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$\mathcal{PT}$ symmetric models with nonlinear pseudosupersymmetry

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By applying the higher order Darboux algorithm to an exactly solvable non-Hermitian $\mathcal{PT}$ symmetric potential, we obtain a hierarchy of new exactly solvable non-Hermitian $\mathcal{PT}$ symmetric potentials with real spectra. It is shown that the symmetry underlying the potentials so generated and the original one is *nonlinear pseudosupersymmetry*. We also show that this formalism can be used to generate a larger class of new solvable potentials when applied to non-Hermitian systems.

I. INTRODUCTION

There are not many exactly solvable potentials in quantum mechanics. As a result there have always been efforts to enlarge the class of exactly solvable potentials. Some of the different methods which have been used time and again to generate a hierarchy of isospectral potentials are the factorization method of Infeld and Hull,[1] the Darboux algorithm,[2] the method of supersymmetric quantum mechanics (SUSY QM),[3] or the integral transformations of Abraham–Moses–Pursey,[4] etc. Among these methods the Darboux algorithm and the SUSY QM are closely related and these methods have found numerous applications in different areas of theoretical and mathematical physics.[3]

At the same time, the scheme is still narrow as conventional SUSY fails to explain certain phenomena, e.g., the disappearance of the leading Borel singularity of the perturbation correction for the ground state energy of a SUSY theory.[5] In order to explain such behavior and also to widen the scope of SUSY QM, an idea was put forward to extend SUSY to higher orders.[6] We recall that in the conventional intertwining technique, two one-dimensional Schrödinger Hamiltonians H and $\widetilde{H}$ are intertwined by means of differential operators L as

$$\widetilde{H}L = LH, \quad HL^{\dagger} = L^{\dagger}\widetilde{H}. \tag{1}$$

If L is of the first order in derivatives, the standard SUSY QM, with supercharges built of first order Darboux transformation operators, and the factorization method are recovered. On the other hand, if higher order differential operators are involved in the construction of L, it is variously referred to as *polynomial SUSY*,[6] or *nonlinear SUSY*,[7] or *higher order SUSY* (*n-SUSY*),[8,9] or $\mathcal{N}$*-fold SUSY*,[5,10] the study of which has attracted the attention of a lot of researchers in recent times.[5–10] Contrary to standard SUSY, anticommutator of the supercharges no longer coincides with the Hamiltonian in general. Instead, it becomes a polynomial of the Hamiltonian in degree N, and is sometimes referred to as the *Mother Hamiltonian*.[5,10]

Furthermore, the equivalence between an Nth order Darboux transformation and a chain of N first order Darboux transformation is well established.[9] Every chain of N first order Darboux transformation creates a chain of exactly solvable Hamiltonians $h_0 \to h_1 \to \cdots \to h_N$. Hence the intertwining operator $L^{(N)}$ between the initial Hamiltonian h_0 and the final Hamiltonian h_N can always be presented as a product of N first order Darboux transformation operators between every two juxtaposed Hamiltonians $h_0, h_1, \ldots, h_N$,

$$L^{(N)} = L_N L_{N-1} \cdots L_2 L_1, \quad h_p L_p = L_p h_{p-1}, \quad p = 1,2,\ldots,N. \tag{2}$$

In conventional higher order SUSY, h_0 and h_N are essentially self-adjoint Hermitian operators in a Hilbert space, with square integrable eigenfunctions. If all the intermediate potentials $V_1(x)$, $V_2(x), \ldots, V_{N-1}(x)$ are real valued functions in their common domain of definition (a,b), the chain is called *reducible*, and the Nth order Darboux transformation is called *reducible* as well. Additionally, if all the intermediate potentials are free of singularities in (a,b), the chain and the corresponding transformation are called *completely reducible*. When at least one intermediate potential is a complex valued function, the chain and the corresponding transformation are called *irreducible*.

At the same time, non-Hermitian Hamiltonians have made an important place for themselves in the recent development of quantum mechanics, because of their intrinsic interest[11] and possible applications.[12] It is well known by now that a non-Hermitian $\mathcal{PT}$ symmetric Hamiltonian admits real eigenvalues if the eigenfunctions, too, respect the $\mathcal{PT}$ invariance (the so-called unbroken $\mathcal{PT}$ symmetry), whereas the eigenvalues occur as complex conjugate pairs if $\mathcal{PT}$ symmetry is spontaneously broken (in this case the eigenfunctions are no longer $\mathcal{PT}$ invariant). For such non-Hermitian $\mathcal{PT}$ symmetric Hamiltonians,

$$\mathcal{PT}H = H\mathcal{PT}, \tag{3}$$

where $\mathcal{P}$ stands for the *space inversion* operator and $\mathcal{T}$ denotes *time reversal*,

$$\begin{aligned} \mathcal{P}&: \quad x \to -x, \quad p \to -p, \\ \mathcal{T}&: \quad x \to x, \quad p \to -p, \quad i \to -i \end{aligned} \tag{4}$$

