Constraining Active Contour Evolution via Lie Groups of Transformation

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***Abstract*—We present a novel approach to constraining the evolution of active contours used in image analysis. The proposed approach constrains the final curve obtained at convergence of curve evolution to be related to the initial curve from which evolution begins through an element of a desired Lie group of plane transformations. Constraining curve evolution in such a way is important in numerous tracking applications where the contour being tracked in a certain frame is known to be related to the contour in the previous frame through a geometric transformation such as translation, rotation, or affine transformation, for example. It is also of importance in segmentation applications where the region to be segmented is known up to a geometric transformation. Our approach is based on suitably modifying the Euler-Lagrange descent equations by using the correspondence between Lie groups of plane actions and their Lie algebras of infinitesimal generators, and thereby ensures that curve evolution takes place on an orbit of the chosen transformation group while remaining a descent equation of the original functional. The main advantage of our approach is that it does not necessitate any knowledge of nor any modification to the original curve functional and is extremely straightforward to implement. Our approach therefore stands in sharp contrast to other approaches where the curve functional is modified by the addition of geometric penalty terms. We illustrate our algorithm on numerous real and synthetic examples.**

***Index Terms*—Active contours, curve evolution equations, Lie groups, tracking.**

I. Introduction

THIS paper addresses the problem of curve evolution, with applications to tracking and segmentation in image sequences [11]–[13], [19]. Curve evolution equations are usually obtained as Euler-Lagrange descent equations of a curve functional $E : \gamma \mapsto E(\gamma) \in \mathbb{R}$ tailored to a particular application [2]. Starting from an initial curve γ_0, a curve evolution equation prescribes the construction of a one-parameter family $(\gamma_t)_{t \in \mathbb{R}^+}$ of curves (with $\gamma_t|_{t=0} = \gamma_0$) such that the curve $\gamma_\infty = \lim_{t \to \infty} \gamma_t$ obtained at convergence is a local minimum of the curve functional. In many applications of interest such as tracking, there may be *a priori* knowledge concerning the geometric relation between γ_0 and γ_∞; In particular, this *a priori* knowledge could dictate that γ_0 and γ_∞ be related, up to reparametrization, by an arbitrary transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ in a certain family of transformations. For example, it may be known *a priori* that γ_0 and γ_∞ should be related by a translation, or by a Euclidean transformation. Due to image noise and clutter, however, and depending on the particular curve functional from which the curve evolution equation is derived, the curve γ_∞ obtained at convergence of the evolution may not have the desired geometric relation to the initial curve γ_0.

Deformation of shape in a more generalized framework can be accomplished via region information. Active shape models are used to recognize shape deformations given the shape model and in the estimation of shape deformations using a point distribution model [6]. Region based approaches for tracking shapes in color images are given in [7]. Segmentation and tracking applications are also demonstrated applying nonlinear shape statistics using available training data [8].

The question is then how to suitably modify the curve evolution equation so that the resulting equations remain descent equations of the original curve functional while simultaneously ensuring that the curve obtained at convergence has the desired geometric relation to the initial curve. A solution to this problem has been proposed in [1], whereby the curve functional is extended by the addition of penalty terms which try to bias the minimum of the functional toward a curve with the desired geometric properties. In other words, the original curve functional E is changed to $E + \lambda E_p$, with $E_p : \gamma \to E_p(\gamma)$ penalizing deviations of γ from the desired geometry. While such penalty terms can be easily defined for simply parametrized shapes such as circles and ellipses, it is not clear how to define them for arbitrary planar shapes. Thus, this approach is feasible only in very restricted cases. Furthermore, even in these cases, it is not clear how the penalty terms should be weighted in comparison to the original energy functional, that is, how the coefficient λ should be chosen.

We propose a novel and straightforward solution to the problem of geometrically constraining curve evolution in the case where the geometric relation between γ_0 and γ_∞ is given by elements of a finite-dimensional Lie group [16] of plane transformations.[1] This is the case with most applications of curve evolution, with the Lie groups of interest being the group of translations, the group of rotations, the group of Euclidean transformations, as well as the group of affine transformations. This allows us to use the correspondence between Lie groups and their Lie algebras in order to reduce the original problem to one of basic linear algebra. The main advantage of our

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[1]It is important to note that the problem we are addressing in this paper is radically different from the problem of defining group-invariant flows [9], [15], i.e., flows which commute with elements of particular groups of transformation.

approach is that it does not necessitate any knowledge of or modification to the original curve functional from which the original curve evolution equations were obtained. Rather, only the curve evolution equation is modified, in a very straightforward way, all the while ensuring both that the initial and final curves are related as desired, and that the modified curve evolution equation remains a descent equation on the original curve functional. Such a technique has already been proposed for tracking the long time behavior of dynamical systems which are known to obey certain symmetries [14]. A related idea also appears in [4], [5], where 3-D structures are tracked by using the Lie algebra of the Special Euclidean group in $\mathbb{R}^3$ and the Adjoint representation of the group to transform the model between video frames. The theory presented in Sections II and III, in conjunction with the results shown in Section IV, demonstrate the efficacy of the proposed curve evolution approach.

II. Curve Evolution Equations and Lie Transformation Groups

A. Basic Curve Evolution Equations

Consider a smooth functional $E : \Gamma \to \mathbb{R}$, where Γ is the family of smooth closed plane curves $\gamma : \mathbf{I} \to \mathbb{R}^2, s \mapsto \gamma(s)$, $\mathbf{I}$ is a compact interval of $\mathbb{R}$, and s is the arc parameter (not necessarily arc length). We restrict ourselves to functionals of the form

$$\gamma \mapsto E(\gamma) = \int_{\mathbf{I}} L(s, \gamma(s), \dot{\gamma}(s), \ddot{\gamma}(s), \ldots)\, ds \tag{1}$$

where L is a smooth function, and $\dot{\gamma}$ (resp. $\ddot{\gamma}, \ldots$) denotes the first (resp. second, ...) derivative of γ with respect to s. This is the general form of the curve functionals most used in image processing applications [10]. We are interested in finding the curve (or those curves) in Γ which (locally) minimizes E. To perform this minimization, γ is embedded in a family $(\gamma_t)_{t\in\mathbb{R}^+}$ of curves, and this family is constructed so as to satisfy the evolution equation

$$\begin{aligned} \frac{d\gamma_t}{dt} &= -\frac{\delta E}{\delta\gamma}(\gamma_t), \quad t \geq 0 \\ \gamma_t|_{t=0} &= \gamma_0 \end{aligned} \tag{2}$$

where γ_0 is the initial curve, and where $(\delta E/\delta\gamma)(\gamma_t)$ is the functional derivative of E with respect to γ at γ_t [2]. $(\delta E/\delta\gamma)(\gamma_t)$ is a vector tangent to the space Γ at the point $\gamma_t \in \Gamma$, i.e., an element of the tangent space $T_{\gamma_t}(\Gamma)$. An element of the tangent space $T_\gamma(\Gamma)$ to Γ at a particular curve $\gamma \in \Gamma$ is given by a smooth vector field along γ, that is, by a mapping $X : \mathbf{I} \to \mathbb{R}^2, s \to X(\gamma(s))$. Thus, for each value of the arc parameter $s \in \mathbf{I}$ of γ, $X(\gamma(s))$ is a vector in $\mathbb{R}^2$. This allows us to define an inner product $\langle .,. \rangle_\gamma$ on $T_\gamma(\Gamma)$ as follows:

$$\langle X, Y \rangle_\gamma = \int_{\mathbf{I}} \langle X(\gamma(s)), Y(\gamma(s)) \rangle_{\mathbb{R}^2}\, ds, \quad \forall X, Y \in T_\gamma(\Gamma)$$

where $\langle .,. \rangle_{\mathbb{R}^2}$ is the Euclidean inner product on $\mathbb{R}^2$. The tangent vector $(\delta E/\delta\gamma)(\gamma_t)$ is defined as the unique element of $T_{\gamma_t}(\Gamma)$ which satisfies the relation

$$\frac{d}{dt}E[\gamma_t] = \left\langle \frac{\delta E}{\delta\gamma}(\gamma_t), \frac{d\gamma_t}{dt} \right\rangle_{\gamma_t}. \tag{3}$$

It follows from (2) and (3) that

$$\begin{aligned} \frac{d}{dt}E[\gamma_t] &= \left\langle \frac{\delta E}{\delta\gamma}(\gamma_t), \frac{d\gamma_t}{dt} \right\rangle_{\gamma_t} \\ &= -\left\langle \frac{\delta E}{\delta\gamma}(\gamma_t), \frac{\delta E}{\delta\gamma}(\gamma_t) \right\rangle_{\gamma_t} \leq 0 \end{aligned}$$

by the positive-definiteness of the inner product. As a result the mapping $t \mapsto E(\gamma_t)$ is monotonically decreasing. Equation (2) is thus a descent equation, and the family $(\gamma_t)_{t\in\mathbb{R}^+}$ of curves a minimizing family for the functional E. The goal of constructing such a family is to compute the curve $\gamma_\infty = \lim_{t\to\infty}\gamma_t$, which, if it exists, is a critical point, and hopefully a local minimum, of E.

