

With the help of a Galois field $G\,F(s)$, we can construct a finite projective geometry of m dimensions in the following manner :—

Any ordered set of $m+1$ elements

$$(x_0, x_1, x_2, \ldots\ldots\ldots\ldots x_m) \qquad \ldots \quad (2)$$

where the x_i's belong to $G\,F(s)$ and are not all simultaneously zero, may be termed a point of our projective geometry $P\,G(m, s)$, it being implicit that the set $(y_0, y_1, \ldots\ldots y_m)$ represents the same point as $(x_0, x_1, \ldots\ldots x_m)$ when and only when there exists a non-zero element a of $G\,F(s)$ such that $y_i = a\,x_i (i=0, 1, \ldots\ldots\ldots m)$. We may speak of $(x_0, x_1, \ldots\ldots\ldots x_m)$ as the co-ordinates of the point. It may readily be shown that the number of points in $P\,G(m, s)$ is exactly

$$s^m + s^{m-1} + \ldots\ldots\ldots + s^2 + s + 1 = \frac{s^{m+1}-1}{s-1} \qquad \ldots \quad (3)$$

where $s = p^n$

All the points which satisfy a set of $m-l$ $(l < m)$ independent linear homogeneous equations

$$\left.\begin{array}{l} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1m}x_m = 0 \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2m}x_m = 0 \\ \quad\ldots \qquad \ldots \qquad\qquad \ldots \qquad \ldots \\ a_{m-l,0}x_0 + a_{m-l,1}x_1 + a_{m-l,2}x_2 + \ldots\ldots\ldots + a_{m-l,m}x_m = 0 \end{array}\right\} \qquad \ldots \quad (4)$$

may be said to form a l-dimensional sub-space, or briefly, a l-flat in P G(m, s). The equations may be said to represent this flat. Clearly, any other set of $m-l$ independent equations, which can be obtained by linear combinations of the equations (4), will have the same set of solutions, and will consequently represent the same l-flat. A 0-flat is of course identical with a point, and we shall, following the usual nomenclature, call a 1-flat a line and a 2-flat a plane.

The number of l-flats in P G(m, s) can be shown to be

$$\phi(m, l, s) = \frac{(s^{m+1}-1)(s^{m}-1)\ldots\ldots\ldots\ldots(s^{m-l+1}-1)}{(s^{l+1}-1)(s^{l}-1)\ldots\ldots\ldots\ldots(s-1)} \qquad \ldots \quad (5)$$

where, as before, $s=p^n$. It may be noticed that

$$\phi(m, l, s) = \phi(m, m-l-1, s) \qquad \ldots \quad (6)$$

It is also convenient to put formally $\phi(m, -1, s) = 1$ ... (7)

(1·3) *The Finite Euclidean Geometry E G(m, s).*

Again, any ordered set of m elements $(x_1, x_2, \ldots\ldots\ldots x_m)$ belonging to G F(s) may be called a point of the finite m-dimensional Euclidean Geometry E G(m, s), where the points $(x_1, x_2, \ldots\ldots\ldots x_m)$ and $(y_1, y_2, \ldots\ldots\ldots y_m)$ are identical when and only when $x_i = y_i$ $(i = 1, 2, \ldots\ldots\ldots, m)$.

The number of points in E G(m, s) is evidently s^m, where $s = p^n$.

All the points satisfying a set of $m-l$ $(l<m)$ consistent and independent linear equations

$$\left.\begin{array}{llllll} a_{10} & +a_{11}x_1 & +a_{12}x_2 & +\ldots\ldots\ldots & +a_{1m}x_m & =0 \\ a_{20} & +a_{21}x_1 & +a_{22}x_2 & +\ldots\ldots\ldots & +a_{2m}x_m & =0 \\ \ldots & \ldots & & & \ldots & \ldots \\ a_{m-l,0} & +a_{m-l,1}x_1 & +a_{m-l,2}x_2 & +\ldots\ldots\ldots & +a_{m-l,m}x_m & =0 \end{array}\right\} \qquad \ldots \quad (8)$$

may be said to constitute a l-flat of E G(m, s) represented by the equations (8). Any other set of $m-l$ consistent and independent linear equations, obtained by linear combinations of (8), represents the same l-flat. The number of l-flats in E G(m, s) is

$$\phi(m, l, s) - \phi(m-1, l, s). \qquad \ldots \quad (9)$$

(1·4). *Relation between* P G(m, s) *and* E G(m, s).

The $(m-1)$-flat $x_0 = 0$ of P G(m, s) may be conventionally called the $(m-1)$-flat at infinity. Points lying on this flat may be called the points at infinity, and the other points may be called finite points. Since the x_0-coordinate of a finite point is non-zero, we can, by dividing all the coordinates by x_0, express the coordinates of a finite point in the unique form $(1, x_1, x_2, \ldots x_m)$. Let this point correspond to the point $(x_1, x_2, \ldots x_m)$ of E G(m, s). This establishes a (1,1) correspondence between the finite points of P G(m, s) and those of E G(m, s). Again, a l-flat of P G(m, s) $(l<m)$ may be said to be entirely at infinity if all its points are points at infinity. All other l-flats are spoken of as finite l-flats. To any finite l-flat of P G(m, s) given by the equations (4), let there

correspond the l-flat of E G(m, s) given by the equations (8). It is easily seen that the equations (8) are consistent when and only when the l-flat of P G(m, s) given by the equations (4) is finite. It thus follows that there exists a (1,1) correspondence between the finite l-flats of P G(m, s) and the l-flats of E G(m, s), the finite points of the l-flats of P G(m, s) corresponding to the points of the l-flat of E G(m, s). The geometry E G(m, s) is thus derivable from P G(m, s) by cutting out all the points lying at infinity and the l-flats lying wholly at infinity.