The reality of the spectrum may be attributed to the so-called η-pseudo Hermiticity of the non-Hermitian Hamiltonian[13]

$$H^\dagger = \eta H \eta^{-1}, \tag{5}$$

where η is a linear, invertible, Hermitian operator. Several non-Hermitian Hamiltonians, whether possessing $\mathcal{PT}$ invariance or not, have been identified as η pseudo-Hermitian under $\eta = e^{-\theta p}$, where θ is real, and $p = -i(d/\mathrm{d}x)$, or $\eta = e^{-\phi(x)}$, where $\phi(x)$ is some gaugelike transformation. We note that for $\mathcal{PT}$ symmetric Hamiltonians, η may simply be taken as the parity operator $\mathcal{P}$, whereas for conventional Hermitian Hamiltonians, $\eta = 1$.

Moreover, the square integrability of the wave functions is no longer a prerequisite for non-Hermitian Hamiltonians. Instead, the orthonormalization of the wave function for Hermitian quantum mechanics

$$\int \Psi_m^* \Psi_n \, \mathrm{d}x = \delta_{m,n} \tag{6}$$

is replaced by[14]

$$\int [\mathcal{CPT}\Psi_m] \Psi_n \, \mathrm{d}x = \delta_{m,n}, \tag{7}$$

where $\mathcal{C}$ plays the role of a linear charge operator, obeying the relationship

$$[\mathcal{C},H]=0, \quad [\mathcal{C},\mathcal{PT}]=0 \tag{8}$$

and has the property $\mathcal{C}^2=1$. In the position representation $\mathcal{C}$ is given as

$$\mathcal{C}(x,y)=\sum_n \psi_n(x)\psi_n(y) \tag{9}$$

and the completeness relation gets modified to

$$\sum_n [\mathcal{CPT}\psi_n(x)]\psi_n(y)=\delta(x-y). \tag{10}$$

While nonlinear SUSY for $N=2$ has been investigated widely for Hermitian Hamiltonians,[5–10] such studies have not been carried out as yet for non-Hermitian Hamiltonians. Motivated by the importance of such systems in the recent development of quantum mechanics, our aim in the present work is to generalize the concept of nonlinear SUSY to include non-Hermitian quantum systems. In analogy with the first order systems, where the partner Hamiltonians $H_\pm$ of non-Hermitian systems were found to be related through *pseudosupersymmetry*,[13,15] it will be shown that the underlying symmetry between the isospectral partners h_0 and h_N is a generalization of $\mathcal{N}$ SUSY and may be called *nonlinear pseudosupersymmetry*. The nature of the intermediate Hamiltonians as well as the corresponding wave functions will also be investigated.

The organization of the paper is as follows. For the sake of completeness, in Sec. II we briefly outline conventional nonlinear SUSY for Hermitian quantum mechanics. In Sec. III we describe a similar framework for non-Hermitian Hamiltonians and show that the underlying symmetry for the potentials produced by higher order Darboux algorithm is nonlinear pseudosupersymmetry. Some explicit examples are given in Secs. IV and V, while Sec. VI is devoted to a conclusion.

II. NONLINEAR SUSY FOR HERMITIAN HAMILTONIANS

In the conventional first order supersymmetric quantum mechanics, if a given solvable Hamiltonian

$$H=-\frac{d^2}{dx^2}+V(x) \tag{11}$$

possesses a discrete spectrum of bound states E_n, $n=0,1,2,\ldots$, together with the square-integrable eigenfunctions $\psi_n(x)$, then a pair of first-order operators L_0 and $L_0^\dagger$ can be constructed from the ground state ψ_0, given by

$$L_0=\frac{d}{dx}+W_0(x), \quad L_0^\dagger=-\frac{d}{dx}+W_0(x), \tag{12}$$

where

$$W_0(x)=-[\ln \psi_0(x)]' \tag{13}$$

such that L_0 and $L_0^\dagger$ play the role of intertwining operators for the initial and final Hamiltonians H and $\widetilde{H}$, respectively,

$$\widetilde{H}L_0=L_0H, \quad HL_0^\dagger=L_0^\dagger\widetilde{H} \tag{14}$$

with

$$H=L_0^\dagger L_0, \quad \widetilde{H}=L_0L_0^\dagger. \tag{15}$$

Simple straightforward algebra shows that the partner potentials $V(x)$ and $\widetilde{V}(x)$ can be expressed as