In image processing applications of curve evolution, a closed plane curve is usually represented by an N-tuple $(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)$ of points $\vec{p}_i \in \mathbb{R}^2$, yielding a polygonal approximation to the desired curve. The space Γ_N of such N-tuples can then be identified with the finite-dimensional vector space $\mathbb{R}^{2N}$, and the curve evolution (2) in Γ is replaced with an evolution equation in Γ_N:

$$\begin{aligned} \frac{d\vec{p}_i}{dt}(t) &= \vec{F}_i\left((\vec{p}_j(t))_{j=1}^N\right), \quad t \geq 0, \quad i = 1, \ldots, N \\ \vec{p}_i|_{t=0} &= \vec{p}_{i,0} \end{aligned} \tag{4}$$

where the functions $\vec{F}_i$ are obtained by spatial discretization of the expression for $(\delta E)/(\delta\gamma)$ in (2). Here again, the N-tuple $(\vec{F}_1((\vec{p}_j(t))_{j=1}^N), \vec{F}_2((\vec{p}_j(t))_{j=1}^N), \ldots, \vec{F}_N((\vec{p}_j(t))_{j=1}^N))$ is a vector tangent to $\Gamma_N \simeq \mathbb{R}^{2N}$ at the point $(\vec{p}_1(t), \vec{p}_2(t), \ldots, \vec{p}_N(t))$ of $\mathbb{R}^{2N}$, which associates to $\vec{p}_i(t) \in \mathbb{R}^2$ the vector $\vec{F}_i((\vec{p}_j(t))_{j=1}^N) \in \mathbb{R}^2$. Not surprisingly, the space $T_\gamma(\Gamma_N)$ of all vectors tangent to Γ_N at the point $\gamma \in \Gamma_N$ can be identified with $\mathbb{R}^{2N}$ as well. Let then $X = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N), Y = (\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_N) \in T_\gamma(\Gamma_N) \simeq \mathbb{R}^{2N}$ (with $\vec{x}_i, \vec{y}_i \in \mathbb{R}^2$ for all $i = 1, \ldots, N$); the inner product $\langle .,. \rangle$ on $T_\gamma(\Gamma_N)$ is defined as follows:

$$\langle X, Y \rangle = \sum_{i=1}^N \langle \vec{x}_i, \vec{y}_i \rangle_{\mathbb{R}^2}.$$

The evolution (4) is further discretized temporally as well, yielding the following discrete evolution equation:

$$\begin{aligned} \vec{p}_i((k+1)\Delta t) &= \vec{p}_i(k\Delta t) - \Delta t \vec{F}_i\left((\vec{p}_j(k\Delta t))_{j=1}^N\right), \\ & \qquad k \in \mathbb{N}, \quad i = 1, \ldots, N \\ \vec{p}_i|_{k=0} &= \vec{p}_{i,0} \end{aligned} \tag{5}$$

where Δt is the temporal discretization step.

B. Group Actions, Orbits, and Infinitesimal Generators

Let G be a (finite-dimensional) Lie group of transformations acting on $\mathbb{R}^2$. We refer the interested reader to [16]–[18] for a detailed introduction to the theory of Lie groups. Assume G acts on $\mathbb{R}^2$ via a smooth map $G \times \mathbb{R}^2 \to \mathbb{R}^2, (g, \vec{p}) \to g \cdot \vec{p}$ such that $e \cdot \vec{p} = \vec{p}$ for all $\vec{p} \in \mathbb{R}^2$, where e is the identity element of G, and $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $g_1, g_2 \in G, p \in \mathbb{R}^2$. The action of G on $\mathbb{R}^2$ induces an action of G on Γ given by the smooth map $G \times \Gamma \to \Gamma, (g, \gamma) \mapsto g \cdot \gamma$, where the curve $g \cdot \gamma$ is defined by $g \cdot \gamma : \mathbf{I} \to \mathbb{R}^2, s \mapsto (g \cdot \gamma(s))$. The orbit of γ under the action of G is the subset $G \cdot \gamma$ of Γ defined as

$$G \cdot \gamma = \{g \cdot \gamma \in \Gamma \mid g \in G\}.$$

The meaning of the orbit $G \cdot \gamma$ is clear: It is the set of all plane curves obtained by applying all the transformations in G to the curve γ.

Let now $h :]-\epsilon, \epsilon[\to G, t \mapsto h(t)$ be a smooth curve in G (with $\epsilon > 0$ arbitrary) with $h(0) = e$. Let $x = (d/dt)_{t=0} h(t)$; x is an element of the tangent space $T_e(G)$ of G at e. The smooth curve h in G induces a smooth curve $h \cdot \gamma :]-\epsilon, \epsilon[\to G \cdot \gamma, t \to h(t) \cdot \gamma$ in the orbit $G \cdot \gamma$ of γ. The vector $X(x) \in T_\gamma(\Gamma)$ defined by the vector field $s \mapsto (d/dt)|_{t=0}[h(t) \cdot \gamma(s)]$ on γ is thus an element of the tangent space $T_\gamma(G \cdot \gamma)$ of the orbit $G \cdot \gamma$ at γ. Since $G \cdot \gamma \subset \Gamma, T_\gamma(G \cdot \gamma)$ is a vector subspace of $T_\gamma(\Gamma)$, and since G is finite-dimensional, $T_\gamma(G \cdot \gamma)$ is finite-dimensional as well. Thus, to each $x \in T_e(G)$ there corresponds a unique vector $X(x) \in T_\gamma(G \cdot \gamma) \subset T_\gamma(\Gamma)$. Furthermore, the mapping $X : x \mapsto X(x)$ is linear, and for each $Y \in T_\gamma(G \cdot \gamma)$, there exists a $x \in T_e(G)$ such that $Y = X(x)$. Therefore, the tangent space to $G \cdot \gamma$ at γ is given by

$$T_\gamma(G \cdot \gamma) = \{X(x) \in T_\gamma(\Gamma) \mid x \in T_e(G)\}.$$

C. Curve Evolution on Orbits of Lie Transformation Groups

Assume we are given curve evolution (2) corresponding to the minimization of the functional E and assume we know *a priori* that the initial curve $\gamma_0 \in \Gamma$ and the final curve $\gamma_\infty \in \Gamma$ should be related, up to reparametrization, via a transformation in G; that is, there exists $g \in G$ and a monotonically increasing diffeomorphism $\phi : \mathbf{I} \to \mathbf{I}$ (called a *reparametrization* of $\mathbf{I}$) such that $\gamma_\infty \circ \phi = g \cdot \gamma_0$. The question is how to incorporate this *a priori* information in the evolution equation (2) *without* assuming any knowledge of the functional E; in other words, we wish to suitably modify (2) such that the following two requirements are met.

1) $t \mapsto E \circ \gamma_t$ should be a decreasing function of t, that is, the resulting evolution equation should remain a descent equation on the functional E.
2) γ_∞ and γ_0 should be related, up to reparametrization, by a transformation of the Lie group G.