We thus see that E G(m, s) can be regarded as a portion of P G(m, s), the latter being derived from the former by the adjunction of the elements at infinity. Thus in the following pages, when dealing with the finite elements of P G(m, s), we shall, for convenience, write their equations and co-ordinates, etc., as if they belonged to E G(m, s).

For further details on finite geometries, reference may be made to Carmichael[4].

§2. Connexion of degrees of freedom with Finite Geometry

Let us consider a factorial design s^m, involving m factors, each at s levels. Any treatment combination can then be represented by a symbol of the form

$$(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_m),$$

x_i denoting the level of the i-th factor in the treatment combination. Now x_i can assume s possible values. These values we now identify with the elements of the Galois field GF(s), so that every element represents a level. Now $(x_1, x_2, \ldots\ldots\ldots x_m)$ can be taken as the coordinates of a point in E G(m, s) which, as explained, corresponds to the finite point $(1, x_1, x_2, \ldots\ldots\ldots\ldots x_m)$ of P G(m, s). Thus there is a (1,1) correspondence between the s^m treatments and the s^m finite points of the projective geometry P G(m, s).

Let O be the point $(1, 0, 0, \ldots\ldots\ldots\ldots 0)$ of P G(m, s) and X_i the point for which $x_0=0$, $x_1=0, \ldots\ldots$ $x_{i-1}=0$, $x_i=1$, $x_{i+1}=0, \ldots\ldots$ $x_m=0$. Then the lines OX_1, $OX_2, \ldots\ldots$ $\ldots\ldots OX_m$ play the same part as the axes of reference in ordinary geometry, O being the origin. The points $X_i (i=1, 2, \ldots\ldots\ldots m)$ are, of course, at infinity. Then the simplex $X_1, X_2, \ldots\ldots\ldots\ldots X_m$ will be termed the *fundamental simplex*, since it occupies a key position in the further development of our theory. The points X_i $(i=1, 2, \ldots\ldots\ldots m)$ may be called its vertices or zero cells; the lines $X_i X_j$ $(i, j=1, 2, \ldots\ldots\ldots m; i \neq j)$ its edges or one-cells; the triangles $X_i X_j X_k$ $(i, j, k=1, 2, \ldots\ldots\ldots m; i\neq j\neq k)$ its 2-cells, and, in general, the $(k-1)$-dimensional partial simplexes formed from any k of the m points $X_1 X_2 \ldots\ldots X_m$ $(k<m)$ may be called its $(k-1)$-cells.

Through any $(m-2)$-flat at infinity there will pass a pencil of s parallel finite $(m-1)$-flats, each containing s^{m-1} finite points. These will divide the s^m treatments into s sets of s^{m-1} treatments each, if the treatments corresponding to the s^{m-1} points in any one of these $(m-1)$-flats are considered as belonging to the same set. The contrast between these sets represents $s-1$ degrees of freedom. We shall speak of these degrees of freedom as belonging to the pencil of $(m-1)$-flats considered this pencil being determined by the $(m-2)$-flat at infinity which may be called the vertex of this pencil.

Now the number of $(m-2)$-flats in the $(m-1)$-flat at infinity is $s^{m-1}+s^{m-2}+\ldots+s^2+s+1$. To each of these there corresponds a pencil of s finite $(m-1)$-flats with $s-1$ degrees of freedom. Thus the total number of degrees of freedom carried by these pencils is s^m-1, which we know to be the total number of degrees of freedom for all treatment comparisons.

Consider the pencil $x_i=\alpha_j$ $(j=0, 1, \ldots\ldots s-1;\ i \text{ fixed})$, α_j $(j=0, 1, \ldots\ldots s-1)$ denoting, as already explained in (1·1) of §1, elements of G F(s). Each $(m-1)$-flat of this pencil passes through the $(m-2)$-cell $X_1, X_2, \ldots\ldots X_{i-1}, X_{i+1} \ldots X_m$ of the fundamental simplex. The $s-1$ degrees of freedom corresponding to this pencil are none other than the degrees of freedom corresponding to the i-th main effect. This is so because the set corresponding to the $(m-1)$-flat $x_i=\alpha_j$ $(i, j \text{ fixed})$ is constituted by just those points for which the ith coordinate is α_j, *i.e.*, all those treatment combinations in which the i-th factor has the level α_j. Varying j, we now see that the contrast between the sets generated by the pencil $x_i=\alpha_j$ $(i \text{ fixed};\ j=0, 1, \ldots\ldots s-1)$ is the contrast between the various levels of the i-th factor, summed up over all the other factors, *i.e.*, the main effect of the i-th factor. Thus every $(m-2)$-cell of the fundamental simplex serves as the vertex of a parallel pencil of s $(m-1)$-flats, corresponding to a certain main effect. It will be seen that there are just m of these $(m-2)$-cells and also m main effects, each having $s-1$ degrees of freedom. Thus the total number of degrees of freedom, *i.e.*, $m(s-1)$, enumerated in this manner is just the number of degrees of freedom corresponding to the main effects.

