ISITRA: A generalized way of signal decomposition and reconstruction

Yumnam Kirani Singh, Swapan Kumar Parui *

Computer Vision and Pattern Recognition Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700108, India

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Abstract

A new generalized way of signal decomposition and reconstruction entitled ISITRA is proposed. It is similar to the 2-channel filter bank scheme. In ISITRA, all the filters are obtained from a real vector. ISITRA allows decomposition of a signal into (1) an approximation and a detail, (2) two details and (3) two approximations. The latter two cases are not generally possible in the filter bank scheme. The choice of filter coefficients in ISITRA is much simpler and more arbitrary compared to that in the existing schemes. This allows one to find better filter coefficients for different applications. One can straight achieve an image compression ratio of 8:1 without doing any coding by modifying the range of pixel values in the decomposed components. One can also find a better set of decomposition and reconstruction filters than the commonly used Daubechies' wavelet filters of length 4. ISITRA is simpler and computationally marginally better than even the computationally efficient polyphase filter bank scheme.

Keywords: Signal decomposition; Perfect reconstruction; Polyphase scheme; Convolution; Filter banks; Wavelets; Multi-resolution analysis; Compression

1. Introduction

Processing of signals is done in various ways according to our need. Fourier analysis has long been used as a classical tool for the signal analysis purpose. The drawback of

* Corresponding author. Fax: +91 33 2577 3035.
E-mail address: swapan@www.isical.ac.in (S.K. Parui).

Fourier analysis is its unsuitability of analyzing transient signals because of lack of time resolution in frequency domain. This led to the development of a small windowed function, known as Gabor transform in which mainly a Gaussian function is taken as a window function and its size is fixed. Gabor transform has poor performance as regards resolution because of its fixed window size. Then came signal analysis using variable windows with the development of wavelets and filter bank. In both cases the window size is varying because of the so-called multi-resolution technique. The logic behind the wavelet transform is to find a set of filter coefficients for decomposition and reconstruction of a signal from a pre-determined function known as wavelet. The history of wavelet is much older but it became popular for various application areas only after 1988, when Ingrid Daubechies [4] constructed compactly supported orthonormal wavelet using QMF (quadrature mirror filter) [3] properties of filter bank that led to the development of the current discrete wavelet transform (DWT).

In the strict sense, DWT is not a direct discrete version of the continuous wavelet transform (CWT) which Grossman and Morlet [7] introduced for multi-resolution signal representation through dilation and translation of a basis function known as mother wavelet. In DWT, there are two basis functions respectively known as scaling function (or father wavelet) and wavelet function (or mother wavelet). It seems to be a mystery why scaling function would be required in DWT when there is no requirement of the same in CWT. Also, the term "scaling function" is a misnomer, as the function has nothing to do with scaling operation. Another mystery in DWT is the issue of inversibility or perfect reconstruction. Theoretically, perfect reconstruction of CWT is given on admissibility condition [4], even though it is not possible to get perfect reconstruction in CWT in practice. What makes the perfect reconstruction possible in DWT when the same is not possible in CWT? Because the wavelet filters are of QMF type which is a sufficient condition for perfect reconstruction. So, the obvious reason for introducing scaling function in DWT is to establish a relation between wavelet and filter bank, which at the same time would allow perfect reconstruction. The main advantage of such an establishment is the possibility of finding short length perfect reconstruction wavelet filters in the filter bank scheme. We investigate the conditions for perfect reconstruction in 2-channel filter bank scheme and find them to be dependent on how reconstruction is performed. That is, different reconstruction schemes give rise to different conditions for perfect reconstruction. The existing perfect reconstruction theory of 2-channel filter bank is also one possible way of reconstruction. The transform ISITRA (Indian Statistical Institute Transform) proposed in this paper provides a generalized model for a subband decomposition and reconstruction scheme.

ISITRA is a new generalized transform for signal decomposition and reconstruction. The decomposition procedure is similar to that of DWT. But unlike in the case of DWT or filter bank decomposition scheme where a signal is always decomposed into an approximation and a detail, here a signal can be decomposed into an approximation and a detail or into two details or into two approximations. By "an approximation" we mean a decomposed component having less local variations in the values of its data elements (usually obtained with a low pass filter) and by "a detail" we mean such a component having high local variations in the values of its data elements (usually obtained with a high pass filter). Also, the coefficients of the filters can be chosen quite arbitrarily, a unique feature of ISITRA. There is a general condition known as the member condition to be satisfied by a decomposition filter and its corresponding reconstruction filter to ensure perfect reconstruction. We call a set of decomposition filters and their corresponding reconstruction filters satisfying the member condition, a PRF (perfect reconstruction filter) set. Whether a PRF set is of QMF type or not depends on how one chooses the filter coefficients to form a framework of a PRF set. A PRF set that can be obtained from a non-zero vector is known as a PRF set of ISITRA. We will discuss in detail PRF sets of lengths 2 and 4 of ISITRA and give a generalization of it so that longer PRF sets can also be found. In this paper, bold letters denote vectors or matrices and others scalars.

Computationally, ISITRA is significantly better than the direct 2-channel filter bank scheme and is marginally better than the computationally efficient polyphase filter bank scheme. The rest of the paper is divided into five sections. Section 2 briefly describes the decomposition schemes of filter banks and ISITRA and makes a comparative study of their computational efficiencies. The decomposition and reconstruction schemes of ISITRA are described in Section 3. The properties and advantages of ISITRA are discussed with examples in Section 4. A discussion is given in Section 5 and conclusions are given in Section 6.

2. Filter bank scheme and ISITRA

We will briefly discuss here the single stage 2-channel filter bank scheme which may be useful for a better understanding of the proposed transform ISITRA. In the filter bank scheme, a signal is decomposed into two components using a high pass filter and a low pass filter. Then each of the filtered outputs is down sampled by a factor of 2 as shown in Fig. 1a, to give the outputs. The decomposition filters need to be of QMF or CQF (conjugate quadrature filters) type to ensure perfect reconstruction. The down-sampling operations performed in the decomposition process are useful for data reduction. In most of the literature on filter bank, it is mentioned that the down-sampling introduces an aliasing effect. But that is not the case as explained in [10] and later in Section 3. The process of down-sampling is required for both perfect reconstruction and multi-resolution analysis. The decomposition scheme in Fig. 1a performs computation of unnecessary terms in the filtering operations, which would be simply discarded in the succeeding down-sampling process. As filter bank is usually studied using z-transform, signals and filters are given in z-domain.

Suppose the input signal $X(z)$ is a polynomial of degree $m-1$ and the decomposition filters are polynomials of degree $L-1$. The arithmetic operations in the computation of a filtered output are $(L+m-1)L$ multiplication and $(L+m-1)(L-1)$ addition op-

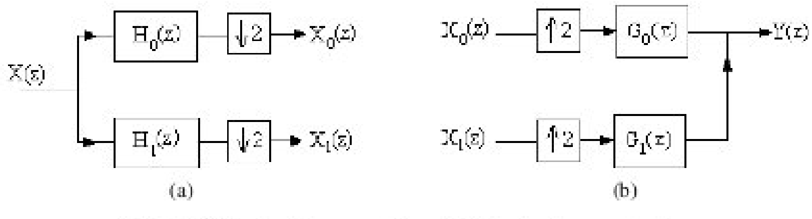

Fig. 1. (a) Filter bank decomposition. (b) Filter bank reconstruction.

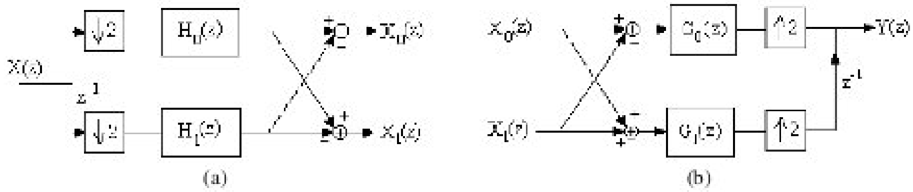

Fig. 2. (a) Polyphase decomposition. (b) Polyphase reconstruction.