Note that the first requirement guarantees that the modified evolution equation still minimizes the functional E and thus continues to solve the original problem for which the functional E was intended. The second requirement is that γ_∞ lie in the union $\bigcup_\phi G \cdot (\gamma_0 \circ \phi)$ of orbits, where the union is taken over all reparametrizations ϕ of $\mathbf{I}$. Note in particular that if $\gamma_\infty \in G \cdot \gamma_0$, then the second requirement is met.

Our approach hinges on the following proposition:

Proposition 1: If $\gamma_t \in G \cdot \gamma_0$ for all $t \in \mathbb{R}^+$ then $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_t)$ for all $t \in \mathbb{R}^+$. Conversely, if Γ is finite dimensional, then $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_t)$ for all $t \in \mathbb{R}^+$ implies $\gamma_t \in G \cdot \gamma_0$ for all $t \in \mathbb{R}^+$.

Proof: See Appendix. □

Remark 1: The importance of Proposition 1 lies in the fact that it converts the original difficult problem of verifying whether or not a curve lies in the orbit of another curve into a family of tractable problems, each consisting of verifying whether a given vector lies in a certain vector space.

Remark 2: To use the full force of Proposition 1, we shall assume henceforth that Γ is finite-dimensional. This restricts in no way the practical applications of the results that follows, since, as was noted following (4), in all implementations of curve evolution equations, the space of curves can be identified with $\mathbb{R}^{2N}$ for some positive integer N, and hence is finite-dimensional. For simplicity of notation however, we shall use the notation of (2).

Consider the tangent space $T_{\gamma_t}(\Gamma)$ to Γ at the curve γ_t. Recall that the tangent space $T_{\gamma_t}(G \cdot \gamma_t)$ is a finite-dimensional vector subspace of $T_{\gamma_t}(\Gamma)$, for all $t \in \mathbb{R}^+$. Let $\pi_{\gamma_t} : T_{\gamma_t}(\Gamma) \to T_{\gamma_t}(G \cdot \gamma_t)$ be the projection operator defined with respect to the inner product $\langle \cdot, \cdot \rangle_{\gamma_t}$. We have the following result:

Proposition 2: Let $E : \Gamma \to \mathbb{R}$ be a smooth functional. Let $(\gamma_t)_t$ be a one-parameter family of smooth closed plane curves satisfying the evolution equation

$$\frac{d\gamma_t}{dt} = -\pi_{\gamma_t}\left(\frac{\delta E}{\delta \gamma}(\gamma_t)\right),$$
$$\gamma_t|_{t=0} = \gamma_0. \tag{6}$$

Then

1) $\gamma_\infty \in G \cdot \gamma_0$, and
2) $t \mapsto E \circ \gamma_t$ is a decreasing function of t.

Proof: We have $\pi_{\gamma_t}((\delta E/\delta \gamma)(\gamma_t)) \in T_{\gamma_t}(G \cdot \gamma_t)$ for all $t \in \mathbb{R}^+$, and hence $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_t)$ for all $t \in \mathbb{R}^+$; the fact that $\gamma_\infty \in G \cdot \gamma_0$ thus follows from Proposition 1. To show that $t \mapsto E \circ \gamma_t$ is a decreasing function of t, we write:

$$\begin{aligned}\frac{d(E \circ \gamma_t)}{dt} &= \left\langle \frac{\delta E}{\delta \gamma}(\gamma_t), \frac{d\gamma_t}{dt} \right\rangle_{\gamma_t} \\ &= \left\langle \frac{\delta E}{\delta \gamma}(\gamma_t), -\pi_{\gamma_t}\left(\frac{\delta E}{\delta \gamma}(\gamma_t)\right) \right\rangle_{\gamma_t} \\ &= -\left\langle \frac{\delta E}{\delta \gamma}(\gamma_t), \pi_{\gamma_t}\left(\frac{\delta E}{\delta \gamma}(\gamma_t)\right) \right\rangle_{\gamma_t}.\end{aligned}$$

Since the projection π_{γ_t} is defined with respect to the inner product $\langle \cdot, \cdot \rangle_{\gamma_t}$ on $T_{\gamma_t}(\Gamma)$, we have

$$\langle X - \pi_{\gamma_t}(X), \pi_{\gamma_t}(Y) \rangle_{\gamma_t} = 0, \quad \forall X, Y \in T_{\gamma_t}(\Gamma)$$

that is,

$$\langle X, \pi_{\gamma_t}(Y) \rangle_{\gamma_t} = \langle \pi_{\gamma_t}(X), \pi_{\gamma_t}(Y) \rangle_{\gamma_t}, \quad \forall X, Y \in T_{\gamma_t}(\Gamma).$$

We thus obtain,

$$\frac{d(E \circ \gamma_t)}{dt} = -\left\langle \pi_{\gamma_t}\left(\frac{\delta E}{\delta \gamma}(\gamma_t)\right), \pi_{\gamma_t}\left(\frac{\delta E}{\delta \gamma}(\gamma_t)\right) \right\rangle_{\gamma_t} \leq 0$$

and hence $t \mapsto E \circ \gamma_t$ is a decreasing function of t. □

The evolution equation (6) fulfills both of the requirements we had imposed on constraining curve evolution via transformation groups. The tangent space $T_\gamma(G \cdot \gamma)$ to the orbit $G \cdot \gamma$ at γ is the vector space spanned by the infinitesimal generators of the group action of G at γ. By virtue of Proposition 2 therefore, projecting the velocity vector of a curve onto the Lie algebra of infinitesimal generators of a group action yields an evolution which minimizes the desired functional while taking place on the desired orbit. In Section III, we compute π_γ for some of the Lie groups we shall consider.

It is important to note that constraining γ_t to evolve on the orbit $G \cdot \gamma_0$ of γ_0 under the action of the Lie group G is equivalent to estimating a one-parameter family $(g_t)_{t \in \mathbb{R}^+} \subset G$ such that $\gamma_t = g_t \cdot \gamma_0$ for all t. Indeed, expressing $E \circ \gamma_t$ as $E \circ (g_t \cdot \gamma_0)$, the gradient descent equation for g_t can be written:

$$\frac{dg_t}{dt} = -\frac{\delta(E \circ (g_t \cdot \gamma_0))}{\delta g_t} = -\left\langle \frac{\delta E}{\delta \gamma}, \frac{\delta(g_t \cdot \gamma_0)}{\delta g_t} \right\rangle_{\gamma_t}.$$

This is a descent equation in the Lie group G itself, and not in the space of all plane curves. As G is finite-dimensional, and is typically small-dimensional, it can be locally parametrized by a small number of parameters; the descent equation in G then corresponds to estimating these few parameters which define the transformation g_∞ relating γ_0 and γ_∞.

III. Lie Groups of Plane Transformations

We consider a (discretized) curve γ to be an N-tuple $(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)$ of points $\vec{p}_i = (x_i, y_i) \in \mathbb{R}^2$; an element of the tangent space to Γ at γ is then given by an N-tuple $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N)$ of vectors $\vec{v}_i \in \mathbb{R}^2$. We also assume given a one-parameter family $(\gamma_t)_{t \geq 0}$ of curves evolving according to (4). This has the advantage of illustrating very concretely the computation of the projection operator π_γ. Recall that the tangent space $T_\gamma(\Gamma)$ is endowed with the following inner product:

$$\langle X, Y \rangle = \sum_{i=1}^{N} \langle \vec{v}_i, \vec{w}_i \rangle_{\mathbb{R}^2}$$

where $X = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N)$, $Y = (\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_N)$ are elements of $T_\gamma(\Gamma)$. In what follows, all projection operators π_γ are computed with respect to this inner product.