Now let us consider pencils with equations of the form

$$x_i+\alpha_u\, x_j=\alpha_r \quad (r=0, 1, \ldots\ldots\ldots s-1\ ;\ u, i, j \text{ fixed}) \qquad \ldots \quad (10)$$

The $(m-1)$-flats of this pencil all pass through the $(m-3)$-cell $X_1, X_2, \ldots\ldots\ldots X_{i-1}$ $X_{i+1} \ldots\ldots\ldots X_{j-1}\, X_{j+1} \ldots\ldots\ldots X_m$ of the fundamental simplex. We now proceed to consider the sets into which this pencil decomposes the s^m treatments.

Let us, in particular, consider the set corresponding to the $(m-1)$-flat

$$x_i+\alpha_u\, x_j=\alpha_r \quad (u, r \text{ fixed}) \qquad \ldots \quad (11)$$

of this pencil. To every value of x_j there corresponds a definite value of x_i, different values of x_i corresponding to different values of x_j. Hence x_i varies over all the possible s values as x_j assumes all the s possible values.

With each of the s pairs of values of (x_i, x_j), we associate all the possible combinations of values of the other $m-2$ variates $x_1, x_2, \ldots\ldots x_{i-1}, x_{i+1}, \ldots x_{j-1}\, x_{j+1} \ldots\ldots\ldots\ldots x_m$, thus obtaining the s^{m-1} treatments corresponding to this flat. Giving r all values from 0 to $s-1$, we obtain the s sets of treatment combinations into which the pencil (10) partitions the s^m treatment combinations. As all combinations of the $m-2$ factors other than the i-th and the j-th, but not those of any of the $m-1$ factors, occur in each of these s sets, the contrast between them gives $s-1$ degrees of freedom for the first order interaction between the i-th and the j-th factors.

Let

$$x_i+\alpha_{u'}\, x_j=\alpha_{r'} \quad (u' \text{ fixed} \neq u;\ r'=0, 1, \ldots\ldots\ldots\ldots s-1) \qquad \ldots \quad (12)$$

be the equations of another parallel pencil of $(m-1)$-flats. Arguing as before, the contrast between the sets given by these also represents $s-1$ degrees of freedom for the interaction between the ith and the jth factors. Now, for r and r' fixed, the linear equations

$$x_i+\alpha_u\ x_j=\alpha_r \qquad \dots \quad (13)$$

$$x_i+\alpha_{u'}\ x_j=\alpha_{r'} \qquad \dots \quad (14)$$

have a unique solution. As r' is varied from 0 to $s-1$, s solutions in all are obtained which must all be different. Thus the s pairs of values of (x_i, x_j) given by (13) occur once each in each of the s different sets represented by (12). It immediately follows that the s^{m-1} treatment combinations corresponding to (13) are distributed s^{m-2} each in the sets represented by (12). Varying r from 0 to $s-1$, we see that each of the s sets of s^{m-1} treatments into which the s^m treatments are split up by the pencil (10) occur equally among the sets given by the pencil (12), and *vice versa*. Thus the $s-1$ degrees of freedom corresponding to the pencil (10) are orthogonal to the $s-1$ degrees of freedom corresponding to the pencil (12). Varying u and $u'(u\neq u')$ over all possible values, we see that the pencils given by the equations

$$x_i+\alpha_u\ x_j=\alpha_r(u=1, \dots\dots s-1\,;\ r=0\ 1,\dots\dots\dots s-1) \qquad \dots \quad (15)$$

yield $(s-1)^2$ degrees of freedom, which constitute the totality of the degrees of freedom corresponding to the first order interaction of the i-th and j-th factors.

Next, let us consider pencils with equations of the form

$$x_i+\alpha_u\ x_j+\alpha_v\ x_k=\alpha_r\ (i,\ j,\ k,\ u,\ v \text{ fixed};\ r=0,\ 1,\dots\dots\dots s-1) \qquad \dots \quad (16)$$

The $(m-1)$-flats of this pencil all pass through the $(m-4)$-cell $X_1\, X_2\dots\dots\dots X_{i-1}$ $X_{i+1}\dots\dots\dots X_{j-1}\, X_{j+1}\dots\dots\dots X_{k-1}\, X_{k+1}\dots\dots\dots X_m$ of the fundamental simplex.

Let us now obtain the s sets into which the different $(m-1)$-flats of this pencil partition the s^m treatment combinations. Keeping r fixed, we see that to every pair of values of (x_j, x_k), there will correspond a definite value of x_i. To every value of the triplet (x_i, x_j, x_k) we associate all possible combinations (s^{m-3} in number) of the remaining $m-3$ variates. The number of values which the triplet can assume being s^2, we get the s^{m-1} treatment combinations corresponding to one of the $(m-1)$-flats of the pencil (16). Changing r from 0 to $s-1$, we obtain s such sets. As only all combinations of the $m-3$ factors, other than the i-th, j-th and k-th, but not those of any larger number of factors, occur in each of these s sets, the $s-1$ degrees of freedom corresponding to the pencil (16) are the degrees of freedom belonging to the second order interaction between the i-th, j-th and k-th factors.