$$V(x) = W_0^2(x) - W_0'(x), \tag{16}$$

$$\widetilde{V}(x) = W_0^2(x) + W_0'(x) = V(x) + 2W_0'(x). \tag{17}$$

The eigenfunctions $\psi(x)$ and $\widetilde{\psi}(x)$ of H and $\widetilde{H}$ are interrelated through L_0 and $L_0^\dagger$:

$$L_0\psi_0(x) = 0, \quad L_0(x)\psi_i(x) = \frac{W_{0,i}(x)}{\psi_0(x)}\alpha\ \widetilde{\psi}_i(x), \quad L_0^\dagger\widetilde{\psi}_i(x)\alpha\ \psi_i(x), \quad i = 1,2,\ldots , \tag{18}$$

where $W_{0,i}(x)=\{\psi_0(x)\psi_i'(x)-\psi_0'(x)\psi_i(x)\}$ is the Wronskian of $\psi_0(x)$ and $\psi_i(x)$. The concise algebraic form of spectral equivalence is given by the superalgebra for the partners H and $\widetilde{H}$, and the supercharges Q and $Q^\dagger$,

$$Q = \begin{pmatrix} 0 & L_0 \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ L_0^\dagger & 0 \end{pmatrix}, \tag{19}$$

$$\mathcal{H} = \{Q,Q^\dagger\} = \begin{pmatrix} H & 0 \\ 0 & \widetilde{H} \end{pmatrix} = \begin{pmatrix} L_0^\dagger L_0 & 0 \\ 0 & L_0 L_0^\dagger \end{pmatrix} \tag{20}$$

satisfying the relations

$$\{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0, \quad [Q,\mathcal{H}] = [Q^\dagger,\mathcal{H}] = 0. \tag{21}$$

Thus H and $\widetilde{H}$ are isospectral except for the lowest eigenvalue E_0 which is missing in $\widetilde{H}$, as $\widetilde{\psi}_0$ is not normalizable.

To generalize standard SUSY to higher order, the supercharges are built of higher order intertwining operators.[9] The two Hamiltonians h_0 and h_N are intertwined through an Nth order differential operator $L^{(N)}$, as

$$L^{(N)}h_0 = h_N L^{(N)}, \quad h_0 L^{(N)\dagger} = L^{(N)\dagger}h_N, \tag{22}$$

where h_0 and h_N are self-adjoint operators. The proper eigenfunctions ψ_i of the original Hamiltonian h_0 are known exactly, $h_0\psi_i = E_i\psi_i$. Any such operator $L^{(N)}$ can always be presented in the form known as Crum–Krein formula[16]

$$L^{(N)} = W^{-1}(u_1,u_2,\ldots,u_N)\begin{vmatrix} u_1 & u_2 & \cdots & 1 \\ u'_1 & u'_2 & \cdots & \dfrac{d}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(N)} & u_2^{(N)} & \ldots & \dfrac{d^N}{dx^N} \end{vmatrix}, \tag{23}$$

where $W(u_1,u_2,\ldots,u_N)$ stands for the usual symbol for the Wronskian of the functions u_1,u_2, …, u_N. The functions u_i $(i=1,2,\ldots,N)$ called the transformation functions are eigenfunctions of h_0, $h_0u_i=\alpha_iu_i$, and they need not necessarily satisfy any physical boundary condition. The final potential has the form

$$V_N(x) = V(x) - 2\frac{d^2}{dx^2}\ln W(u_1, u_2, \ldots, u_N) \tag{24}$$

and will be free of singularities whenever the Wronskian is nodeless, which in turn requires that only consecutive eigenfunctions of h_0 must be considered.[9] The eigenfunctions $\psi(x)$ and $\tilde{\psi}(x)$ of h_0 and h_N are connected by the intertwiners $L^{(N)}$ and $L^{(N)\dagger}$ as

$$\tilde{\psi}_i(x) = L^{(N)}\psi_i(x) = \frac{W_{j,j+1,\ldots,j+N,i}(x)}{W_{j,j+1,\ldots,j+N}(x)}, \tag{25}$$

where $W_{j,j+1,\ldots,j+N,i}(x)$ and $W_{j,j+1,\ldots,j+N}(x)$ are the Wronskians of the eigenfunctions of h_0 associated with the corresponding subindices. Thus if $\psi_i(x)$ is an eigenfunction of h_0 with energy E_i, then $\tilde{\psi}_i(x)$ is an eigenfunction of h_N with the same energy E_i. Evidently