A. The Group of Plane Translations

The most basic nontrivial group of transformations of the plane is the group of plane translations. This group is isomorphic to $(\mathbb{R}^2, +)$, and an element (a, b) of it acts on $\mathbb{R}^2$ by

$$(a, b) \cdot (x, y) = (x + a, y + b).$$

In this section, we describe how to constrain active contour evolution equations so that the evolving contour remain on the orbit, under the group of plane translations, of the initial contour. Let then $G \sim \mathbb{R}^2$ be this group. Let $\tau \mapsto (a(\tau), b(\tau))$ be a one-parameter family of G with $(a(0), b(0))$ the identity element $(0, 0)$. Then

$$\left. \frac{d[(a(\tau), b(\tau)) \cdot (x, y)]}{d\tau} \right|_{\tau=0} = \left. \frac{da(\tau)}{d\tau} \right|_{\tau=0} (1, 0) + \left. \frac{db(\tau)}{d\tau} \right|_{\tau=0} (0, 1).$$

The tangent space $T_\gamma(G \cdot \gamma)$ of $G \cdot \gamma$ at $\gamma = (\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)$ is thus two-dimensional, and spanned by the vectors

$$\begin{cases} A = ((1, 0), (1, 0), \ldots, (1, 0)) \\ B = ((0, 1), (0, 1), \ldots, (0, 1)). \end{cases}$$

Let $X = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N) \in T_{\gamma_t}(\Gamma)$ be a vector tangent to Γ at γ_t; the projection operator $\pi_{\gamma_t} : T_{\gamma_t}(\Gamma) \to T_{\gamma_t}(G \cdot \gamma_t)$ associated to G is given by $X \mapsto \pi_{\gamma_t}(X) = \alpha_t A + \beta_t B$, where $\alpha_t, \beta_t \in \mathbb{R}$ are such that

$$\langle \alpha_t A + \beta_t B - X, \alpha_t A + \beta_t B - X \rangle = \sum_{i=1}^{N} \|(\alpha_t, \beta_t) - \vec{v}_i\|^2$$

is minimized. The necessary conditions

$$\frac{\partial}{\partial \alpha_t} \sum_{i=1}^{N} \|(\alpha_t, \beta_t) - \vec{v}_i\|^2 = \frac{\partial}{\partial \beta_t} \sum_{i=1}^{N} \|(\alpha_t, \beta_t) - \vec{v}_i\|^2 = 0$$

for a minimum easily yield

$$(\alpha_t, \beta_t) = \frac{1}{N} \sum_{i=1}^{N} \vec{v}_i.$$

Thus, in order to constrain evolution equation (4) to evolution on the orbit of γ_0 under the group of plane translations, (4) should be replaced with

$$\begin{cases} \frac{d\vec{p}_i}{dt}(t) = \frac{1}{N} \sum_{l=1}^{N} \vec{F}_l\left((\vec{p}_j(t))_{j=1}^{N}\right), \\ \vec{p}_i|_{t=0} = \vec{p}_{i,0}, \end{cases} \quad t \geq 0, \quad i = 1, \ldots, N$$

and, correspondingly, (5) with

$$\begin{cases} \vec{p}_i((k+1)\Delta t) = \vec{p}_i(k\Delta t) + \frac{\Delta t}{N} \sum_{l=1}^{N} \vec{F}_l\left((\vec{p}_j(k\Delta t))_{j=1}^{N}\right), \\ \vec{p}_i|_{k=0} = \vec{p}_{i,0}, \end{cases} \quad k \in \mathbb{N}, \quad i = 1, \ldots, N.$$

B. The Group of Plane Rotations

In this section, we describe how to constrain active contour evolution equations so that the evolving contour remain on the orbit, under the group of plane rotations, of the initial contour. The group G of plane rotations is isomorphic to the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and an element $\theta \in G$ acts on $\mathbb{R}^2$ by

$$\theta \cdot (x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

Let $\tau \mapsto \theta(\tau)$ be a one-parameter family of G with $\theta(0)$ the identity element 0. Then

$$\left. \frac{d[\theta(\tau) \cdot (x, y)]}{d\tau} \right|_{\tau=0} = \left. \frac{d\theta(\tau)}{d\tau} \right|_{\tau=0} (-y, x).$$

The tangent space $T_\gamma(G\cdot\gamma)$ of $G\cdot\gamma$ at $\gamma = (\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)$ (with $\vec{p}_i = (x_i, y_i)$) is thus one-dimensional, and spanned by the vector

$$C = ((-y_1, x_1), (-y_2, x_2), \ldots, (-y_N, x_N)).$$

Let $X = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N) \in T_{\gamma_t}(\Gamma)$ be a vector tangent to Γ at γ_t, with $\vec{v}_i = (v_i^{(1)}, v_i^{(2)})$ for $i = 1, \ldots, N$; the projection operator $\pi_{\gamma_t} : T_{\gamma_t}(\Gamma) \to T_{\gamma_t}(G\cdot\gamma_t)$ associated to G is given by $X \mapsto \pi_{\gamma_t}(X) = \omega_t C$, where $\omega_t \in \mathbb{R}$ is such that

$$\langle \omega_t C - X, \omega_t C - X\rangle = \sum_{i=1}^{N} \|\omega_t(-y_i, x_i) - \vec{v}_i\|^2$$

is minimized. The necessary conditions for a minimum easily yield

$$\omega_t = \frac{\sum_{i=1}^{N}\left(x_i v_i^{(2)} - y_i v_i^{(1)}\right)}{\sum_{i=1}^{N}(x_i^2 + y_i^2)}.$$

To constrain evolution (2) to evolution on the orbit of γ_0 under the group of plane rotations therefore, (4) should be replaced with

$$\begin{cases} \frac{d\vec{p}_i}{dt}(t) = \frac{\sum_{i=1}^{N}\left(x_i v_i^{(2)} - y_i v_i^{(1)}\right)}{\sum_{i=1}^{N}(x_i^2+y_i^2)}(-y_i(t), x_i(t)), \\ \vec{p}_i|_{t=0} = \vec{p}_{i,0}, \end{cases} \quad t \geq 0, \quad i = 1, \ldots, N$$

and, correspondingly, (5) with (see first equation at the bottom of the page) where $\vec{F}_l((\vec{p}_j(t))_{j=1}^N) = (v_l^{(1)}, v_l^{(2)})$ for $l = 1, \ldots, N$.

C. *The Group of Euclidean Plane Transformations*

In this section, we describe how to constrain active contour evolution equations so that the evolving contour remain on the orbit, under the group of plane Euclidean transformations, of the initial contour. The group G of Euclidean plane transformations is the semi-direct product of the group of plane rotations and the group of plane translations, and an element $(a, b, \theta) \in G$ acts on $\mathbb{R}^2$ by

$$(a, b, \theta)\cdot(x, y) = (x\cos\theta - y\sin\theta - a, x\sin\theta + y\cos\theta - b).$$

It follows from the above that the tangent space $T_\gamma(G\cdot\gamma)$ of $G\cdot\gamma$ at $\gamma = (\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N)$ (with $\vec{p}_i = (x_i, y_i)$) is three-dimensional, and spanned by the vectors

$$\begin{cases} A = ((1, 0), (1, 0), \ldots, (1, 0)) \\ B = ((0, 1), (0, 1), \ldots, (0, 1)) \\ C = ((-y_1, x_1), (-y_2, x_2), \ldots, (-y_N, x_N)) \end{cases}$$

Let $X = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N) \in T_{\gamma_t}(\Gamma)$ be a vector tangent to Γ at γ_t, with $\vec{v}_i = (v_i^{(1)}, v_i^{(2)})$ for $i = 1, \ldots, N$; the projection operator $\pi_{\gamma_t} : T_{\gamma_t}(\Gamma) \to T_{\gamma_t}(G\cdot\gamma_t)$ associated to G is given by $X \mapsto \pi_{\gamma_t}(X) = \alpha_t A + \beta_t B + \omega_t C$, where $\alpha_t, \beta_t, \omega_t \in \mathbb{R}$ are such that

$$\langle \alpha_t A + \beta_t B + \omega_t C - X, \alpha_t A + \beta_t B + \omega_t C - X\rangle = \sum_{i=1}^{N} \|(\alpha_t - \omega_t y_i, \beta_t + \omega_t x_i) - \vec{v}_i\|^2$$

is minimized. An easy calculation shows that $\alpha_t, \beta_t, \omega_t$ are given by

$$\begin{pmatrix}\alpha_t \\ \beta_t \\ \omega_t\end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{N}\sum_i y_i \\ 0 & 1 & \frac{1}{N}\sum_i x_i \\ -\frac{1}{N}\sum_i y_i & \frac{1}{N}\sum_i x_i & \frac{1}{N}\sum_i (x_i^2 + y_i^2) \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{N}\sum_i v_i^{(1)} \\ \frac{1}{N}\sum_i v_i^{(2)} \\ \frac{1}{N}\sum_i \left(x_i v_i^{(2)} - y_i v_i^{(1)}\right) \end{pmatrix}$$