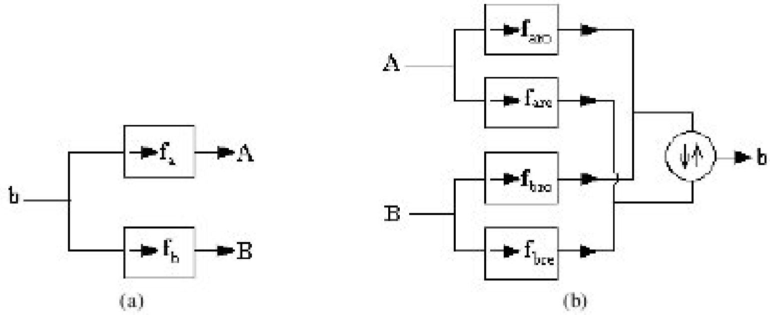

Fig. 3. (a) ISITRA decomposition. (b) ISITRA reconstruction.

erations. In the reconstruction process, the decomposed components are up-sampled by 2 as in Fig. 1b, thus making their length $L + m - 1$ before filtering with the reconstruction filters. The total numbers of multiplication and addition operations in the reconstruction process are $(2L + m - 2)2L$ and $(2L + m - 2)(2L - 1)$, respectively.

The computationally efficient polyphase scheme [2,6] of 2-channel filter bank is shown in Fig. 2. In the decomposition process, the signal is first divided into two, even and odd, components using two down-samplers, one coupled with a delay operator. Here, the total number of addition (as well as multiplication) operations needed in the computation of $X_0(z)$ or $X_1(z)$ is $(L + m/2 - 1)L$. During reconstruction, the decomposed components of length $L + m/2 - 1$ are first added and subtracted and then passed through the reconstruction filters to give outputs of length $2L + m/2 - 2$. The computation of the reconstructed signal $Y(z)$ requires $4L + m - 4$ multiplications and as many additions.

ISITRA is similar to the filter bank scheme in its decomposition scheme as shown in Fig. 3a. It is studied in discrete time domain. A signal vector $\mathbf{b}$ of length m is decomposed into two component vectors $\mathbf{A}$ and $\mathbf{B}$ using the decomposition filter coefficients $\mathbf{f}_a$ and $\mathbf{f}_b$ of length L. The arrow symbol that precedes the filter denotes advancing of one index which avoids performing extra multiplication operations in the filtering process. In other words, we skip alternate data points in the computation of a decomposed component in the filtering process. The length of each decomposed component then is $m/2 + L/2 - 1$ and hence the computation of a decomposed component requires $L(m/2 + L/2 - 1)$ multiplications and $(L - 1)(m/2 + L/2 - 1)$ additions. The computational advantage, though marginal, will be more for longer signal and decomposition filters. Also, no down-sampling is required, it is done in the filtering process itself.

Table 1
Number of multiplication and addition operations in different subband schemes

Subband scheme	Total number of		Extra operations required
	Multiplications	Additions	
1. Direct FB	$2L(3L+2m-3)$	$2L(3L+2m-5)-3m+4$	Up-sampling, down-sampling
2. Polyphase	$2L(3L+m-1)+m-4$	$2L(3L+m-2)+m-2$	Up-sampling, down-sampling, delay operators
3. ISITRA	$2L(L+m-2)$	$2(L-1)(L+m-2)$	None

The reconstruction scheme of ISITRA, shown in Fig. 3b, also requires marginally less computation. Here, we do not up sample the decomposed components. Instead, we do the filtering operation with only the halves of the reconstruction filters. Let $\mathbf{f}_{\mathrm{aro}}$ denote the odd parts and $\mathbf{f}_{\mathrm{are}}$, the even parts of the reconstruction filter $\mathbf{f}_{\mathrm{ar}}$. Similar is the case for the other reconstruction filter $\mathbf{f}_{\mathrm{br}}$. Addition of the outputs of the two odd filters gives the odd indexed components of the reconstructed signal. And addition of the outputs of the two even filters gives the even indexed components of the reconstructed signal. It can be seen that the numbers of multiplication and addition operations required in the computation of odd components or even components of the reconstructed signal are respectively $L(m/2+L/2-1)$ and $(L-1)(m/2+L/2-1)$. The odd and even components are simply intermerged to obtain the reconstructed signal. It is denoted by the symbol ⊕ in the figure. In actual computation this requires no arithmetic operation. A summary of the overall arithmetic operations required during decomposition and reconstruction processes in direct filter bank, polyphase and ISITRA schemes is given in Table 1. As compared to the polyphase filter bank scheme, ISITRA is simpler and requires marginally less amount of computation.

In both filter banks (direct and polyphase), the equality of $X(z)$ and $Y(z)$ depends on how the decomposition filters $H_0(z)$, $H_1(z)$ and the reconstruction filters $G_0(z)$, $G_1(z)$ are chosen. With a proper choice of decomposition and reconstruction filters, it is possible to make $X(z)$ equal to $Y(z)$. In that case, the corresponding filter bank scheme is known as perfect reconstruction filter bank. Several researchers worked on the perfect reconstruction theory of filter banks. Mention may be made of Akansu and Hadad [1], Smith and Barnwell [12], Mintzer [9], Vaidyanathan et al. [13–15], and Vetterli et al. [16–18].

Later on scientists working on wavelets established a relation between wavelet theory and filter bank [4,5,8]. For that purpose, they introduced an extra function known as scaling function to correspond to the low pass filter in filter bank. With this introduction the inversible discrete wavelet transforms can be developed based on the perfect reconstruction theory of filter bank. Perfect reconstruction theory of 2-channel filter bank involves reconstruction filters. However, reconstruction filters are not explicitly studied regarding their roles in perfect reconstruction in DWT, nor any explicit wavelet functions are mentioned that would correspond to the reconstruction filters. There are certain conditions to be satisfied by wavelet filters. The decomposition filters $\mathbf{h}_0$, corresponding to the scaling function, and $\mathbf{h}_1$, corresponding to wavelet function, should satisfy the following conditions:

$$\begin{cases} \sum \mathbf{h}_0(n) = \sqrt{2}, \\ \sum \mathbf{h}_1(n) = 0. \end{cases} \tag{1}$$

Condition (1) alone is neither a necessary condition nor a sufficient condition for perfect reconstruction. For orthogonal wavelet filters, the following relation is required to ensure perfect reconstruction:

$$\begin{cases} \mathbf{h}_1(n) = (-1)^n \mathbf{h}_0(L+1-n), \\ \mathbf{g}_0(n) = \mathbf{h}_0(L+1-n), \\ \mathbf{g}_1(n) = \mathbf{h}_1(L+1-n). \end{cases} \tag{2}$$

Condition (2) is generally known as QMF relation in filter bank community [1,6,14]. Usually the first equation in condition (2) is used to specify the QMF relation. But for perfect reconstruction, the first equation is not sufficient. The relation of the reconstruction filters with the decomposition filters is also required and hence the reconstruction filters $\mathbf{g}_0$ and $\mathbf{g}_1$ are also given in condition (2).

The conditions given above are sufficient conditions to get perfect reconstruction in 2-channel filter bank scheme. But they are not a necessary condition for perfect reconstruction. In Section 3, we will find various decomposition and reconstruction filters which do not satisfy the above conditions but give perfect reconstruction in ISITRA. Also, it is very difficult to find filters that satisfy the above conditions as none of the above relations suggests anything about how to find perfect reconstruction filters. We summarize the drawbacks of the filter bank or DWT as follows:

(1) There is no rational justification for why filters need to be of QMF type and why there are always to be one approximation and one detail decomposed components.
(2) There is no exact mathematical expression for a decomposed component in time domain.
(3) Finding perfect reconstruction filters is not straightforward.
(4) Perfect reconstruction issues are not well addressed. There are several possible ways of perfect reconstruction in the 2-channel filter bank. Only one of them is addressed.
(5) The condition for perfect reconstruction usually studied in z-domain is not very helpful for finding filter coefficients.

ISITRA takes care of these drawbacks [11] which will also be clear from the following sections.

