

# Effective-Mass Schrödinger Equation and Generation of Solvable Potentials

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## ABSTRACT

A one-dimensional Schrödinger equation with position-dependent effective mass in the kinetic energy operator is studied in the framework of an  $so(2, 1)$  algebra. New mass-deformed versions of Scarf II, Morse and generalized Pöschl-Teller potentials are obtained. Consistency with an intertwining condition is pointed out.

**Keywords:** Schrödinger equation, position-dependent effective mass,  $so(2, 1)$ , intertwining.

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Significant attention has been focussed on the issue of position-dependent-effective-mass (PDEM) quantum Hamiltonians and their impact on the construction of soluble quantum systems. Interest in PDEM stems from its physical relevance in problems of compositionally graded crystals [1], quantum dots [2], liquid crystals [3], etc., where the need for a varying mass has long been felt. Indeed the appearance of PDEM is well known in the energy density functional approach to the nuclear many-body problem in the context of nonlocal terms of the accompanying potential [4, 5, 6]. Exact solutions of the PDEM Schrödinger equation have also been reported by extending the methods of coordinate transformation and supersymmetric quantum mechanics [7–20]. In particular, for the free-particle case we have recently found [21] by exploiting the intertwining relation that for an appropriate choice of the mass function the problem can be completely solved leading to normalizable bound states.

The purpose here is to study the PDEM within an  $so(2, 1)$  algebra and obtain new mass-deformed solutions of the underlying functions characterizing the  $so(2, 1)$ , which are natural counterparts of those in the constant-mass case. We also realize that a class of solutions exists for which the effective potential induced by  $so(2, 1)$  coincides with the one provided by a first-order intertwining condition.

The PDEM kinetic energy operator has a wide range of forms. In the following we adopt von Roos' scheme [22], which has the advantage of a builtin Hermiticity. It is given by

$$\hat{T} = \frac{1}{4} \left[ m^\alpha(x) \hat{p} m^\beta(x) \hat{p} m^\gamma(x) + m^\gamma(x) \hat{p} m^\beta(x) \hat{p} m^\alpha(x) \right], \quad (1)$$

where  $\hat{p} (\equiv -i\hbar \frac{d}{dx})$  is the momentum operator,  $m(x)$  is the position-dependent mass and the parameters  $\alpha, \beta, \gamma$  are tied by the condition  $\alpha + \beta + \gamma = -1$ . We have shown elsewhere [21] that if we set  $m(x) = m_0 M(x)$ ,  $M(x)$  being dimensionless, and use the identity

$$\begin{aligned} M^\alpha \frac{d}{dx} M^\beta \frac{d}{dx} M^\gamma + M^\gamma \frac{d}{dx} M^\beta \frac{d}{dx} M^\alpha &= 2 \frac{d}{dx} \frac{1}{M} \frac{d}{dx} - (\beta + 1) \frac{M''}{M^2} \\ &+ 2 [\alpha(\alpha + \beta + 1) + \beta + 1] \frac{M'^2}{M^3}, \end{aligned} \quad (2)$$

the primes denoting derivatives with respect to the variable  $x \in (-\infty, \infty)$ , then the ambiguity parameters  $\alpha, \beta, \gamma$  get shifted to an overall effective potential energy

term. With  $\hbar = 2m_0 = 1$ , we get, for a given potential  $V(x)$ , a modified Schrödinger equation that reads

$$H\psi(x) \equiv \left[ -\frac{d}{dx} \frac{1}{M(x)} \frac{d}{dx} + V_{\text{eff}}(x) \right] \psi(x) = E\psi(x), \quad (3)$$

in which  $V_{\text{eff}}(x)$  also depends upon  $M(x)$  and its derivatives:

$$V_{\text{eff}}(x) = V(x) + \frac{1}{2}(\beta + 1) \frac{M''}{M^2} - [\alpha(\alpha + \beta + 1) + \beta + 1] \frac{M'^2}{M^3}. \quad (4)$$