Let

$$x_i+\alpha_{u'}\ x_j+\alpha_{v'}\ x_k=\alpha_{r'}\ (u',\ v' \text{ fixed};\ r'=0,\ 1,\dots\dots\dots s-1) \qquad \dots \quad (17)$$

represent another parallel pencil of $(m-1)$-flats. The $(s-1)$-degrees of freedom corresponding to it will evidently represent the degrees of freedom for the second order interaction between the i-th, j-th and k-th factors. We shall now show that the s^{m-1}

treatments given by any of the flats of the pencil (16) are distributed equally in groups of s^{m-2} among the sets given by the s $(m-1)$-flats of the pencil (17). Considering the two linear equations

$$x_i + \alpha_u x_j + \alpha_v x_k = \alpha_r \ (u,\ v,\ r \text{ fixed}) \qquad \dots \quad (18)$$

$$x_i + \alpha_{u'} x_j + \alpha_{v'} x_k = \alpha_{r'} \ (u',\ v',\ r' \text{ fixed}) \qquad \dots \quad (19)$$

it will be seen that to every value of x_i there corresponds one unique solution for (x_j, x_k), and the solutions corresponding to the s different values of x_i are all different. We thus obtain in this case s values of the triplet (x_i, x_j, x_k) satisfying both (18) and (19). To each of these may be associated any one of the s^{m-3} possible combinations of values of the other $m-3$ variates, giving in all s^{m-2} common treatment combinations.

From here it follows, as before, that the $s-1$ degrees of freedom represented by the pencil (16) are orthogonal to the $s-1$ degrees of freedom corresponding to (17), another pencil of the same type. Varying u and v over the values $1, 2, \dots\dots\dots\dots s-1$, we obtain $(s-1)^2$ different pencils of this type, the $(s-1)^3$ degrees of freedom corresponding to which represent the totality of the degrees of freedom corresponding to the second order interaction between the i-th, j-th and k-th factors.

In the same way, it may be shown that the $s-1$ degrees of freedom given by the contrast between the s sets of treatment combinations into which the s^m treatments are divided up by the pencil of $(m-1)$-flats represented by the equation

$$x_{i_0} + \alpha_{u_1} x_{i_1} + \alpha_{u_2} x_{i_2} + \dots + \alpha_{u_k} x_{i_k} = \alpha_r \ (i_0,\ i_1,\ i_2,\ \dots i_k,\ u_1,\ u_2, \dots u_k \text{ fixed};\ r = 0,\ 1, \dots s-1) \quad (20)$$

belong to the k-th order interaction. Also, by arguing as before, it is easily seen that the two sets of $s-1$ degrees of freedom corresponding to any two pencils of this type are mutually orthogonal. Every $(m-1)$-flat of this pencil passes through the $(m-k-2)$-cell of the fundamental simplex, obtained by excluding the $k+1$ points X_{i_0}, $X_{i_1}, \dots\dots X_{i_k}$ from among the vertices of the fundamental simplex. As each of $u_1, u_2 \dots\dots$ $\dots\dots u_k$ can assume $s-1$ different values, the total number of pencils of the type (20) is $(s-1)^k$, which give $(s-1)^{k+1}$ degrees of freedom corresponding to the k-th order interaction between the i_0-th, i_1-th$\dots\dots\dots i_k$-th factors.

The degrees of freedom corresponding to the highest order interaction correspond to pencils of the type

$$x_1 + \alpha_{u_2} x_2 + \alpha_{u_3} x_3 + \dots\dots + \alpha_{u_m} x_m = \alpha_r \ (u_2,\ u_3, \dots u_m \text{ fixed};\ r = 0,\ 1, \dots\dots s-1) \qquad \dots \quad (21)$$

Since each of $u_2, u_3 \dots\dots\dots u_m$ can vary over $s-1$ possible values, there are $(s-1)^{m-1}$ such pencils, yielding $(s-1)^m$ degrees of freedom for the highest order interaction. The $(m-2)$-flats at infinity which are the vertices of these pencils pass clear of the fundamental simplex, *i.e.*, they do not pass through any vertex of the fundamental simplex. In general, a k-flat at infinity may be said to pass clear of the fundamental simplex if it does not cut any $(m-k-2)$-cell of the simplex.

§3. The Principle of Generalized Interaction

(3·1) We now proceed to enunciate the principle of generalized interaction in the general case of a s^m factorial arrangement.