3. Decomposition and reconstruction schemes in ISITRA

Let $\mathbf{b} = [b_1, b_2, b_3, \ldots, b_m]$ be a discrete signal and $\mathbf{f}_a$ and $\mathbf{f}_b$ be the decomposition filters. Let us consider the following decomposition:

$$\mathbf{A}(n) = \sum_{i=1}^{L} \mathbf{f}_a(i)\mathbf{b}(2n+1-i), \tag{3}$$

$$\mathbf{B}(n) = \sum_{i=1}^{L} \mathbf{f}_b(i)\mathbf{b}(2n+1-i) \tag{4}$$

for $n = 1, 2, \ldots, M$, where $\mathbf{A}$ and $\mathbf{B}$ are the decomposed components of $\mathbf{b}$ and L, an even integer, is the length of the filter and the length of each of the decomposed components is $M = \lceil m/2 \rceil + L/2 - 1$. If we decompose a signal into two components using filter of length 2, then the length of each of the decomposed components will be $\lceil m/2 \rceil$.

Equations (3) and (4) are the general decomposition equations. The choice of values of the decomposition filters depends on the desired nature of its decomposed components. Using appropriate reconstruction filters $\mathbf{f}_{\mathrm{ar}}$ and $\mathbf{f}_{\mathrm{br}}$, reconstruction of the original signal from the decomposed components is possible. How this is done is described below.

3.1. PRF2 sets of ISITRA

Let us first consider the decomposition and reconstruction schemes of ISITRA with filters of length 2. The decomposition equations obtained from Eqs. (3) and (4) by putting $L = 2$ are

$$\mathbf{A}(n) = \mathbf{f}_{\mathrm{a}}(1)\mathbf{b}(2n) + \mathbf{f}_{\mathrm{a}}(2)\mathbf{b}(2n-1), \tag{5}$$

$$\mathbf{B}(n) = \mathbf{f}_{\mathrm{b}}(1)\mathbf{b}(2n) + \mathbf{f}_{\mathrm{b}}(2)\mathbf{b}(2n-1). \tag{6}$$

The reconstruction scheme using Fig. 3b can be written as

$$\mathbf{f}_{\mathrm{ar}}(1)\mathbf{A}(n) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{B}(n) = \big\{\mathbf{f}_{\mathrm{ar}}(1)\mathbf{f}_{\mathrm{a}}(1) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{f}_{\mathrm{b}}(1)\big\}\mathbf{b}(2n) + \big\{\mathbf{f}_{\mathrm{ar}}(1)\mathbf{f}_{\mathrm{a}}(2) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{f}_{\mathrm{b}}(2)\big\}\mathbf{b}(2n-1), \tag{7}$$

$$\mathbf{f}_{\mathrm{ar}}(2)\mathbf{A}(n) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{B}(n) = \big\{\mathbf{f}_{\mathrm{ar}}(2)\mathbf{f}_{\mathrm{a}}(1) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{f}_{\mathrm{b}}(1)\big\}\mathbf{b}(2n) + \big\{\mathbf{f}_{\mathrm{ar}}(2)\mathbf{f}_{\mathrm{a}}(2) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{f}_{\mathrm{b}}(2)\big\}\mathbf{b}(2n-1). \tag{8}$$

From Eqs. (7) and (8) we can solve for $\mathbf{b}(2n)$ and $\mathbf{b}(2n-1)$ in two different ways.

First method. Putting

$$\mathbf{f}_{\mathrm{ar}}(1)\mathbf{f}_{\mathrm{a}}(2) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{f}_{\mathrm{b}}(2) = 0 \tag{9}$$

in Eq. (7), we can write

$$\mathbf{b}(2n) = \frac{\mathbf{f}_{\mathrm{ar}}(1)\mathbf{A}(n) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{B}(n)}{\mathbf{f}_{\mathrm{ar}}(1)\mathbf{f}_{\mathrm{a}}(1) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{f}_{\mathrm{b}}(1)} \tag{10}$$

provided

$$\mathbf{f}_{\mathrm{ar}}(1)\mathbf{f}_{\mathrm{a}}(1) + \mathbf{f}_{\mathrm{br}}(1)\mathbf{f}_{\mathrm{b}}(1) \neq 0. \tag{11}$$

Again putting

$$\mathbf{f}_{\mathrm{ar}}(2)\mathbf{f}_{\mathrm{a}}(1) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{f}_{\mathrm{b}}(1) = 0 \tag{12}$$

in Eq. (8), we can write

$$\mathbf{b}(2n-1) = \frac{\mathbf{f}_{\mathrm{ar}}(2)\mathbf{A}(n) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{B}(n)}{\mathbf{f}_{\mathrm{ar}}(2)\mathbf{f}_{\mathrm{a}}(2) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{f}_{\mathrm{b}}(2)} \tag{13}$$

provided

$$\mathbf{f}_{\mathrm{ar}}(2)\mathbf{f}_{\mathrm{a}}(2) + \mathbf{f}_{\mathrm{br}}(2)\mathbf{f}_{\mathrm{b}}(2) \neq 0. \tag{14}$$

If we find $\mathbf{f}_a$, $\mathbf{f}_b$, $\mathbf{f}_{ar}$ and $\mathbf{f}_{br}$ of length 2 such that conditions (9), (11), (12) and (14) are satisfied, we would be able to get back the original signal $\mathbf{b}$ from its decomposed components. Equations (9) and (12) together are known as member condition I and inequalities (11) and (14) are known as member condition II. Member conditions I and II together are known as the member condition. Any set $\{\mathbf{f}_a, \mathbf{f}_b, \mathbf{f}_{ar}, \mathbf{f}_{br}\}$ whose members satisfy the member condition will be known as a perfect reconstruction filter (PRF) set. Since the members of a PRF set here are of length 2, it is called a PRF2 set. If the members of a PRF set are of length L, it is referred to as a PRFL set. We can easily find numerous PRF sets that satisfy the above member condition. For example, for any non-zero vector $\mathbf{f} = [f_1, f_2]$ of any two non-negative real numbers f_1 and f_2, the vectors $\mathbf{f}_a = [f_1, f_2]$, $\mathbf{f}_b = [f_1, -f_2]$, $\mathbf{f}_{ar} = \mathbf{f}_a$ and $\mathbf{f}_{br} = \mathbf{f}_b$ form a PRF2 set. Also, we can see in what follows that the elements of vectors in a PRF2 set $\{\mathbf{f}_a, \mathbf{f}_b, \mathbf{f}_{ar}, \mathbf{f}_{br}\}$ can be obtained by simply changing the sign and/or the order of the elements of the vector $\mathbf{f}$. The above PRF2 set is not used in the wavelet and filter bank community because of the difference in the reconstruction process.

However, we may find PRF2 sets which have similar characteristics as the ones used in discrete wavelet or filter bank scheme, using the reconstruction scheme given below.

Second method. Putting

$$\mathbf{f}_{ar}(1)\mathbf{f}_a(1) + \mathbf{f}_{br}(1)\mathbf{f}_b(1) = 0 \tag{15}$$

in Eq. (7), we can write

$$\mathbf{b}(2n-1) = \frac{\mathbf{f}_{ar}(1)\mathbf{A}(n) + \mathbf{f}_{br}(1)\mathbf{B}(n)}{\mathbf{f}_{ar}(1)\mathbf{f}_a(2) + \mathbf{f}_{br}(1)\mathbf{f}_b(2)} \tag{16}$$

provided

$$\mathbf{f}_{ar}(1)\mathbf{f}_a(2) + \mathbf{f}_{br}(1)\mathbf{f}_b(2) \neq 0. \tag{17}$$

Again, putting

$$\mathbf{f}_{ar}(2)\mathbf{f}_a(2) + \mathbf{f}_{br}(2)\mathbf{f}_b(2) = 0 \tag{18}$$

in Eq. (8), we can write

$$\mathbf{b}(2n) = \frac{\mathbf{f}_{ar}(2)\mathbf{A}(n) + \mathbf{f}_{br}(2)\mathbf{B}(n)}{\mathbf{f}_{ar}(2)\mathbf{f}_a(1) + \mathbf{f}_{br}(2)\mathbf{f}_b(1)} \tag{19}$$

provided

$$\mathbf{f}_{ar}(2)\mathbf{f}_a(1) + \mathbf{f}_{br}(2)\mathbf{f}_b(1) \neq 0. \tag{20}$$

Equations (15) and (18) form member condition I and inequalities (17) and (20) form member condition II. Our aim here is to find PRF sets that satisfy the member condition (i.e., both member condition I and member condition II). We may find various filters of PRF sets of length 2 which can be used for decomposition and perfect reconstruction of a signal. We consider below only a few of them which may be used for derivation of other PRF sets.