Consider now the generators of the  $so(2, 1)$  algebra represented by

$$\begin{aligned} J_0 &= -i \frac{\partial}{\partial \phi}, \\ J_{\pm} &= e^{\pm i \phi} \left[ \pm \frac{1}{\sqrt{M}} \frac{\partial}{\partial x} + F(x) \left( i \frac{\partial}{\partial \phi} \mp \frac{1}{2} \right) + G(x) \right], \end{aligned} \quad (5)$$

where  $\phi$  is an auxiliary variable and we restrict to  $M(x) > 0$ . The commutation relations of  $so(2, 1)$ , namely  $[J_+, J_-] = -2J_0$ ,  $[J_0, J_{\pm}] = \pm J_{\pm}$ , imply that the functions  $F$  and  $G$  be constrained by the equations

$$F' = \sqrt{M}(1 - F^2), \quad G' = -\sqrt{M}FG. \quad (6)$$

With

$$\begin{aligned} J_0 |k\mu\rangle &= \mu |k\mu\rangle, \\ J^2 |k\mu\rangle &= k(k-1) |k\mu\rangle, \end{aligned} \quad (7)$$

where  $|k\mu\rangle = \chi_{k\mu}(x)e^{i\mu\phi}$  are basis functions appropriate to an  $so(2, 1)$  irreducible representation of the type  $D_k^+$  and  $\mu$  takes on values  $k, k+1, k+2, \dots$ , the Casimir operator  $J^2 = J_0^2 \mp J_0 - J_{\pm}J_{\mp}$ , when expanded, gives

$$\left[ -\frac{1}{\sqrt{M}} \frac{d}{dx} \frac{1}{\sqrt{M}} \frac{d}{dx} + V_{\mu} \right] \chi = -\left(k - \frac{1}{2}\right)^2 \chi, \quad (8)$$

in which  $V_{\mu}$  describes a one-parameter family of potentials

$$V_{\mu} = \frac{1}{\sqrt{M}} \left[ \left( \frac{1}{4} - \mu^2 \right) F' + 2\mu G' \right] + G^2. \quad (9)$$

From (8), it follows that the functions  $\chi(x) \equiv \chi_{k\mu}(x)$  are the eigenfunctions of different Hamiltonians but conform to the same energy level.

Equation (8) is in a direct one-to-one correspondence with the von-Roos-generated PDEM form (3) if we transform  $\chi(x) \rightarrow [M(x)]^{-1/4}\psi(x)$ . We have

$$\left[ -\frac{d}{dx} \frac{1}{M(x)} \frac{d}{dx} + \frac{M''}{4M^2} - \frac{7M'^2}{16M^3} + V_\mu \right] \psi = -\left(k - \frac{1}{2}\right)^2 \psi, \quad (10)$$

suggesting the identifications

$$V_{\text{eff}}(x) = \frac{M''}{4M^2} - \frac{7M'^2}{16M^3} + V_\mu, \quad (11)$$

$$E = -\left(k - \frac{1}{2}\right)^2, \quad (12)$$

along with

$$V(x) = \left[ \alpha(\alpha + \beta + 1) + \beta + \frac{9}{16} \right] \frac{M''}{M^3} - \frac{1}{4}(2\beta + 1) \frac{M''}{M^2} + V_\mu \quad (13)$$

on using the expression (4) for  $V_{\text{eff}}(x)$ .

The set of equations (10) – (13) extends the realization of  $so(2, 1)$  to the PDEM case. In particular, Eq. (13) states that for a given potential  $V(x)$  placed in a suitable mass environment,  $so(2, 1)$  as a potential algebra can be realized for it characterized by  $V_\mu$  and supporting the same set of energy eigenvalues (12). Of course, in the constant-mass case,  $V(x)$  reduces to  $V_\mu$ , which is as it should be.