Given any two of the pencils of $(m-1)$-flats symbolised by the equations

$$x_{i_0} + a_{u_1} x_{i_1} + a_{u_2} x_{i_2} + \dots\dots + a_{u_r} x_{i_r} = a_t$$
$$(i_0, i_1, \dots, i_r, u_1, u_2, \dots, u_r \text{ fixed}; \; t = 0, 1, \dots, s-1) \quad \dots \quad (22)$$

$$x_{j_0} + a_{v_1} x_{j_1} + a_{v_2} x_{j_2} + \dots\dots + a_{v_{r'}} x_{j_{r'}} = a_{t'}$$
$$(j_0, j_1, j_2, \dots, j_{r'}, v_1, v_2, \dots, v_{r'} \text{ fixed}; \; t' = 0, 1, \dots, s-1) \quad \dots \quad (23)$$

each representing $s-1$ degrees of freedom for a main effect or an interaction in a s^m factorial design, their generalized interaction is given by the $(s-1)^2$ degrees of freedom belonging to the main effects and interactions represented by the $s-1$ pencils of which the equations are

$$(x_{i_0} + a_{u_1} x_{i_1} + a_{u_2} x_{i_2} + \dots\dots + a_{u_r} x_{i_r})$$
$$+ a_\lambda (x_{j_0} + a_{v_1} x_{j_1} + a_{v_2} x_{j_2} + \dots\dots + a_{v_{r'}} x_{j_{r'}}) = a_\mu ,$$
$$(\lambda = 1, 2, \dots\dots s-1; \; \mu = 0, 1, \dots\dots s-1) \quad \dots \quad (24)$$

Thus any two given pencils, either representing $s-1$ degrees of freedom for a main effect or an interaction in a s^m factorial design, determine as their generalized interaction, $s-1$ other pencils, each representing a main effect or an interaction. It thus follows that in the general case of a s^m factorial arrangement, there subsists an internal configurational symmetry so that if any suitably selected m of the pencils be called the main effects, the remaining pencils will represent all the interactions in their entirety.

This procedure is best carried out in successive stages. We may start by taking any two pencils and and calling them two main effects. These will then, by their generalized interaction, fix $s-1$ other pencils which will denote the interaction of the two main effects. There being $l \left(= \dfrac{s^m - 1}{s-1} \right)$ pencils in all, we may term any of the remaining $l-s-1$ pencils as a third main effect. The latter will, by its generalised interaction with each of the former $s+1$ pencils, fix s^2-1 further pencils, the totality of the number of pencils fixed at this stage being s^2+s+1. Out of the remaining $l-s^2-s-1$, pencils left at this stage, we may designate any as the fourth main effect. Then, again by the principle of generalized interaction, s^3-1 further pencils will be specified, the total number of these now being s^3+s^2+s+1. This process may be repeated until the m-th main effect has been specified, when all the $s^{m-1}+s^{m-2}+\dots\dots+s^2+s+1$ pencils will have been exhausted.

(3·2) A concrete example may now given to further elucidate the procedure. Let $s=3$, $m=3$, so that we have a $3 \times 3 \times 3$ factorial design. The three factors may be taken to be nitrogen, potash and superphosphate. The following are the equations to the pencils representing the main effects and interactions, N K(1), N K(2) denoting the pairs of degrees of freedom constituting the totality of degrees of freedom for the interaction NK, with a similar notation for other degrees of freedom.

TABLE 2. EQUATIONS TO PENCILS CORRESPONDING TO MAIN EFFECTS AND INTERACTIONS IN A 3^3 DESIGN.

Serial No.	Main effects and interactions	Equations to pencils	New set of main effects and interactions
1	N	$x=\alpha_u \quad (u=0, 1, 2)$	R
2	K	$y=\alpha_u \quad (u=0, 1, 2)$	R S (2)
3	P	$z=\alpha_u \quad (u=0, 1, 2)$	S T (2)
4	N K (1)	$x+\alpha_1 y=\alpha_u \quad (u=0, 1, 2)$	S
5	N K (2)	$x+\alpha_2 y=\alpha_u \quad (u=0, 1, 2)$	R S (1)
6	K P (1)	$y+\alpha_1 z=\alpha_u \quad (u=0, 1, 2)$	R T (2)
7	K P (2)	$y+\alpha_2 z=\alpha_u \quad (u=0, 1, 2)$	R S T(1)
8	N P (1)	$z+\alpha_1 x=\alpha_u \quad (u=0, 1, 2)$	R S T(3)
9	N P (2)	$z+\alpha_2 x=\alpha_u \quad (u=0, 1, 2)$	R S T(2)
10	N K P(1)	$x+\alpha_1 y+\alpha_1 z=\alpha_u (u=0, 1, 2)$	T
11	N K P(2)	$x+\alpha_1 y+\alpha_2 z=\alpha_u (u=0, 1, 2)$	S T (1)
12	N K P(3)	$x+\alpha_2 y+\alpha_1 z=\alpha_u (u=0, 1, 2)$	R S T(4)
13	N K P(4)	$x+\alpha_2 y+\alpha_2 z=\alpha_u (u=0, 1, 2)$	R T (1)

Now let the 1st pencil denote a new main effect R and the 4th, the main effect S. Then, by the principle of generalized interaction, the 5th and 2nd pencils will represent the interactions RS(1) and RS(2) respectively. Finally, take the pencil (10) to represent the third main effect T, so that the 13th and 6th pencils represent the interactions RT(1) and RT(2) respectively and the 11th and 3rd pencils the interactions ST(1) and ST(2). Finally, from the generalized interactions of T with RS(1) and RS(2), we find that the 7th, 9th, 8th and 12th pencils represent respectively RST(1), RST(2), RST(3) and RST(4). The corresponding change in the nomenclature of treatment combinations may be easily effected.