Case 1. $\mathbf{f}_a = [f_1, f_2]$ and $\mathbf{f}_b = [f_1, -f_2]$, where f_1 and f_2 are any two non-zero real numbers. Let us find the corresponding reconstruction filters. From Eq. (15), we have $\mathbf{f}_{ar}(1) = -\mathbf{f}_{br}(1)$. That is, the first elements of $\mathbf{f}_{ar}$ and $\mathbf{f}_{br}$ should have equal magnitude but opposite signs. There is no restriction on their magnitude. Again from Eq. (18), we have $\mathbf{f}_{ar}(2) = \mathbf{f}_{br}(2)$. So the possible reconstruction filters are (1) $\mathbf{f}_{ar} = [f_1, f_2]$, $\mathbf{f}_{br} = [-f_1, f_2]$; (2) $\mathbf{f}_{ar} = [f_1, f_1]$, $\mathbf{f}_{br} = [-f_1, f_1]$; (3) $\mathbf{f}_{ar} = [f_2, f_1]$, $\mathbf{f}_{br} = [-f_2, f_1]$; (4) $\mathbf{f}_{ar} = [f, f]$, $\mathbf{f}_{br} = [-f, f]$ for any non-zero real number f and so on. Also, the member condition II is valid in all the above reconstruction filters. In other words, if we decompose a signal using $\mathbf{f}_a = [f_1, f_2]$ and $\mathbf{f}_b = [f_1, -f_2]$, we can reconstruct the signal from its decomposed components using various reconstruction filters.

Case 2. $\mathbf{f}_a = [f_1, f_2]$ and $\mathbf{f}_b = [-f_2, f_1]$. From Eq. (15), we have $f_1\mathbf{f}_{ar}(1) = f_2\mathbf{f}_{br}(1)$ from which a possible solution is $\mathbf{f}_{ar}(1) = f_2$ and $\mathbf{f}_{br}(1) = f_1$. Again, from Eq. (18), we have $f_2\mathbf{f}_{ar}(2) = -f_1\mathbf{f}_{br}(2)$ from which a possible solution is $\mathbf{f}_{ar}(2) = -f_1$, $\mathbf{f}_{br}(2) = f_2$ or $\mathbf{f}_{ar}(2) = f_1$, $\mathbf{f}_{br}(2) = -f_2$. So two possible reconstruction filters are (1) $\mathbf{f}_{ar} = [f_2, -f_1]$, $\mathbf{f}_{br} = [f_1, f_2]$ and (2) $\mathbf{f}_{ar} = [f_2, f_1]$, $\mathbf{f}_{br} = [f_1, -f_2]$. As in Case 1, the member condition II is valid in both the above reconstruction filters. Here the PRF set is similar to that of the PRF set of QMF type. When $f_1 = f_2 = 1/\sqrt{2}$, the PRF set becomes the PRF set of Haar's wavelet filter, the only known wavelet filter of length 2. We cannot have any values of f_1, f_2 other than $1/\sqrt{2}$ because of the restriction in Eq. (1). But here we can use any two real numbers and use them for decomposition and perfect reconstruction, which is an interesting feature of ISITRA.

Case 3. $\mathbf{f}_a = [f_1, -f_2]$ and $\mathbf{f}_b = [f_2, -f_1]$. We get $\mathbf{f}_{ar} = [-f_2, f_1]$ and $\mathbf{f}_{br} = [f_1, -f_2]$ as one of the four reconstruction filter pairs of the above decomposition filters. In all these cases, the denominator terms are either $f_1^2 - f_2^2$ or $f_2^2 - f_1^2$, implying an additional condition that $f_1 \neq f_2$. Both the decomposed components here are details or high frequency components of the original signal. But the perfect reconstruction is possible since the above choices of decomposition and the reconstruction filters form a PRF set with the condition $f_1 \neq f_2$.

Case 4. $\mathbf{f}_a = [f_1, f_2]$ and $\mathbf{f}_b = [f_2, f_1]$. It is similar to Case 3. However, both the decomposed components here correspond to approximations or low frequency components of the original signal.

As mentioned above there are various ways in which a signal can be decomposed using different filters of length 2. For example, one can choose filter coefficients of length 2, which can be used for decomposition of a signal into two details or into two approximations or into a detail and an approximation depending on the application. The decomposition filters used in Cases 1, 3 and 4 are not of QMF type, and still we can get perfect reconstruction from the decomposed components. Thus, we see that to make perfect reconstruction possible, it is not necessary that the filters used for decomposition and reconstruction be of QMF type.

3.2. PRF4 sets of ISITRA

In this section, we will discuss how various types of PRF4 sets can be obtained for decomposition and perfect reconstruction of a signal.

When $L = 4$, the decomposition equations (3) and (4) can be written as

$$\mathbf{A}(n) = \mathbf{f}_a(1)\mathbf{b}(2n) + \mathbf{f}_a(2)\mathbf{b}(2n-1) + \mathbf{f}_a(3)\mathbf{b}(2n-2) + \mathbf{f}_a(4)\mathbf{b}(2n-3), \tag{21}$$

$$\mathbf{B}(n) = \mathbf{f}_b(1)\mathbf{b}(2n) + \mathbf{f}_b(2)\mathbf{b}(2n-1) + \mathbf{f}_b(3)\mathbf{b}(2n-2) + \mathbf{f}_b(4)\mathbf{b}(2n-3). \tag{22}$$

The method of reconstruction of the original signal from the decomposed components is given below. For simplicity and ease of computation, let us define the following for $L = 4$:

$$\mathbf{A}_1(n) = \sum_{i=1}^{L/2} \mathbf{f}_{ar}(2i-1)\mathbf{A}(n+1-i), \tag{23}$$

$$\mathbf{A}_2(n) = \sum_{i=1}^{L/2} \mathbf{f}_{ar}(2i)\mathbf{A}(n+1-i), \tag{24}$$

$$\mathbf{B}_1(n) = \sum_{i=1}^{L/2} \mathbf{f}_{br}(2i-1)\mathbf{B}(n+1-i), \tag{25}$$

$$\mathbf{B}_2(n) = \sum_{i=1}^{L/2} \mathbf{f}_{br}(2i)\mathbf{B}(n+1-i) \tag{26}$$

for $n = 1, 2, \ldots, M$ (M, as defined earlier, is $\lceil m/2 \rceil + L/2 - 1$). Here, $\mathbf{A}(i)$, $\mathbf{B}(i)$ terms are taken as zero for $i \leqslant 0$.