With $\vec{F}_l((\vec{p}_j(t))_{j=1}^N)$ in (4) and (5) being written as $\vec{F}_l((\vec{p}_j(t))_{j=1}^N) = (v_l^{(1)}, v_l^{(2)})$ for $l = 1, \ldots, N$, and with $\alpha_t, \beta_t, \omega_t$ given as above, constraining evolution (4) to evolution on the orbit of γ_0 under the group of Euclidean plane transformations is performed by replacing (4) with

$$\begin{cases} \frac{d\vec{p}_i}{dt}(t) = (\alpha_t - \omega_t y_i(t), \beta_t + \omega_t x_i(t)), \\ \vec{p}_i|_{t=0} = \vec{p}_{i,0}, \end{cases} \quad t \geq 0, \quad i = 1, \ldots, N$$

and, correspondingly, (5) with (see second equation at the bottom of the page).

D. *The Group of Affine Plane Transformations*

In this section, we describe how to constrain active contour evolution equations so that the evolving contour remain on the orbit, under the group of affine plane transformations,

$$\begin{cases} \vec{p}_i((k+1)\Delta t) = \vec{p}_i(k\Delta t) - \Delta t \frac{\sum_{i=1}^{N}\left(x_i v_i^{(2)} - y_i v_i^{(1)}\right)}{\sum_{i=1}^{N}(x_i^2+y_i^2)}(-y_i(t), x_i(t)), \quad k \in \mathbb{N}, \quad i = 1, \ldots, N \\ \vec{p}_i|_{k=0} = \vec{p}_{i,0} \end{cases}$$

$$\begin{cases} \vec{p}_i((k+1)\Delta t) = \vec{p}_i(k\Delta t) + \Delta t(\alpha_t - \omega_t y_i(kt), \beta_t + \omega_t x_i(kt)), \quad k \in \mathbb{N}, \quad i = 1, \ldots, N \\ \vec{p}_i|_{k=0} = \vec{p}_{i,0}. \end{cases}$$

of the initial contour. The group G of affine plane transformations is the semi-direct product of the group $GL(2,\mathbb{R})$ of invertible two-by-two real matrices and of $\mathbb{R}^2$, and an element $(a,b,c,d,e,f) \in G$ (where $ad-bc \neq 0$) acts on $\mathbb{R}^2$ by

$$(a,b,c,d,e,f)\cdot(x,y) = (ax+by+e, cx+dy+f)$$

The tangent space $T_\gamma(G\cdot\gamma)$ of $G\cdot\gamma$ at $\gamma = (\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_N)$ (with $\vec{p}_i = (x_i, y_i)$) is thus six-dimensional, and spanned by the vectors

$$\begin{cases} A = ((1,0),(1,0),\ldots,(1,0)) \\ B = ((0,1),(0,1),\ldots,(0,1)) \\ U_1 = ((x_1,0),(x_2,0),\ldots,(x_N,0)) \\ U_2 = ((y_1,0),(y_2,0),\ldots,(y_N,0)) \\ V_1 = ((0,x_1),(0,x_2),\ldots,(0,x_N)) \\ V_2 = ((0,y_1),(0,y_2),\ldots,(0,y_N)) \end{cases}$$

Let $X = (\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_N) \in T_{\gamma_t}(\Gamma)$ be a vector tangent to Γ at γ_t, with $\vec{v}_i = (v_i^{(1)}, v_i^{(2)})$ for $i = 1,\ldots,N$; the projection operator $\pi_{\gamma_t} : T_{\gamma_t}(\Gamma) \to T_{\gamma_t}(G\cdot\gamma_t)$ associated to G is given by $X \mapsto \pi_{\gamma_t}(X) = \alpha_t A + \beta_t B + \alpha_t^{(1)} U_1 + \alpha_t^{(2)} U_2 + \beta_t^{(1)} V_1 + \beta_t^{(2)} V_2$, where $\alpha_t, \beta_t, \alpha_t^{(1)}, \beta_t^{(1)}, \alpha_t^{(2)}, \beta_t^{(2)}$ are such that

$$\sum_{i=1}^{N} \left\| \left(\alpha_t + \alpha_t^{(1)} x_i + \alpha_t^{(2)} y_i, \beta_t + \beta_t^{(1)} x_i + \beta_t^{(2)} y_i\right) - \vec{v}_i \right\|^2$$

is minimized. An easy calculation shows that $\alpha_t, \beta_t, \alpha_t^{(1)}, \beta_t^{(1)}, \alpha_t^{(2)}, \beta_t^{(2)}$ are given by solving the overdetermined systems

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \cdots & \cdots & \cdots \\ 1 & x_N & y_N \end{pmatrix} \begin{pmatrix} \alpha_t \\ \alpha_t^{(1)} \\ \alpha_t^{(2)} \end{pmatrix} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \\ \cdots \\ v_N^{(1)} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \cdots & \cdots & \cdots \\ 1 & x_N & y_N \end{pmatrix} \begin{pmatrix} \beta_t \\ \beta_t^{(1)} \\ \beta_t^{(2)} \end{pmatrix} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \\ \cdots \\ v_N^{(2)} \end{pmatrix}.$$

With $\vec{F}_l((\vec{p}_j(t))_{j=1}^N)$ in (4) and (5) being written as $\vec{F}_l((\vec{p}_j(t))_{j=1}^N) = (v_l^{(1)}, v_l^{(2)})$ for $l = 1,\ldots,N$, and with α_t, $\beta_t, \alpha_t^{(1)}, \beta_t^{(1)}, \alpha_t^{(2)}, \beta_t^{(2)}$ given by solving the above linear overdetermined systems, constraining evolution equation (4) to evolution on the orbit of γ_0 under the group of affine plane transformations is performed by replacing (4) with (equation at the bottom of the page) and (5) with the corresponding temporally discretized equation, as illustrated in the previous cases.

IV. Experimental Results

In our implementation of curve evolution on orbits of Lie transformation groups, we adopt the curve functional

$$\gamma \mapsto E[\gamma] = \frac{1}{2}\int_0^1 \{\mu\|\dot{\gamma}(s)\|^2 + \nu\|\ddot{\gamma}(s)\|^2\}\, ds + \int_0^1 L_{\text{ext}}(\gamma(s))\, ds \quad (7)$$

which is of the same form as the functional in (1) (with $\mathrm{I} = [0,1]$) and is similar to the functional proposed in [10]. Whereas the first integral on the right-hand side of (7) represents the internal energy of the active contour due to stretching and bending, the second integral represents the image-dependent (i.e., external) energy of the snake, and the Lagrangian L_{ext} is defined as the distance of the active contour from the local gradient maxima of the image function. The positive coefficients μ and ν control the elasticity and stiffness, respectively, of the active contour. It should be noted that using the Lie group approach, the active contour becomes less sensitive to the standard active contour parameters that represent stretching and bending. Although these "smoothness" terms are still retained, the final contour determined by the transformation is not sensitive to minor variations in the parameters. As such, the contours evolved by Lie groups of transformation are more robust to parameter selection.