(3·3) We now consider the special case of a s^m factorial design, where $s=2$, and proceed to demonstrate that the principle of generalized interaction, as defined by Barnard[1] in the case of the 2^m design, comes out as a particular case of our more generalized definition. The following are the equations to the pencils representing the various main effects and interactions:

TABLE 3. EQUATIONS TO PENCILS CORRESPONDING TO MAIN EFFECTS AND INTERACTIONS IN A 2^m DESIGN

Main effects and interactions	Equations to pencils
A_1	$x_1 = 0, 1$
A_2	$x_2 = 0, 1$
$\vdots$	$\vdots$
A_m	$x_m = 0, 1$
$A_1 \; A_2$	$x_1 + x_2 = 0, 1$
$A_1 \; A_3$	$x_1 + x_3 = 0, 1$
$\vdots \quad \vdots$	$\vdots$
$A_{m-1} \; A_m$	$x_{m-1} + x_m = 0, 1$
$A_1 \; A_2 \; A_3$	$x_1 + x_2 + x_3 = 0, 1$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$A_1 \; A_2 \; A_3 \; A_4$	$x_1 + x_2 + x_3 + x_4 = 0, 1$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$A_1 \; A_2 \ldots\ldots A_m$	$x_1 + x_2 + \ldots\ldots + x_m = 0, 1$

Let us consider the generalized interaction of the two pencils

$$x_1 + x_2 + \ldots\ldots\ldots\ldots + x_r = 0, 1 \quad (r < m) \qquad \ldots \quad (25)$$

$$x_{r+1} + x_{r+2} + \ldots\ldots \ldots\ldots + x_p = 0, 1 \quad (p \leqslant m) \qquad \ldots \quad (26)$$

This is immediately seen to be

$$x_1 + x_2 + \ldots\ldots\ldots + x_r + x_{r+1} + \ldots\ldots \ldots x_p = 0, 1 \qquad \ldots \quad (27)$$

Thus it appears that the interaction of $A_1 \, A_2 \ldots\ldots A_r$ with $A_{r+1} \, A_{r+2} \ldots\ldots\ldots A_p$ is $A_1 \, A_2 \ldots\ldots\ldots A_p$, in accordance with Barnard's definition. Again, consider the two pencils

$$x_1 + x_2 + \ldots\ldots\ldots x_r = 0, 1 \qquad \ldots \quad (28)$$

$$x_1 + x_2 + \ldots\ldots\ldots x_r + \ldots\ldots\ldots + x_{r+p} = 0, 1 \qquad \ldots \quad (29)$$

where $0 < r$, $0 < p$, $r + p \leq m$.

Their generalized interaction is given by the pencil

$$x_{r+1} + x_{r+2} + \ldots\ldots\ldots + x_{r+p} = 0, 1 \qquad \ldots \quad (30)$$

From this it follows that the generalized interaction of $A_1 A_2 \ldots\ldots\ldots A_r$ and $A_1 A_2 \ldots\ldots A_{r,p}$ is given by $A_{r+1} A_{r+2} \ldots\ldots\ldots A_{r+p}$, in accordance with the definition given by Barnard. From this and the preceding results, it appears that the definition of the generalized interaction in the general case of a s^m factorial arrangement reduces, when s is equal to 2, to that given by Barnard[1] in the case of the 2^m design.

§4. Formation of Confounded Arrangements in a s^m Design in Blocks of s^{m-1} Plots

The principle of generalized interaction may now be utilized in enumerating the various possible types of confounding by studying the relation in which 0-flats, 1-flats, 2-flats, 3-flats,.........$(m-2)$-flats in the $(m-1)$-flat at infinity stand to the fundamental simplex.

Each $(m-2)$-flat in the $(m-1)$-flat at infinity constitutes the vertex of a parallel pencil of s $(m-1)$-flats, which, as already seen above, partition the s^m treatment combinations into s sets of s^{m-1} treatment combinations each. Thus to each $(m-2)$-flat at infinity are associated $s-1$ degrees of freedom. If it constitutes one of the m $(m-2)$-cells of the fundamental simplex, the $s-1$ degrees of freedom belong to a main effect. If, however, it is one of the ${}_mC_2(s-1)$ $(m-2)$-flats representing the first order interaction, passing through the $(m-3)$-cells of the fundamental simplex, the $s-1$ degrees of freedom are clearly those of a first order interaction. In general, the degrees of freedom belong to a k-th order interaction if the given $(m-2)$-flat is one of the k-th order $(m-2)$-flats, ${}_mC_{k+1}$ $(s-1)^k$ in number, passing through the ${}_mC_{k+1}$ $(m-k-2)$-cells of the simplex. The number of $(m-2)$-flats in the $(m-1)$-flat at infinity being $\frac{s^m-1}{s-1}$, these are also the different number of ways in which confounded arrangements in blocks of s^{m-1} plots may be formed, confounding $s-1$ degrees of freedom.