A couple of observations are in order:

(a) A change of variable

$$u(x) = \int^x \sqrt{M(t)} dt \quad (14)$$

allows one to avoid explicit presence of  $\frac{1}{\sqrt{M}}$  factor in the generators (5). Consequently the conditions (6) read

$$\dot{\tilde{F}} = 1 - \tilde{F}^2, \quad \dot{\tilde{G}} = -\tilde{F}\tilde{G}, \quad (15)$$

where the overhead dot indicates a derivative with respect to the variable  $u(x)$  and  $(\tilde{F}, \tilde{G})$  are transformed  $(F, G)$  under (14). The forms (15) are similar to those for the conventional constant-mass case, for which it is known [23,24] that there exist at most three classes of realizations of the functions  $\tilde{F}$  and  $\tilde{G}$  depending on the sign of  $\tilde{\omega} = (\tilde{F}^2 - 1)/\tilde{G}^2$ :

$$\tilde{\omega} = -\frac{1}{b^2} < 0: \quad \tilde{F}(u) = \tanh(u - c), \quad \tilde{G}(u) = b \operatorname{sech}(u - c),$$

$$\begin{aligned}\tilde{\omega} = 0: \quad \tilde{F}(u) &= \pm 1, & \tilde{G}(u) &= be^{\mp u}, \\ \tilde{\omega} = \frac{1}{b^2} > 0: \quad \tilde{F}(u) &= \coth(u - c), & \tilde{G}(u) &= b \operatorname{cosech}(u - c),\end{aligned}\tag{16}$$

where  $b$  and  $c$  are constants. Hence there would be three classes of realizations for (5) too, depending on the sign of  $\omega = (F^2 - 1)/G^2$  according to which

$$\begin{aligned}\omega = -\frac{1}{b^2} < 0: \quad F(x) &= \tanh[u(x) - c], & G(x) &= b \operatorname{sech}[u(x) - c], \\ \omega = 0: \quad F(x) &= \pm 1, & G(x) &= be^{\mp u(x)}, \\ \omega = \frac{1}{b^2} > 0: \quad F(x) &= \coth[u(x) - c], & G(x) &= b \operatorname{cosech}[u(x) - c],\end{aligned}\tag{17}$$

where  $u$  is known as a function of  $x$  from (14) for some mass function  $M(x)$ . For instance, a plausible choice of the latter could be

$$M(x) = \left(1 + \frac{q}{1+x^2}\right)^2, \quad q > 0,\tag{18}$$

which has yielded interesting results [18] with respect to the shape-invariant condition. In (18),  $q$  may be treated as a deformation parameter so that when  $q \rightarrow 0$ ,  $M(x) \rightarrow 1$ . The transformation (14) gives, because of (18), the relationship

$$u(x) = x + q \tan^{-1} x.\tag{19}$$

It implies that as  $x \rightarrow \pm\infty$ , a small deformation has an insignificant effect on the variable  $x$ .

(b) When  $F^2 \neq 1$ , we can eliminate  $M(x)$  to obtain from (6)

$$F^2 + \delta G^2 = 1,\tag{20}$$

where  $\delta$  is subject to the restriction  $\delta > 0$  if  $F^2 < 1$  and  $\delta < 0$  if  $F^2 > 1$ . The case  $F^2 = 1$  is incorporated for  $\delta = 0$ . Consistency with (17) demands that we set  $\delta = \frac{1}{b^2}$  for  $F^2 < 1$  and  $\delta = -\frac{1}{b^2}$  for  $F^2 > 1$ .

We can employ the machinery [24] of  $so(2, 1)$  to determine the wave functions  $\chi(\equiv \chi_{k\mu})$ . First the operator relation  $J_- \chi_0 e^{ik\phi} = 0$  is solved for  $\chi_0 = \chi_{kk}$  and then  $\chi_n = \chi_{k, k+n}$  are calculated for  $n = 1, 2, \dots$  by evaluating  $J_+^n \chi_0 e^{ik\phi}$ . To obtain solutions for the same potential  $V_k$  we need to replace  $k$  by  $k-1, k-2, \dots$  in the expressions for  $\chi_1, \chi_2, \dots$ , respectively. We then arrive at a chain of solutions

$$\chi_0 \sim G^{k-\frac{1}{2}} \exp\left(\int \sqrt{MG} dx\right),$$

$$\chi_1 \sim [G - (k-1)F] G^{k-\frac{3}{2}} \exp\left(\int \sqrt{MG} dx\right), \quad (21)$$

$$\chi_2 \sim \{2[G - (k-1)F][G - (k-2)F] - (k-2)\} G^{k-\frac{5}{2}} \exp\left(\int \sqrt{MG} dx\right),$$

and in a similar way the higher ones. These eigenfunctions correspond to the same potential  $V_k$ , given by (9) for  $\mu = k$ . In view of (20),  $V_k$  can be expressed as

$$V_k = \left[1 + \delta\left(\frac{1}{4} - k^2\right)\right] G^2 - 2kFG. \quad (22)$$