The Euler-Lagrange descent equation corresponding to the functional (7) is given by

$$\frac{d\gamma_t}{dt}(s) = \mu\ddot{\gamma}_t(s) - \nu\gamma_t^{(4)}(s) - \nabla L_{\text{ext}}(\gamma_t(s)) \quad (8)$$

where $\gamma^{(4)}$ denotes the fourth derivative of γ with respect to the arc parameter s. Spatial and temporal discretizations of (8) yield a representation of the curve γ as a finite ordered set of points $(\vec{p}_i)_{i=1}^N$ in $\mathbb{R}^2$, and yield an evolution equation of the form

$$\vec{p}_i^{(k+1)} = \vec{p}_i^{(k)} + \vec{v}_i^{(k)} \quad (9)$$

where the superscript denotes the iteration index (see (5)). Assuming a temporal discretization step of Δt, the initial curve γ_0 is thus represented by the ordered set of points $(\vec{p}_i^{(0)})$, while the curve $\gamma_{k\Delta t}$, corresponding to iteration k of the evolution, is represented by the ordered set $(\vec{p}_i^{(k)})$.

A. Synthetic Results

We first demonstrate the basic concept of active contour evolution on Lie group orbits through a set of synthetic images. The synthetic experiments are followed by experiments on real image sequences. We adopt the discretized version (9) of curve evolution equation (8) as our benchmark, and we shall call it *unconstrained curve evolution*. In all experiments, the energy functional parameters μ and ν are assigned values 0.01 and 0,

$$\begin{cases} \frac{d\vec{p}_i}{dt}(t) = \left(\alpha_t + \alpha_t^{(1)} x_i(t) + \alpha_t^{(2)} y_i(t), \beta_t + \beta_t^{(1)} x_i(t) + \beta_t^{(2)} y_i(t)\right) & t \geq 0, \quad i = 1,\ldots,N \\ \vec{p}_i|_{t=0} = \vec{p}_{i,0} \end{cases}$$

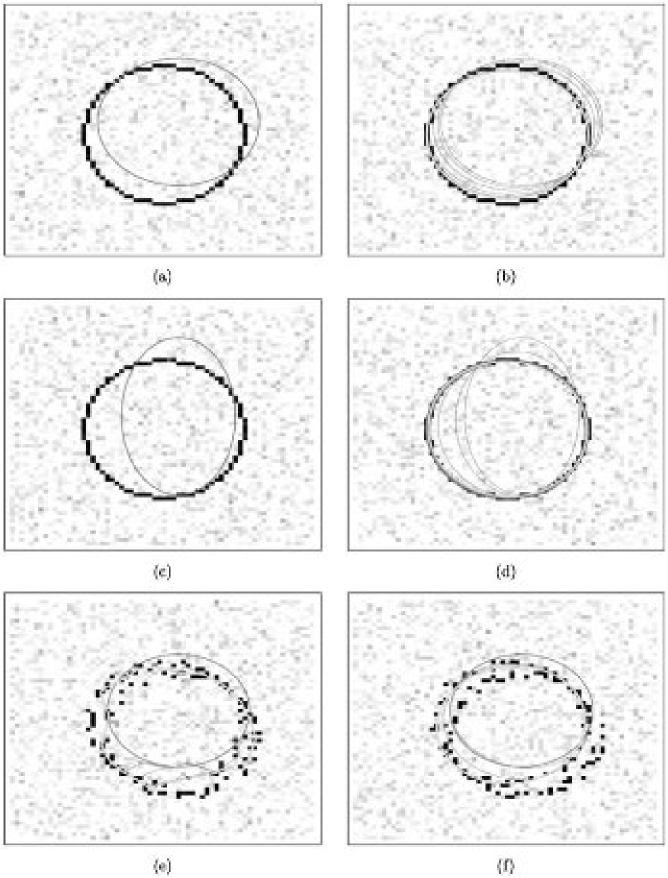

Fig. 1. Capturing an ellipse: (a) initial contour; (b) evolution on translation group orbit; (c) initial contour; (d) evolution on affine group orbit; (e) unconstrained evolution for capturing noisy ellipse; (f) evolution on affine group orbit for capturing noisy ellipse.

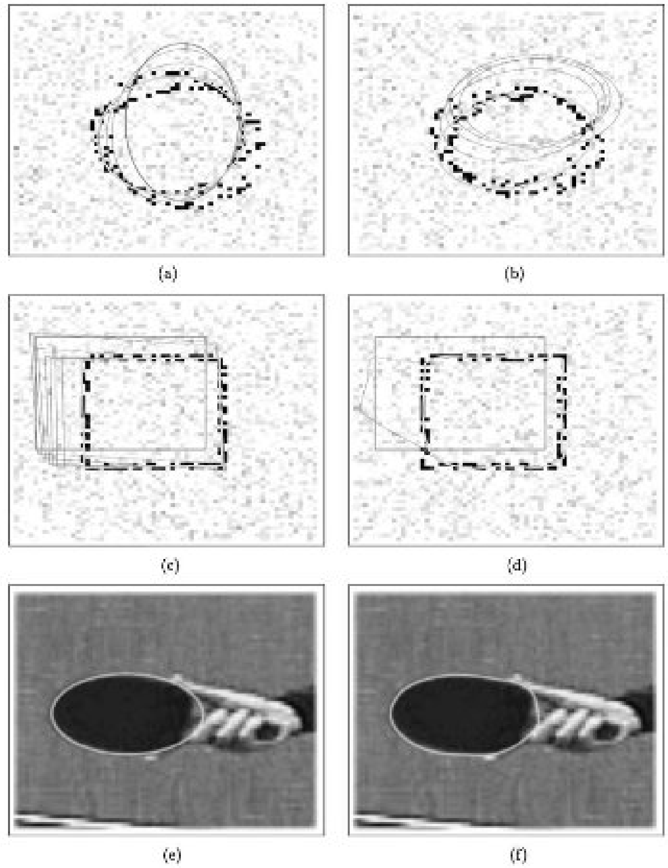

Fig. 2. (a) Evolution on affine group orbit for capturing noisy ellipse; (b) evolution on Euclidean group orbit for capturing noisy ellipse; (c) evolution on affine group orbit for capturing noisy rectangle; (d) unconstrained evolution for capturing noisy rectangle; (e) evolution on affine group orbit for capturing paddle; (f) unconstrained evolution for capturing paddle.

respectively. In all the images shown, a red thin contour represents either the initial curve (γ_0) or the intermediate curves $(\gamma_{k\Delta t}, k = 0, 1, \ldots)$ (shown every 50 iterations, i.e., for $k = 0, 50, 100, \ldots$), while a thick green contour represents the contour at convergence (γ_∞). The stopping criterion we have used is as follows: Curve evolution stops whenever there is less than one pixel maximum displacement between the current and the previous contours at all contour points.

Fig. 1(a) is constructed from the binary image of a dark ellipse (the target) on a uniform white background by the addition of zero-mean white Gaussian noise of normalized variance .01. The thin red contour represents the initial position of the snake (γ_0). Clearly, the initial contour γ_0 and the target ellipse differ only by a translation; hence, constraining the snake γ_t to evolve on the orbit of γ_0 under the group of translations should yield a contour γ_∞ at convergence that coincides with the target ellipse. The result of this constrained evolution is as expected and is shown in Fig. 1(b). Note that convergence is achieved even if the initial axes are different in length. Consider now Fig. 1(c); the underlying image is similar to that in Fig. 1(b), but the initial contour γ_0 is now defined so as to be unrelated to the target ellipse by a mere translation. Thus, constraining the snake γ_t to evolve on the orbit of γ_0 under the group of translations would not yield a contour γ_∞ at convergence that coincides with the target ellipse. However, γ_0 being an ellipse as well, constraining the snake γ_t to evolve on the orbit of γ_0 under the group of affine transformation allows γ_∞ to coincide with the target ellipse. This is shown in Fig. 1(d). Note that in Fig. 1(b) [resp. 1(d)] the intermediate contours (γ_t) are all related to the initial contour γ_0 by a translation (resp. affine transformation), consistent with evolution on a transformation group orbit.

In Fig. 1(e) and (f), the target ellipse is itself randomly deformed while preserving its rough elliptic shape. Fig. 1(e) shows the result of unconstrained snake evolution, yielding a contour γ_∞ at convergence that has a very irregular shape. Constraining snake evolution to lie on the orbit of γ_0 under the affine group, however, yields an elliptic contour γ_∞ at convergence, as can be seen in Fig. 1(f). Clearly, constraining contour evolution to take place on a transformation group orbit has a regularizing effect on the contour.