Consider now a $(m-3)$-flat at infinity. This is determined by the intersection of two $(m-2)$-flats at infinity, which are the vertices of two pencils of $(m-1)$-flats, determining by their intersection s^2 $(m-2)$-flats in the finite space having the given $(m-3)$-flat at infinity for vertex. As the number of finite points in a flat in P G (m,s) is s^{m-2}, this pencil of $(m-2)$-flats partitions the s^m treatment combinations into s^2 sets of s^{m-2} treatments each, so that to each $(m-3)$-flat at infinity are associated s^2-1 degrees of freedom, 2 $(s-1)$ of which are associated with the two $(m-2)$-flats which fix the $(m-3)$-flat, and the rest with the other $(m-2)$-flats, $s-1$ in number, passing through the $(m-3)$-flat. This also follows from the consideration that the two sets of $s-1$ degrees of freedom associated with the two $(m-2)$-flats at infinity which fix the given $(m-3)$-flat at infinity, determine, by their generalized interaction, $(s-1)^2$ other degrees of freedom associated with the other $(m-2)$-flats at infinity, $s-1$ in number, concurrent with the initial $(m-2)$-flats. The main effects or interactions to which the $s+1$ sets, each of $s-1$ degrees of freedom, belong may be determined by considering, as before, the nature of the $(s+1)$ $(m-2)$-flats in relation to the fundamental simplex. In this case, the number of ways in which different confounded arrangements in blocks of s^{m-2} plots may be obtained is $\frac{(s^m-1)\ (s^{m-1}-1)}{(s^2-1)\ (s-1)}$.

In general, consider a $(m-k-1)$-flat at infinity and the degrees of freedom associated with it. It is fixed as the common $(m-k-1)$-flat of intersection of k independent $(m-2)$-flats at infinity. The pencils of $(m-1)$-flats corresponding to these intersect in the finite plane in s^k $(m-k)$-flats which have the given flat at infinity for vertex. As these constitute the totality of $(m-k)$-flats with the given $(m-k-1)$-flat at infinity as vertex, to each of these $(m-k-1)$-flats at infinity are associated s^k-1 degrees of freedom given by the contrast between the s^k sets of s^{m-k} treatment combinations into which the s^m treatments are split up. Of these, $k(s-1)$ degrees of freedom belong to the main effects or interactions corresponding to the initial k $(m-2)$-flats at infinity, and the remaining $s^k-1-k(s-1)$ degrees of freedom to the main effects or interactions determined by the generalized interactions in their entirety of the k initial main effects or interactions. It appears, therefore, that for the formation of confounded arrangements in the case of a s^m design in s^k sub-blocks, we have to look for a particular $(m-k-1)$-flat at infinity and set down the s^{m-k} treatments occurring in each of the s^k finite $(m-k)$-flats having the given $(m-k-1)$-flat at infinity as vertex. The nature of confounding thus effected would, as above, be deducible from considering the relation in which the totality of the $(m-2)$-flats at infinity, $\dfrac{s^k-1}{s-1}$ in number, passing through the given $(m-k-1)$-flat at infinity stand in relation to the fundamental simplex. Also the number of different ways of dividing up a replication with s^m treatment combinations into s^k sub-blocks is evidently equal to the number of $(m-k-1)$-flats in the $(m-1)$-flat at infinity, which is N, where,

$$N = \frac{(s^m-1)\,(s^{m-1}-1)\,\ldots\ldots\ldots\ldots\,(s^{m-k+1}-1)}{(s^k-1)\,(s^{k-1}-1)\,\ldots\ldots\ldots\ldots\,(s-1)}$$

The totality of the number of ways of getting a s^m design arranged in s^{m-k} plot blocks may be divided up into a number of classes in accordance with the types of the $(m-k-1)$-flats at infinity in relation to the fundamental simplex, each of these different types leading to one particular type of confounding. Among these, the best sets of treatment comparisons which may profitably be confounded are those in which the main effects and first order interactions are affected as little as possible, and will correspond to the $(m-k-1)$-cells, if any, which pass clear of the fundamental simplex (*cf.* § 2). For convenience, we call these the *clear* $(m-k-1)$-cells.

§5. Possible Types of Confounding for s^3 and s^4 Designs in Blocks of s^2 and s Plots

We now proceed to utilize the artifice developed in the preceding article to enumerate all the different types of confounding effected when a s^3 design is arranged in s^2 and s-plot blocks, and also when a s^4 design is arranged in blocks of the same two different sizes. Attention here will only be confined to these, consideration of s^5 designs in s^3-, s^2-, and s-plot blocks, etc., being reserved for a subsequent communication.

(5·1) s^3 *in* s^2*-plot blocks*: In this case, the fundamental simplex reduces to a triangle on the plane at infinity. To each of the s^2+s+1 lines on the plane at infinity will be associated $s-1$ degrees of freedom given by the contrast between the s sets of s^2 treatment combinations each into which the pencil of planes with any of these lines for vertex partitions the s^3 treatment combinations. Clearly, the $s-1$ degrees of freedom may belong to any of the main effects, first order or second order interactions according as the line constitutes one of the sides of the triangle, passes through one of the vertices or goes clear of the fundamental triangle. There are $(s-1)^2$ *clear* lines, pencils corresponding to which give degrees of freedom for the second order interaction.