Corresponding to the three classes of solutions (17) we obtain specifically

$$\begin{aligned} F^2 < 1: \quad V_k &= \left(b^2 - k^2 + \frac{1}{4}\right) \operatorname{sech}^2(u-c) - 2kb \operatorname{sech}(u-c) \tanh(u-c), \\ F^2 = 0: \quad V_k &= b^2 e^{\mp 2u} \mp 2kbe^{\mp u}, \end{aligned} \quad (23)$$

$$F^2 > 1: \quad V_k = \left(b^2 + k^2 - \frac{1}{4}\right) \operatorname{cosech}^2(u-c) - 2kb \operatorname{cosech}(u-c) \coth(u-c),$$

where  $u$  may depend on  $x$  as in (19). The three potentials above can be looked upon as mass-deformed versions of Scarf II, Morse and generalized Pöschl-Teller ones, respectively. The accompanying eigenfunctions are obtained from (21) with the corresponding substitution of the functions  $F$  and  $G$ .

Finally let us demonstrate that  $V_{\text{eff}}$  in (11) for  $\mu = k$  coincides with the intertwining-led effective potential provided by the condition  $\eta H = H_1 \eta$ , where the intertwining operator  $\eta$  is taken in the first-order form  $\eta = A(x) \frac{d}{dx} + B(x)$  [21]. The intertwining relation implies that if  $E_n$  ( $n = 0, 1, 2, \dots$ ) are the bound-state eigenvalues of  $H$  then those of  $H_1$ , having an associated potential  $V_{1,\text{eff}}$ , are  $E_{1,n} = E_{n+1}$  if  $\eta\psi_0 = 0$ , where  $\psi_0$  is the ground-state eigenfunction of  $H$ . Plugging in  $H$  from (3) we get

$$A(x) = M^{-1/2}, \quad (24)$$

$$V_{\text{eff}}(x) = \lambda + B^2 - (AB)'. \quad (25)$$

Note that  $V_{1,\text{eff}}(x)$  depends upon  $V_{\text{eff}}(x)$  through the relation  $V_{1,\text{eff}}(x) = V_{\text{eff}} + 2AB' - AA''$ . In (25),  $\lambda$  denotes some integration constant.

The effective potentials given by the expression (11) for  $\mu = k$  and by equation (25) coincide for  $B(x)$  given by

$$B(x) = -\frac{M'}{4M^{3/2}} + f(x), \quad (26)$$

provided  $f$  satisfies

$$f^2 - \frac{1}{\sqrt{M}} f' = V_k - \lambda. \quad (27)$$

Equation (27) can be readily solved by using  $V_k$  in (22) and looking for solutions of the type  $f = \zeta F + \sigma G$ , where  $\zeta$  and  $\sigma$  are two constants to be determined. We obtain the following class of solutions

$$f_{\pm}(x) = \left(\pm k - \frac{1}{2}\right) F \mp G, \quad \lambda_{\pm} = -\left(k \mp \frac{1}{2}\right)^2, \quad (28)$$

where  $F$  and  $G$  are given by any one of the set (17). Thus the intertwining approach is consistent with the  $so(2,1)$  algebra for the above solutions of  $f$  and  $\lambda$ .

To conclude, we explored the properties of  $so(2,1)$  in the context of PDEM Schrödinger equation and generated mass-deformed versions of the Scarf II, Morse and generalized Pöschl-Teller potentials. We also sought consistency with the intertwining condition and obtained the associated class of solutions.

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