Simplifying Eqs. (23) and (25), we get

$$\begin{aligned}
\mathbf{A}_1(n+1) + \mathbf{B}_1(n+1) &= \big\{\mathbf{f}_a(1)\mathbf{f}_{ar}(1) + \mathbf{f}_b(1)\mathbf{f}_{br}(1)\big\}\mathbf{b}(2n+2) \\
&+ \big\{\mathbf{f}_a(3)\mathbf{f}_{ar}(1) + \mathbf{f}_a(1)\mathbf{f}_{ar}(3) + \mathbf{f}_b(3)\mathbf{f}_{br}(1) + \mathbf{f}_b(1)\mathbf{f}_{br}(3)\big\}\mathbf{b}(2n) \\
&+ \big\{\mathbf{f}_a(3)\mathbf{f}_{ar}(3) + \mathbf{f}_b(3)\mathbf{f}_{br}(3)\big\}\mathbf{b}(2n-2) + \big\{\mathbf{f}_a(2)\mathbf{f}_{ar}(1) + \mathbf{f}_b(2)\mathbf{f}_{br}(1)\big\}\mathbf{b}(2n+1) \\
&+ \big\{\mathbf{f}_a(4)\mathbf{f}_{ar}(1) + \mathbf{f}_a(2)\mathbf{f}_{ar}(3) + \mathbf{f}_b(4)\mathbf{f}_{br}(1) + \mathbf{f}_b(2)\mathbf{f}_{br}(3)\big\}\mathbf{b}(2n-1) \\
&+ \big\{\mathbf{f}_a(4)\mathbf{f}_{ar}(3) + \mathbf{f}_b(4)\mathbf{f}_{br}(3)\big\}\mathbf{b}(2n-3).
\end{aligned} \tag{27}$$

Similarly, from Eqs. (24) and (26), we have

$$\begin{aligned}
\mathbf{A}_2(n+1) + \mathbf{B}_2(n+1) &= \big\{\mathbf{f}_a(1)\mathbf{f}_{ar}(2) + \mathbf{f}_b(1)\mathbf{f}_{br}(2)\big\}\mathbf{b}(2n+2) \\
&+ \big\{\mathbf{f}_a(3)\mathbf{f}_{ar}(2) + \mathbf{f}_a(1)\mathbf{f}_{ar}(4) + \mathbf{f}_b(3)\mathbf{f}_{br}(2) + \mathbf{f}_b(1)\mathbf{f}_{br}(4)\big\}\mathbf{b}(2n) \\
&+ \big\{\mathbf{f}_a(3)\mathbf{f}_{ar}(4) + \mathbf{f}_b(3)\mathbf{f}_{br}(4)\big\}\mathbf{b}(2n-2) + \big\{\mathbf{f}_a(2)\mathbf{f}_{ar}(2) + \mathbf{f}_b(2)\mathbf{f}_{br}(2)\big\}\mathbf{b}(2n+1) \\
&+ \big\{\mathbf{f}_a(4)\mathbf{f}_{ar}(2) + \mathbf{f}_a(2)\mathbf{f}_{ar}(4) + \mathbf{f}_b(4)\mathbf{f}_{br}(2) + \mathbf{f}_b(2)\mathbf{f}_{br}(4)\big\}\mathbf{b}(2n-1) \\
&+ \big\{\mathbf{f}_a(4)\mathbf{f}_{ar}(4) + \mathbf{f}_b(4)\mathbf{f}_{br}(4)\big\}\mathbf{b}(2n-3).
\end{aligned} \tag{28}$$

Putting

$$\begin{cases} \mathbf{f}_a(1)\mathbf{f}_{ar}(1) + \mathbf{f}_b(1)\mathbf{f}_{br}(1) = 0, \\ \mathbf{f}_a(3)\mathbf{f}_{ar}(1) + \mathbf{f}_a(1)\mathbf{f}_{ar}(3) + \mathbf{f}_b(3)\mathbf{f}_{br}(1) + \mathbf{f}_b(1)\mathbf{f}_{br}(3) = 0, \\ \mathbf{f}_a(3)\mathbf{f}_{ar}(3) + \mathbf{f}_b(3)\mathbf{f}_{br}(3) = 0 \end{cases} \tag{29}$$

in Eq. (25), we can write

$$\mathbf{b}(2n-1) = \frac{\mathbf{A}_1(n+1) + \mathbf{B}_1(n+1)}{\mathbf{f}_a(4)\mathbf{f}_{ar}(1) + \mathbf{f}_a(2)\mathbf{f}_{ar}(3) + \mathbf{f}_b(4)\mathbf{f}_{br}(1) + \mathbf{f}_b(2)\mathbf{f}_{br}(3)} \tag{30}$$

provided

$$\begin{cases} \mathbf{f}_a(2)\mathbf{f}_{ar}(1) + \mathbf{f}_b(2)\mathbf{f}_{br}(1) = 0, \\ \mathbf{f}_a(4)\mathbf{f}_{ar}(1) + \mathbf{f}_a(2)\mathbf{f}_{ar}(3) + \mathbf{f}_b(4)\mathbf{f}_{br}(1) + \mathbf{f}_b(2)\mathbf{f}_{br}(3) \neq 0, \\ \mathbf{f}_a(4)\mathbf{f}_{ar}(3) + \mathbf{f}_b(4)\mathbf{f}_{br}(3) = 0. \end{cases} \tag{31}$$

Again, putting

$$\begin{cases} \mathbf{f}_a(2)\mathbf{f}_{ar}(2) + \mathbf{f}_b(2)\mathbf{f}_{br}(2) = 0, \\ \mathbf{f}_a(2)\mathbf{f}_{ar}(4) + \mathbf{f}_a(4)\mathbf{f}_{ar}(2) + \mathbf{f}_b(2)\mathbf{f}_{br}(4) + \mathbf{f}_b(4)\mathbf{f}_{br}(2) = 0, \\ \mathbf{f}_a(4)\mathbf{f}_{ar}(4) + \mathbf{f}_b(4)\mathbf{f}_{br}(4) = 0 \end{cases} \tag{32}$$

in Eq. (28), we can write

$$\mathbf{b}(2n) = \frac{\mathbf{A}_2(n+1) + \mathbf{B}_2(n+1)}{\mathbf{f}_a(1)\mathbf{f}_{ar}(4) + \mathbf{f}_a(3)\mathbf{f}_{ar}(2) + \mathbf{f}_b(1)\mathbf{f}_{br}(4) + \mathbf{f}_b(3)\mathbf{f}_{br}(2)} \tag{33}$$

provided

$$\begin{cases} \mathbf{f}_a(1)\mathbf{f}_{ar}(2) + \mathbf{f}_b(1)\mathbf{f}_{br}(2) = 0, \\ \mathbf{f}_a(1)\mathbf{f}_{ar}(4) + \mathbf{f}_a(3)\mathbf{f}_{ar}(2) + \mathbf{f}_b(1)\mathbf{f}_{br}(4) + \mathbf{f}_b(3)\mathbf{f}_{br}(2) \neq 0, \\ \mathbf{f}_a(3)\mathbf{f}_{ar}(4) + \mathbf{f}_b(3)\mathbf{f}_{br}(4) = 0. \end{cases} \tag{34}$$

Equations (29) and (32) together are known as member condition I and conditions (31) and (34) as member condition II for filters of length 4. Once we choose a set of decomposition and reconstruction filters of length 4 satisfying the member condition, we can decompose a signal into two components using Eqs. (3) and (4) and reconstruct the original signal from its decomposed components using the reconstruction equations (30) and (33).

For reconstruction above, the odd terms $\mathbf{b}(2n-1)$ are chosen from Eq. (27) and the even terms $\mathbf{b}(2n)$ from Eq. (28). However, one can choose the even terms from Eq. (27) and the odd terms from Eq. (28). In that case, member conditions I and II will be interchanged keeping the inequalities in right places. Also, the reconstruction equations require modifications accordingly.

The decomposition and reconstruction filters should satisfy the member condition in order to make perfect reconstruction possible. The evaluation of the PRF4 set can be done in the same way as in the case of PRF2 sets from member conditions I and II. However, we can find the PRF4 set in the following way as well.

First we choose the filter coefficients in such a way as to satisfy member condition I. Once we find the filters that satisfy member condition I, the actual filter coefficients are then evaluated from the set of conditions of member condition II. All the four equations in member condition II should be such that all the terms in an equation should not be of the same sign (i.e., should not be either all positive or all negative) and at the same time the two inequalities of the condition should hold.

Let

$$\mathbf{f}_a = [-f_1, f_2, f_3, f_4], \qquad \mathbf{f}_{ar} = [f_4, f_3, f_2, -f_1],$$
$$\mathbf{f}_b = [f_4, -f_3, f_2, f_1], \qquad \mathbf{f}_{br} = [f_1, f_2, -f_3, f_4]$$

be a trial framework of PRF4 sets that satisfies member condition I. Now we will test whether this set of filters satisfies the member condition II also. Substituting the values of the decomposition and reconstruction filters in Eqs. (31) and (34), we get $-f_1 f_3 + f_2 f_4 = 0$. And from the inequalities we get $f_1^2 + f_2^2 + f_3^2 + f_4^2 \neq 0$.