Fig. 2(a) shows the result of snake evolution on the orbit of γ_0 under the affine group, as in Fig. 1(f) and with a similar underlying image, but with different initial contour γ_0. Fig. 2(b), on the other hand, shows the result of snake evolution on the orbit of γ_0 under the Euclidean group. In both cases, the contour γ_∞ at convergence captures the target noisy ellipse while retaining its elliptic shape. In Fig. 2(c) and (d) the underlying image is constructed just as in Fig. 1 except that the target is now a dark rectangle which has been randomly distorted. Fig. 2(c) shows the result of constrained snake evolution on the affine group orbit

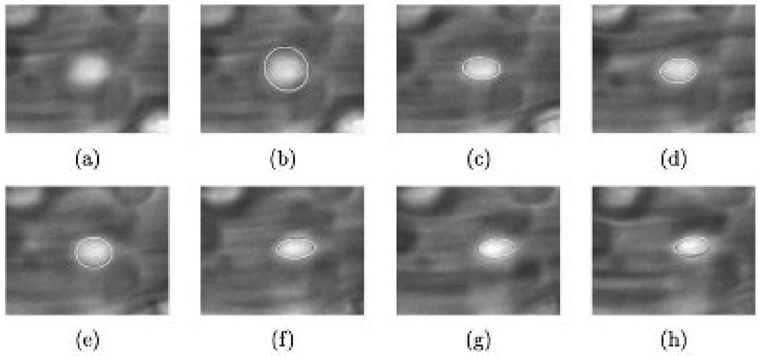

Fig. 3. Results of *in vivo* cell tracking by constrained snake evolution on affine group orbit: (a) initial contour in frame 0; (b) frame 1; (c) frame 3; (d) frame 5; (e) frame 10; (f) frame 15; (g) frame 20; (h) frame 25.

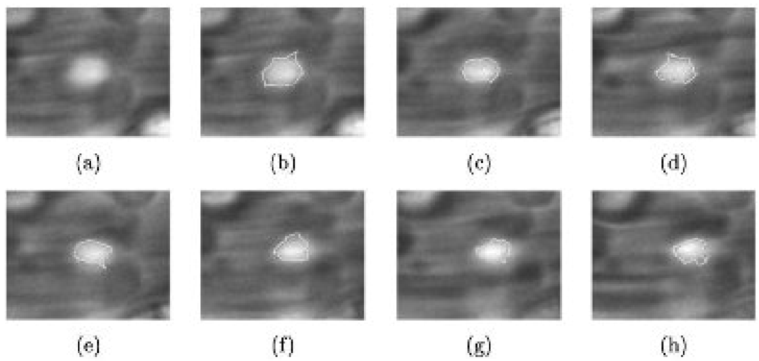

Fig. 4. Results of *in vivo* cell tracking by unconstrained snake evolution: (a) initial contour in frame 0; (b) frame 1; (c) frame 3; (d) frame 5; (e) frame 10; (f) frame 15; (g) frame 20; (h) frame 25.

of an initial contour γ_0 which is itself a rectangle larger than the target rectangle. As can be seen in that same figure, the rectangular shape of the target noisy rectangle is perfectly captured by the contour γ_∞ at convergence. Unconstrained snake evolution on a similar image and starting from a similar initial contour γ_0 yields a contour γ_∞ at convergence which is not rectangular anymore, as can be seen in Fig. 2(d).

To further demonstrate the regularizing effect of contour evolution on transformation group orbits, we have used video frames from a table tennis sequence. The frame to frame coherence that is required for tracking is maintained through our proposed algorithm, as is shown in Fig. 2(e): Here, the contour γ_∞ at convergence is constrained to lie on the orbit of the initial contour γ_0 under the affine group. Unconstrained contour evolution [from the same initial contour as in Fig. 2(e)], on the other hand, yields the result in Fig. 2(f); there, the contour at convergence is distorted, and the distortion worsens as we progress through the sequence. We shall clarify this further when we discuss the tracking examples below.

B. Tracking Results

We use three different sequences from three different application areas to demonstrate the usefulness of constraining snake evolution to transformation group orbits. In each of these sequences, tracking is done over 25 frames, and in each frame (except frame zero), the initial contour from which snake evolution starts is the contour at convergence of snake evolution of the preceding frame.

The first tracking application is that of tracking a white blood cell *in vivo*. Fig. 3 shows the results of tracking using affine group constrained snake evolution, while Fig. 4 shows the results of tracking using unconstrained snake evolution. Not only does our proposed algorithm allow the shape to be captured properly, but the detection of the shape is consistent with the direction of flow within the blood vessel as well. The result of tracking using unconstrained snake evolution violates the known circular shape of the leukocyte and yields boundary localization error. Figs. 3(a) and 4(a) show the initial contours in frame 0, while Figs. 3(b)–(h) and 4(b)–(h) depict the tracking results in frames 1, 3, 5, 10, 15, 20, and 25, respectively.

For the table tennis sequence, in which a paddle is tracked, the results of affine group constrained snake evolution and unconstrained snake evolution are shown in Figs. 5 and 6, respectively. Fig. 5(a) and (b) depict the initial contours in frame 0, while Figs. 5(b)–(g) and 6(b)–(g) show the tracking results in frames 1, 5, 10, 15, 20, and 25, respectively. Fig. 6(g) clearly shows that a simple snake with no additional constraint clearly loses the paddle shape while tracking. The errors in capturing the shape in previous frames are compounded and ultimately force the active contour to drift away from the true boundary. For our proposed algorithm, with the special choice of the affine group, shape fidelity is maintained for the entire 25 frames under observation.

The final image sequence gives an example of tracking of a tank under difficult imaging conditions from infrared video. Figs. 7 and 8 show the results of affine group constrained snake evolution and unconstrained snake evolution, respectively. As in previous examples, Figs. 7(a) and 8(a) show the initial snake in frame 0, while Figs. 7(b)–(g) and 8(b)–(g) show the results of tracking in frames 1, 5, 10, 15, 20, and 25. Note that the

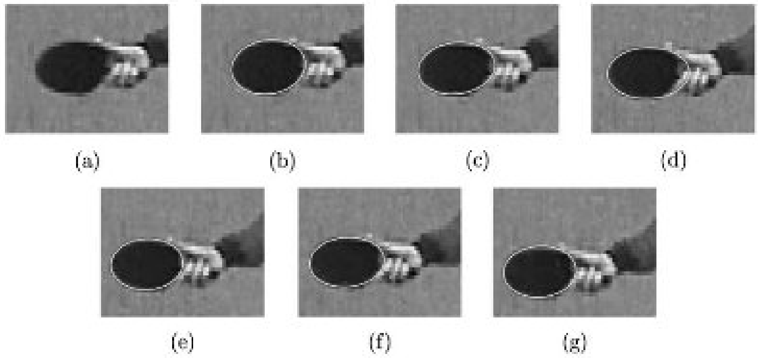

Fig. 5. Results of tracking paddle by constrained snake evolution on affine group orbit: (a) initial contour in frame 0; (b) frame 1; (c) frame 5; (d) frame 10; (e) frame 15; (f) frame 20; (g) frame 25.

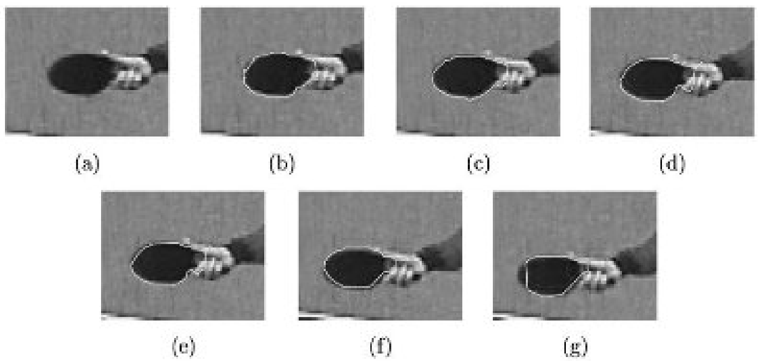

Fig. 6. Results of tracking paddle by unconstrained snake evolution: (a) initial contour in frame 0; (b) frame 1; (c) frame 5; (d) frame 10; (e) frame 15; (f) frame 20; (g) frame 25.