(5·2) s^3 *in* s*-plot blocks*: Each of the s^2+s+1 points in the plane at infinity has s^2-1 degrees of freedom associated with it, given by the contrast between the s^2 sets of s treatment combinations each into which the s^3 treatment combinations are split up by the pencil of s^2 finite lines having any one of the given points for vertex. Now the types of points in relation to the fundamental simplex are: (*i*) Those constituting any of the three vertices of the fundamental triangle, (*ii*) Those lying on any of its three sides, and (*iii*) Clear points. The types of confounding corresponding to these have been presented in Table 4.

TABLE 4. ENUMERATION OF TYPES OF CONFOUNDING FOR A s^3 DESIGN IN s-PLOT BLOCKS

Nature of points	Number	Nature of Confounding			Total
		Main effects	First order interactions	Second order interactions	
Vertices	3	2	$s-1$	...	$s+1$
Points lying on the sides ...	$3(s-1)$	1	1	$s-1$	$s+1$
Clear points	$(s-1)^2$	...	3	$s-2$	$s+1$

(5·3) s^4 *in* s^2*-plot blocks*: In this case, the fundamental simplex is a tetrahedron at infinity, and the possibilities of the different ways of confounding of a s^4 factorial design in s^2-plot blocks are easily deduced by studying the relation of the totality of lines in the hyperplane at infinity to the fundamental tetrahedron.

The total number of lines in the hyperplane at infinity is $(s^2+1)(s^2+s+1)$. These divide themselves up into the following seven classes: (*i*) Lines constituting the edges, 6 in number; (*ii*) Lines lying on a face and passing through a vertex, $12(s-1)$ in number; (*iii*) Lines lying on a face but not through a vertex, their number being $4(s-1)^2$; (*iv*) Lines through a vertex but not on a face, $4(s-1)^2$ in number; (*v*) Lines intersecting only a single edge, their number being $6(s-1)^2$; (*vi*) Lines intersecting two edges but not lying on a face, $3(s-1)^2$ in number, and (*vii*) Clear lines, their number being $(s-1)^2(s-2)$. The annexed table shows the types of confounding corresponding to each of these different types of lines.

TABLE 5. ENUMERATION OF TYPES OF CONFOUNDING FOR A s^4 DESIGN IN s^2-PLOT BLOCKS

Nature of lines	Number	Nature of Confounding				Total
		Main effects	First order interactions	Second order interactions	Third order interactions	
Edges	6	2	$s-1$	...	...	$s+1$
Lines lying on a face and through a vertex	$4 \cdot 3(s-1)$	1	1	$s-1$	...	$s+1$
Lines lying on a face but not through a vertex	$4(s-1)^2$	1	...	1	$s-1$	$s+1$
Lines through a vertex, not on face	$4(s-1)^2$	...	3	$s-2$	...	$s+1$
Lines intersecting a single edge ...	$6(s-1)^3$	...	1	2	$s-2$	$s+1$
Lines intersecting two edges and not on a face	$3(s-1)^2$	...	2	...	$s-1$	$s+1$
Clear lines	$(s-1)^3(s-2)$	..	...	4	$s-3$	$s+1$

(5·4) *s^4 in s-plot blocks*: The different possibilities of confounding s^3-1 degrees of freedom by dividing up s^4 treatment combinations in s^3 sub-blocks, each of s plots, will now be given by the relation which the totality of s^3+s^2+s+1 points in the hyperplane at infinity bear to the fundamental tetrahedron. There are only four types of points: (*i*) Those constituting the vertices, 4 in number; (*ii*) Those lying on edges $6(s-1)$ in number; (*iii*) Those lying on the faces, being $4(s-1)^2$ in number; and, finally, (*iv*) Clear points, $(s-1)^3$ in number. In Table 6 are given the different types of confounding corresponding to each of the above types of points.

TABLE 6. ENUMERATION OF TYPES OF CONFOUNDING FOR s^4 IN s-PLOT BLOCKS

Nature of points	Number	Nature of Confounding				Total
		Main effects	First order interactions	Second order interactions	Third order interactions	
Vertices	4	3	$3(s-1)$	$(s-1)^2$	...	s^2+s+1
Points lying on edges ...	$6(s-1)$	2	s	$2(s-1)$	$(s-1)^2$	s^2+s+1
Points lying on plane faces	$4(s-1)^2$	1	3	$4s-5$	$(s-1)(s-2)$	s^2+s+1
Clear points	$(s-1)^3$	...	6	$4(s-2)$	s^2-3s+3	s^2+s+1

The problem of constructing balanced arrangements in the above and other cases will be investigated in detail in a separate note to be shortly released for publication.

SUMMARY

The problem of confounding in the general symmetrical type of experiment $s \times s \times s \times$ ....$(=s^m)$, where s is a prime positive integer or a power of a prime and m any positive integer, has been considered in this paper and the important principle of generalized interaction in the s^m factorial arrangement has been enunciated,

Barnard's definition of the generalized interaction in the 2^m factorial design being shown to be a particular case of our general definition when $s=2$. This principle has been utilized in demonstrating a general method of forming confounded arrangements in a s^m design in s^k sub-blocks, each of s^{m-k} plots, and the s^3 and s^4 designs have been discussed in their entirety to serve as concrete illustrations, the consideration of other particular cases being reserved for a subsequent communication.

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