Thus, the above set of decomposition and reconstruction filters satisfies member condition II as well and hence form a PRF4 set. The coefficients of the filters are obtained by solving first the equation $-f_1 f_3 + f_2 f_4 = 0$. We can choose any three values arbitrarily and find the remaining one from the equation. As this can be done in infinitely many ways, we can find infinitely many PRF4 sets.

Let us consider another framework of PRF4 sets $\mathbf{f}_a = [f_1, f_2, f_3, f_4]$, $\mathbf{f}_b = [f_4, f_3, -f_2, -f_1]$, $\mathbf{f}_{ar} = [f_4, -f_3, -f_2, f_1]$ and $\mathbf{f}_{br} = [-f_1, f_2, -f_3, f_4]$. This type of PRF4 sets is not possible in wavelet or filter bank because the coefficients of a decomposition filter cannot be all positive. The coefficients here also are found from the member condition which again reduces to $-f_1 f_3 + f_2 f_4 = 0$ and $f_1^2 + f_2^2 + f_3^2 + f_4^2 \neq 0$. Several cases of PRF4 set frameworks are given in Appendix A.

In all cases of the frameworks of PRF4 sets, the actual PRF4 sets are obtained by solving the equation $-f_1 f_3 + f_2 f_4 = 0$. Thus, once we find the values of f_1, f_2, f_3 and f_4 for any case, the same values of the coefficients can also be used in other cases provided the corresponding denominator terms are not zero. As in PRF2 sets, the different cases will give different decomposed components. That is, we may decompose a signal into an approximation and a detail or into two details or into two approximations in the case of length 4 also. We have derived the decomposition and reconstruction equations with the member conditions for filters of lengths 2 and 4. We can obtain decomposition equations and corresponding member conditions for filters of any even length greater than 4 as given in Appendix B.

The decomposition of the signal can be done successively up to a certain desired level on only one or all of the decomposed components at each successive level in order to get multi-resolution signal decomposition.

4. Properties and advantages of ISITRA

In this section, we show, through some examples, certain properties and potential advantages of ISITRA.

4.1. Modification of pixel range in the decomposed components

Let us consider the PRF2 sets of Cases 1 and 2 of Section 3.1. In both these cases, a signal is decomposed into an approximation and a detail. When the two coefficients of

(a) (b) (c)

Fig. 4. (a) Original Barbara image. (b) Compressed image 8:1 using ISITRA filter [0.125, 0.125]. (c) Compressed image 3.5:1 using Haar's wavelet.

Table 2
Reduction in pixel range and its use in compression

Filter type	Pixel range of Appxs	Compression	Average pixel value of RI	MSE	PSNR
[0.5, 0.5]	10–250	4:1	125.81	358.09	22.59
[0.25, 0.25]	3–63	5.3:1	125.83	359.36	22.57
[0.125, 0.125]	1–16	8:1	125.79	379.29	22.34
Haar's wavelet	20–500	3.5:1	125.76	358.01	22.59

a decomposition filter are equal, the PRF2 sets of both cases give essentially the same decomposed components. So, we will consider $\{[f_1, f_1], [-f_1, f_1], [f_1, f_1], [f_1, -f_1]\}$, the framework of PRF2 sets of Case 2 only. The value of f_1 can be any non-zero real number. When $f_1 = 1/\sqrt{2}$, we get the same PRF2 set of Haar's or DB1 wavelet. We cannot choose any other values of f_1 for wavelets filters of length 2 as explained in Case 2 of Section 3.1. But in ISITRA, there is no such restriction. By choosing various appropriate values of f_1, we can control the range of pixel values in the decomposed components. This may be an advantage from the compression point of view. For compression we are interested mainly in the approximation component. We decompose the gray level Barbara image of 128×128 size, shown in Fig. 4a, into four components, one approximation and three details. We retain only the approximation component rounded off to integers after decomposition rejecting the remaining three details. That is, only one fourth of the size of the original image is retained, providing room for compression. If we can reduce the pixel range of the approximation, more compression would be achieved. Table 2 gives the pixel ranges for approximation components (Appxs) using various PRF2 sets. We reconstruct the original signal from only the retained approximation component in each case of the filters, replacing the three details with zeros. The average pixel values, mean square error (MSE) and peak signal to noise ratio (PSNR) of the reconstructed images (RI) for different PRF2 sets are also given in Table 2.

From the table we see that the average pixel values of the lossy reconstructed image are slightly higher for the filters of ISITRA than that for Haar's wavelet filter. But the MSE values are slightly higher for $f_1 < 1/\sqrt{2}$, as a result there is slight decrease in the PSNR values. But we can achieve a considerable amount of compression, as storing of the

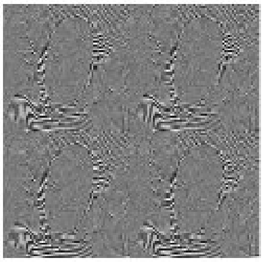

Fig. 5. Decomposition of Barbara image into details.

approximation components with small pixel range requires less number of bits. Thus, we get a compression ratio of about 8:1, if we use only the approximation component whose pixel range is 1–16. The lossy reconstructed image, which has been compressed 8 times, is shown in Fig. 4b, and the one in the case of Haar's wavelet is shown in Fig. 4c.

4.2. *Decomposition of an image into all detail components*

In the reconstruction theory of ISITRA, what is required for perfect reconstruction is to find a set of decomposition and reconstruction filters that satisfy the member condition. Consider the framework of PRF2 sets of Case 3 of Section 3.1. If we choose $\mathbf{f}_a = [0.125, -0.124]$ and $\mathbf{f}_b = [0.124, -0.125]$ as the decomposition filters, we get all the decomposed components as details. The perfect reconstruction of the original signal from its decomposed components is possible using appropriate reconstruction filters, quite contrary to the popular belief that reconstruction is impossible without an approximation component. The reason why this is possible is that the decomposition and the reconstruction filters satisfy the member condition. For PRF2 sets, it is found that the more close the magnitudes of the filter coefficients are the more details are obtained after decomposition. By the term "more details" we mean the components having more local variations, and more and more distinguishable (global) features of the original image are lost in these components. This may have potential application in encryption as we can hide the global features of the original signal in the detail components. The same Barbara image, when decomposed with the above decomposition filters gives all detail subimages as shown in Fig. 5. Note here that the two coefficients cannot be equal, in order to make perfect reconstruction possible.

4.3. *Decomposition of an image into all approximation components*

From above it is clear that we can decompose a signal into details and get a perfect reconstruction from them. The reverse is also possible. That is, we can decompose a signal into all approximation components and get back the original signal from these components. Let us consider the framework of PRF2 set of Case 4 of Section 3.1. Suppose $\mathbf{f}_a = [0.125, 0.124]$ and $\mathbf{f}_b = [0.124, 0.125]$ are two decomposition filters. If we decompose a signal using these decomposition filters, then the decomposed components will all be approximation components. Perfect reconstruction is possible with appropriate recon-

Fig. 6. Decomposition of Barbara image into approximations.

Fig. 7. Compressed image (8:1) reconstructed from only one approximation.

struction filters. Experimentally, we find that the more close the values of the two filter coefficients are, more approximation is obtained. In other words, the decomposed approximation components become smoother and smoother. The interesting feature of this is that we can reconstruct the original signal from any one of its approximation components. In that case, all the four approximation components are to be replaced by any one of them before reconstruction. This can be used for lossy compression point of view. If we decompose the same Barbara image with the above decomposition filters, the decomposed components are all approximations as shown in Fig. 6. The ranges of pixel values of the approximation components are 1–16. So if we reconstruct the image from only one of the approximations, we get a compression ratio of 8:1. The reconstructed image has 22.34 as the PSNR value and is shown in Fig. 7 which is comparable to the reconstruction in Fig. 4b. When the pixel range is 3–60, the PSNR value becomes 22.59. The quality of image becomes as good as that in Fig. 4c. In addition we can achieve a compression ratio of 5.3:1 instead of 3.5:1 obtained in Haar's wavelet case.