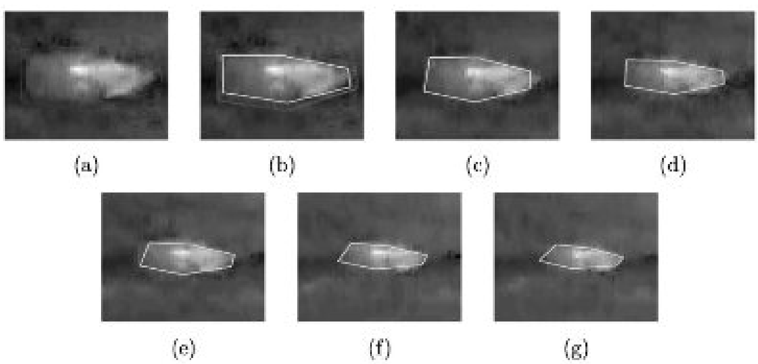

Fig. 7. Results of tracking tank in clutter by constrained snake evolution on affine group orbit: (a) initial contour in frame 0; (b) frame 0; (c) frame 5; (d) frame 10; (e) frame 15; (f) frame 20; (g) frame 25.

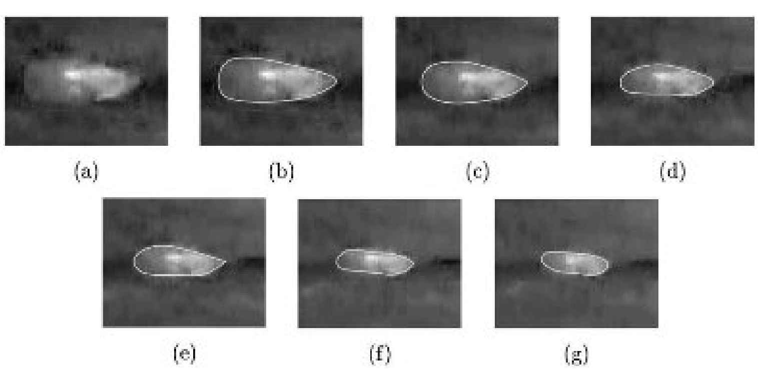

Fig. 8. Results of tracking tank in clutter by unconstrained snake evolution: (a) initial contour in frame 0; (b) frame 0; (c) frame 5; (d) frame 10; (e) frame 15; (f) frame 20; (g) frame 25.

TABLE I
SEGMENTATION ERROR BETWEEN ACTIVE CONTOUR AT CONVERGENCE AND THE GROUND TRUTH. POSITIVE AND NEGATIVE QUANTITIES REPRESENT OVER AND UNDER SEGMENTATION, RESPECTIVELY

Sequence		Frame Number						
		1^{st}	3^{rd}	5^{th}	10^{th}	15^{th}	20^{th}	25^{th}
Blood Cell	Affine	+7.31	-0.78	-1.89	-1.29	-3.43	-2.1	-1.51
	Unconstrained	-10.22	+1.32	-8.02	-7.33	-4.75	-1.99	-1.7
Table Tennis	Affine	+0.43	–	+1.28	−1.93	+0.21	+0.94	+1.18
	Unconstrained	-2.32	–	-2.85	-1.50	-4.39	-2.83	-1.08
Tank	Affine	-2.31	–	-6.74	-5.12	-3.83	-7.37	-12.31
	Unconstrained	+5.37	–	+3.2	+4.2	+1.7	+1.24	-0.46

unconstrained snake responds to spurious details and unreliable image clutter; the constrained snake, on the other hand, preserves its hexagonal shape throughout tracking, providing a target boundary that could be utilized for automatic target recognition.

For the three tracking examples, we provide numerical data to substantiate the validation of active contour evolution on the affine group orbit. Using manually segmented boundaries, we have computed the *segmentation error* for each frame for both the unconstrained active contour evolution and the evolution on the affine group orbit (Table I). A segmentation error of 0% means exact agreement with the "ground truth." Over-segmentation (under-segmentation) occurs when the detected segment is bigger (smaller) than the segment in the ground truth. The segmentation error is expressed as a percentage of excess or less number of pixels with respect to the size of the ground truth segment.

V. CONCLUSION

In this paper, we have presented a novel approach to constrain curve evolution equations to orbits of particular Lie groups of transformation. Such constraints are important in numerous applications of curve evolution where the preservation of certain geometric properties of curves is desired. The approach we have presented makes use of the relation between the group action and the infinitesimal generators of this action. In this way, the original problem of maintaining certain geometrical properties of the curve during its evolution is translated into a straightforward linear algebraic problem. The main advantage of the approach we have presented is that only the curve evolution equation is modified, in a very straightforward way, and no knowledge of or modification to the curve functional from which the curve evolution equation was derived, is assumed. The synthetic image segmentation results demonstrate the shape-preserving noise-resilient properties of the active contour moving on the orbits of Lie groups of transformation. Extending the results to tracking objects in a video sequence, we find that the novel active contour implementation is effective in tracking objects that move and distort according to Lie groups of transformation. Thus, for tracking the same object in a video sequence, the assumptions of evolution on orbits of Lie groups of transformation are well motivated and powerful. In contrast, using the traditional active contour evolution equations leads to detection of objects of arbitrary shape, which may lead to errors during tracking.

APPENDIX

In this Appendix, we provide a Proof of Proposition 1. Assume then that $\gamma_t \in G \cdot \gamma_0$ for all $t \in \mathbb{R}^+$. Then $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_0)$ for all $t \in \mathbb{R}^+$ as well. $\gamma_t \in G \cdot \gamma_0$ implies there exists $g \in G$ such that $\gamma_t = g \cdot \gamma_0$, and hence $G \cdot \gamma_t = G \cdot (g \cdot \gamma_0) = (G \cdot g) \cdot \gamma_0 = G \cdot \gamma_0$, by virtue of the associativity of the group action and the fact that G is a group. Thus $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_0) = T_{\gamma_t}(G \cdot \gamma_t)$ and the statement is proved. Conversely, and assuming Γ is finite-dimensional, assume $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_t)$ for all $t \in \mathbb{R}^+$. Identifying Γ with $\mathbb{R}^{2N}$ for some positive integer N, γ_t can be considered as a point in $\mathbb{R}^{2N}$. Assuming the Lie group G has dimension n and that it acts regularly at γ_t, $T_\gamma(G \cdot \gamma)$ is an n-dimensional vector subspace of $\mathbb{R}^{2N}$ for all $\gamma \in \mathbb{R}^{2N}$ in a small neighborhood of γ_t, yielding a field $\gamma \mapsto T_\gamma(G \cdot \gamma)$ of n-dimensional subspaces in a neighborhood of γ_t, for all $t \in \mathbb{R}^+$. Since these subspaces are spanned by the infinitesimal generators of the Lie group action, and since these infinitesimal generators form a Lie algebra of vector fields, the field $\gamma \mapsto T_\gamma(G \cdot \gamma)$ is completely integrable by virtue of Frobenius' theorem [18]. It is then possible to find local coordinates $(x_1, x_2, \ldots, x_{2N})$ in $\mathbb{R}^{2N}$ in a neighborhood of γ_t for which the orbit $G \cdot \gamma_t$ in that neighborhood is given by $x_{n+1} = \text{const}, x_{n+2} = \text{const}, \ldots x_{2N} = \text{const}$. It then follows from $(d\gamma_t/dt) \in T_{\gamma_t}(G \cdot \gamma_t)$, for all $t \in \mathbb{R}^+$, that $\gamma_t \in G \cdot \gamma_0$ for $t > 0$ small enough. By piecing together local coordinate charts, we deduce the result for $t \in \mathbb{R}^+$. $\square$

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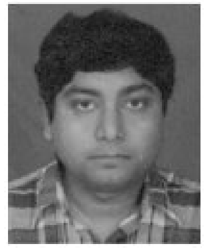

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