4.4. *Finding better PRF4 sets than that of DB2 wavelet filter*

By DB2 wavelet filter, we will mean the popular Daubechies' wavelet filter of length 4. Here we will see, whether we can find a better PRF4 set of ISITRA than that of DB2 wavelet filter. In Table 3 we show the energy content of each subband in a single level decomposition of Barbara image using PRF4 sets of DB2 wavelet filter and ISITRA. The filters in the case of ISITRA are given in non-normalized form for easy entry in the table. In actual decomposition, they are normalized as DB2 wavelet filter is also normalized. Except DB2, the filters are from the following framework of PRF4 sets where the coef-

Table 3
Reconstruction error and energy compaction of DB2 wavelet filters and PRF4 sets of ISITRA

Filter type	Energy contents				MSE	PSNR
DB2	84.916	61.424	65.879	60.726	8.19255e−22	258.996
Reverse DB2	84.917	61.284	65.755	60.786	8.19232e−22	258.996
[4.8, 8, 2, −1.2]	84.919	61.093	65.669	60.777	2.17076e−27	314.764
[27.3, 36.4, 5.2, −3.9]	84.927	60.264	65.280	60.785	6.86492e−28	319.764

Table 4
MSE and PSNR values of the lossy reconstructed image from only the approximation component

Filter type	MSE	PSNR
DB2	360.752	22.558
Reverse DB2	343.906	22.766
[4.8, 8, 2, −1.2]	340.343	22.811
[27.3, 36.4, 5.2, −3.9]	330.881	22.934

ficients can be chosen quite arbitrarily. Consider $\mathbf{f} = [f_1, f_2, f_3, f_4]$ where each entry is positive. From this can be generated the framework of PRF4 sets $\{\mathbf{f}_a = [f_1, f_2, f_3, -f_4], \mathbf{f}_b = [-f_4, f_3, -f_2, -f_1], \mathbf{f}_{ar} = \bar{\mathbf{f}}_a, \mathbf{f}_{br} = \bar{\mathbf{f}}_b\}$, where $\bar{\mathbf{f}}$ denotes the reverse vector of $\mathbf{f}$.

It is found that the energy content (the mean of the squared gray values) of the approximation component is slightly higher for the PRF4 sets of ISITRA than that of DB2 wavelet filter (Table 3). The reverse DB2 in Table 3 denotes the PRF4 set of DB2 where the decomposition and the reconstruction filters are interchanged. The energy contents of the first two detail components for ISITRA are less than the energy contents of the corresponding components for DB2 filter coefficients, with a slight increase of energy in the last detail component. Thus, from the energy contents of the decomposed components, it is seen that the energy compaction is higher in the case of the PRF4 sets of ISITRA.

Lastly, we reject the three detail components and reconstruct the image only from the approximation component (after rounding off to nearest integers), by replacing the three detail components with zeros. We measure the distortion error introduced in the reconstructed image. The MSE and PSNR values of the lossy reconstructed images are shown in Table 4. From the table we see that the PSNR value is lower in the case of DB2 than other filters.

Figure 8a shows the lossy image reconstructed only from the approximation component in the case of the last entry of the PRF4 set of ISITRA in Table 3, and Fig. 8b shows the lossy image reconstructed from the approximation component in the case of DB2 wavelet filter. With respect to both MSE and PSNR values (Table 4), the PRF4 set chosen above performs better for compression than DB2 wavelet filter. From Tables 3 and 4, it is also clear that the PRF4 set of reverse DB2 is better than that of DB2. Hence, it may be advisable to use reverse DB2 instead of DB2, if one is interested in using PRF4 set of DB2.

Also, we test the errors introduced when we reconstruct the original image from all its decomposed components (without rounding off). It is found that the reconstruction error is much higher in the case of DB2. The reconstruction error can be totally eliminated if we use non-normalized PRF4 sets.

(a) (b)

Fig. 8. (a) Image reconstructed from only the approximation component in the case of PRF4 set of ISITRA (last entry in Table 3). (b) Image reconstructed from only the approximation component in the case of DB2 wavelet filters.

5. Discussion

A new transform ISITRA has been developed for signal decomposition and reconstruction. Computational aspects of the decomposition and reconstruction schemes of 2-channel filter bank, its polyphase scheme and ISITRA have been investigated. The polyphase scheme is computationally much better than the usual filter bank scheme and ISITRA is shown to be slightly better than the polyphase scheme. The perfect reconstruction theory of the reconstruction scheme of ISITRA has been described in detail. To make it more easily understandable, the perfect reconstruction scheme is dealt with separately for length 2 and length 4 cases. The process of finding PRF sets is also described. One aim of the paper is to show that we can easily find shorter length perfect reconstruction filters without using wavelet functions. Some of the potential applications of ISITRA are discussed with examples. An important feature of ISITRA is the flexibility in choosing the filter coefficients. There are many ways of choosing the filter coefficients other than that of QMF or CQF type, as discussed in different cases of PRF2 and PRF4 sets. In the case of wavelet, Haar's wavelet filter is the only known filter of length 2. But in the case of ISITRA of length 2, any two non-zero real numbers can be used as filter coefficients. This allows us to keep the pixel values of the decomposed components at any desired range by choosing appropriate filters. Thus, by keeping the pixel values to 1–16, for example, we can get the compression ratio of 8:1 without applying any coding scheme. In the case of wavelet (coefficients of length 4), DB2 is the most commonly used filter. But in case of ISITRA of length 4, any four non-zero real numbers satisfying certain soft conditions (i.e., the member conditions), can be used as the coefficients. Even in wavelet or filter bank scheme, there is no hard and fast rule for choosing filters for compression [19]. But ISITRA offers a larger space of filters and hence there is a possibility of finding a better filter for compression (or for other purposes) than the wavelet scheme. Also, we could find PRF4 sets which give better energy compaction in image decomposition than DB2 wavelet filter. This shows the usefulness of flexibility in choosing arbitrary filter coefficients. Another interesting observation is that if we interchange the decomposition and the reconstruction filters in the PRF4 set of DB2, we can sometimes get better results. We have also provided different cases of PRF4 sets of ISITRA in Appendix A, which may be useful for different applications. A generalized way of finding longer PRF sets is given in Appendix B. This will enable one to find longer PRF sets than the existing longer wavelet filters. Even if one cannot find a better PRF set, one can directly use the known wavelet filters in the decomposition and reconstruction schemes of ISITRA. Also, the normalized PRF sets of ISITRA can also be directly used in place of PRF sets of wavelet filters, if the frameworks of the two PRF sets match. For example, the PRF sets of ISITRA provided in Section 4.4, can be directly used instead of wavelet filters.

6. Conclusions

ISITRA is fast, flexible and easily implementable. It is simple and easily understandable and even a little faster than the computationally efficient polyphase scheme. The flexibility in choosing arbitrary filter coefficients may be of great use in finding better filter coefficients for different applications. It can be used in place of wavelets and filter banks where the latter have applications. Moreover, ISITRA has the capability of decomposing a signal into various ways (into an approximation and a detail or into two details or into two approximations) and may find application in problems where wavelets and filter bank scheme cannot be applied. In other words, its application potential may extend beyond the scope of wavelets and filter banks.

Appendix A. Some possible cases of frameworks of PRF4 sets

Case 1. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} \mathbf{f}_a(i)^2$. The first choice of PRF4 sets given at the end of Section 3.2 belongs to this case.

Case 2. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} (-1)^{i-1} \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [f_1, -f_2, f_3, -f_4]$, $\mathbf{f}_b = [-f_4, f_3, -f_2, f_1]$, $\mathbf{f}_{ar} = [f_4, f_3, f_2, f_1]$ and $\mathbf{f}_{br} = [f_1, f_2, f_3, f_4]$ form a framework of PRF4 sets.

Case 3. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} (-1)^{\lfloor i/2 \rfloor} \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [f_1, f_2, f_3, f_4]$, $\mathbf{f}_b = [f_4, f_3, -f_2, -f_1]$, $\mathbf{f}_{ar} = [f_4, -f_3, -f_2, f_1]$ and $\mathbf{f}_{br} = [-f_1, f_2, -f_3, f_4]$ form a framework of PRF4 sets.

Case 4. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} (-1)^{\lceil i/2 \rceil - 1} \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [-f_1, f_2, f_3, f_4]$, $\mathbf{f}_b = [-f_4, f_3, f_2, f_1]$, $\mathbf{f}_{ar} = [-f_4, -f_3, f_2, -f_1]$ and $\mathbf{f}_{br} = [f_1, f_2, -f_3, f_4]$ form a framework of PRF4 sets.

Case 5. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} (-1)^{\lceil i/2 \rceil} \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [f_1, -f_2, f_3, f_4]$, $\mathbf{f}_b = [-f_4, f_3, -f_2, -f_1]$, $\mathbf{f}_{ar} = [f_4, f_3, f_2, -f_1]$ and $\mathbf{f}_{br} = [f_1, f_2, f_3, -f_4]$ form a framework of PRF4 sets.

Case 6. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4} (-1)^{\lceil (i+1)/2 \rceil} \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [f_1, f_2, f_3, f_4]$,

$\mathbf{f}_b = [f_4, f_3, -f_2, -f_1]$, $\mathbf{f}_{ar} = [-f_4, f_3, f_2, -f_1]$ and $\mathbf{f}_{br} = [f_1, -f_2, f_3, -f_4]$ form a framework of PRF4 sets.

Case 7. The inner product of a decomposition filter and the reverse of its corresponding reconstruction filter is $\sum_{i=1}^{4}(-1)^i \mathbf{f}_a(i)^2$. For example, $\mathbf{f}_a = [-f_1, -f_2, f_3, f_4]$, $\mathbf{f}_b = [f_4, f_3, -f_2, -f_1]$, $\mathbf{f}_{ar} = [f_4, -f_3, -f_2, f_1]$ and $\mathbf{f}_{br} = [f_1, -f_2, -f_3, f_4]$ form a framework of PRF4 sets.

Appendix B. Generalization for arbitrary filter length

In the general case, Eqs. (23)–(26) hold true for any even length L.

Equations (27) and (28) can be written as follows:

$$\begin{aligned} &A_1(n+1) + B_1(n+1) \\ &\quad = \sum_{r=1}^{3}\left[\mathbf{b}(2n+4-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(p) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(p)\right\}\right] \\ &\qquad + \sum_{r=1}^{3}\left[\mathbf{b}(2n+3-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i)\mathbf{f}_{ar}(p) + \mathbf{f}_b(2i)\mathbf{f}_{br}(p)\right\}\right], \quad p = 2r+1-2i, \\ &A_2(n+1) + B_2(n+1) \\ &\quad = \sum_{r=1}^{3}\left[\mathbf{b}(2n+4-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(p) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(p)\right\}\right] \\ &\qquad + \sum_{r=1}^{3}\left[\mathbf{b}(2n+3-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i)\mathbf{f}_{ar}(q) + \mathbf{f}_b(2i)\mathbf{f}_{br}(q)\right\}\right], \quad q = 2r+2-2i, \end{aligned}$$

where $\mathbf{f}(k) = 0$ for any integer k less than 1 or greater than L.

Proceeding in this way, for a filter of length $L = 2N$, where N is a positive integer, we can write the above two equations in a general form as follows:

$$\begin{aligned} &A_1(n+N-1) + B_1(n+N-1) \\ &\quad = \sum_{r=1}^{L-1}\left[\mathbf{b}(2n+L-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(p) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(p)\right\}\right] \\ &\qquad + \sum_{r=1}^{L-1}\left[\mathbf{b}(2n+L-1-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i)\mathbf{f}_{ar}(p) + \mathbf{f}_b(2i)\mathbf{f}_{br}(p)\right\}\right], \\ &A_2(n+N-1) + B_2(n+N-1) \\ &\quad = \sum_{r=1}^{L-1}\left[\mathbf{b}(2n+L-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(q) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(q)\right\}\right] \\ &\qquad + \sum_{r=1}^{L-1}\left[\mathbf{b}(2n+L-1-2r)\sum_{i=1}^{r}\left\{\mathbf{f}_a(2i)\mathbf{f}_{ar}(q) + \mathbf{f}_b(2i)\mathbf{f}_{br}(q)\right\}\right]. \end{aligned}$$

The set of equations (35) forms member condition I:

$$\begin{cases} \sum_{i=1}^{r} \{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(2r+1-2i) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(2r+1-2i)\} = 0, \\ \sum_{i=1}^{r} \{\mathbf{f}_a(2i)\mathbf{f}_{ar}(2r+2-2i) + \mathbf{f}_b(2i)\mathbf{f}_{br}(2r+2-2i)\} = 0, \end{cases}$$
$$r = 1, 2, \ldots, L-1. \tag{35}$$

The set of equations (36) and the set of inequalities (37) form member condition II:

$$\begin{cases} \sum_{i=1}^{r} \{\mathbf{f}_a(2i)\mathbf{f}_{ar}(2r+1-2i) + \mathbf{f}_b(2i)\mathbf{f}_{br}(2r+1-2i)\} = 0, \\ \sum_{i=1}^{r} \{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(2r+2-2i) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(2r+2-2i)\} = 0, \end{cases}$$
$$\forall r \neq N, \tag{36}$$

$$\begin{cases} \sum_{i=1}^{N} \{\mathbf{f}_a(2i)\mathbf{f}_{ar}(L+1-2i) + \mathbf{f}_b(2i)\mathbf{f}_{br}(L+1-2i)\} \neq 0, \\ \sum_{i=1}^{N} \{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(L+2-2i) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(L+2-2i)\} \neq 0. \end{cases} \tag{37}$$

The corresponding reconstruction equations are

$$\mathbf{b}(2n-1) = \frac{\mathbf{A}_1(n+N-1) + \mathbf{B}_1(n+N-1)}{\sum_{i=1}^{N}\{\mathbf{f}_a(2i)\mathbf{f}_{ar}(L+1-2i) + \mathbf{f}_b(2i)\mathbf{f}_{br}(L+1-2i)\}}, \tag{38}$$

$$\mathbf{b}(2n) = \frac{\mathbf{A}_2(n+N-1) + \mathbf{B}_2(n+N-1)}{\sum_{i=1}^{N}\{\mathbf{f}_a(2i-1)\mathbf{f}_{ar}(L+2-2i) + \mathbf{f}_b(2i-1)\mathbf{f}_{br}(L+2-2i)\}}. \tag{39}$$

PRFL sets can be found from the member condition equations (35), (36) and inequalities (37). From the member condition equations, we see that the roles of the decomposition and reconstruction filters can be interchanged without affecting the member condition. That is, once we find a PRF set, we can use the reconstruction filters for decomposition and the decomposition filters for reconstruction. Also, the member conditions I and II can be interchanged. In that case, the reconstruction equations should also be accordingly modified.

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Yumnam Kirani Singh received his Master's degree in electronics science from Guwahati University in 1997. He joined the Computer Vision and Pattern Recognition Unit of the Indian Statistical Institute as a Junior Research Fellow in 1998. Currently, he is a Senior Research Fellow and has recently submitted his Ph.D. thesis in the Indian Statistical Institute. His main areas of research are signal encryption, coding, compression and communication.

Swapan Kumar Parui received his Master's degree in statistics and his Ph.D. degree from the Indian Statistical Institute in 1975 and 1986, respectively. From 1985 to 1987 he held a post-doctoral position in Leicester Polytechnic, UK, working on an automatic industrial inspection project, and from 1988 to 1989 he was a visiting scientist in GSF, Munich, working on biological image processing. He joined the Indian Statistical Institute in 1987 and became a Professor in 1997. Currently, he is the Head of Computer Vision and Pattern Recognition Unit of the Institute. His current research interests include pattern recognition, colour image processing, neural networks, optical character recognition and image compression. Professor Parui has published over 75 papers in journals and conference proceedings.