$$L^{(N)}\psi_i = 0, \quad i = 1,2,\ldots,N. \tag{26}$$

However, for energies E_i $(i=1,2,\ldots,N)$, the corresponding eigenfunctions of h_N,

$$\tilde{\psi}(x)\,\alpha\frac{\psi(x)}{W_{j,j+1,\ldots,j+N}(x)},$$

have growing asymptotics at both infinities. Consequently, these are not physically acceptable solutions of h_N, and the corresponding eigenvalues E_i $(i=1,2,\ldots,N)$ are excluded from the spectrum of h_N. Thus

$$h_N\tilde{\psi}_E = E\tilde{\psi}_E \tag{27}$$

with the exception of the levels $E=E_i$, $i=1,2,\ldots,N$, which will be absent in the spectrum of the new Hamiltonian h_N, as the corresponding eigenfunctions are not square integrable.

It has already been shown[17] that the operator $L^{(N)}$ can always be presented as a product of N first order Darboux transformation operators between every two Hamiltonians $h_0, h_1, \ldots, h_N$,

$$L^{(N)} = L_N L_{N-1}\cdots L_1, \quad h_p L_p = L_p h_{p-1}, \quad p = 1,2,\ldots,N. \tag{28}$$

We note that the final Hamiltonian h_N is Hermitian, although some of the intermediate Hamiltonians h_i could be unphysical, e.g., their associated potentials might contain extra singularities that were not present in the initial one. The supercharges Q_N and $Q_N^\dagger$ are constructed as

$$Q_N = \begin{pmatrix} 0 & L^{(N)} \\ 0 & 0 \end{pmatrix}, \quad Q_N^\dagger = \begin{pmatrix} 0 & 0 \\ L^{(N)\dagger} & 0 \end{pmatrix}. \tag{29}$$

Evidently, Q_N and $Q_N^\dagger$ are nilpotent

$$\{Q_N, Q_N\} = \{Q_N^\dagger, Q_N^\dagger\} = 0. \tag{30}$$

The super-Hamiltonian

$$H_N = \begin{pmatrix} h_0 & 0 \\ 0 & h_N \end{pmatrix} \tag{31}$$

satisfies the relations

$$[Q_N, H_N] = [Q_N^\dagger, H_N] = 0. \tag{32}$$

The anticommutator can be generally expressed by a Nth order polynomial $\mathcal{P}_N$ of the Hamiltonian H_N,

$$\mathcal{H}_N = \{Q_N^\dagger, Q_N\} = \begin{pmatrix} L^{(N)\dagger}L^{(N)} & 0 \\ 0 & L^{(N)}L^{(N)\dagger} \end{pmatrix} = \prod_{k=1}^{N} (H_N - \alpha_k \mathcal{I}), \tag{33}$$

where $\mathcal{I}$ is the 2×2 unit matrix, and

$$L^{(N)\dagger}L^{(N)} = \prod_{k=1}^{N} (h_0 - \alpha_k), \tag{34}$$

$$L^{(N)}L^{(N)\dagger} = \prod_{k=1}^{N} (h_N - \alpha_k). \tag{35}$$

Since the right-hand side of (33) is a polynomial in H_N, it is called nonlinear SUSY or N-fold SUSY. The operator $\mathcal{H}_N$ is termed as the *Mother Hamiltonian* and satisfies the commutation relations[9]

$$[Q_N, \mathcal{H}_N] = [Q_N^\dagger, \mathcal{H}_N] = 0. \tag{36}$$

For $N=1$, N-fold SUSY reduces to standard SUSY.

The most widely studied higher order SUSY is for $N=2$,[8,9] where the formalism reduces to

$$L^{(2)} = L_2 L_1, \tag{37}$$

where

$$L_1 = -\partial_x + (\ln u_1)', \quad L_2 = -\partial_x + (\ln v)', \quad v = L_1 u_2, \tag{38}$$

and the isospectral potential turns out to be

$$\widetilde{V}_2(x) = V(x) - 2\frac{d^2}{dx^2}\ln W_{j,j+1}(x). \tag{39}$$

III. NONLINEAR PSEUDO-SUSY FOR NON-HERMITIAN HAMILTONIANS

In this section we extend the concept of nonlinear or N-fold supersymmetry to non-Hermitian quantum mechanics. Though the Darboux algorithm and (nonlinear) supersymmetric quantum mechanics are equivalent for Hermitian Hamiltonians, the situation is different for non-Hermitian Hamiltonians. However, intertwining operators $A^{(N)}$ and $B^{(N)}$ can still be constructed with the help of Darboux transformation. Analogous to the case of Hermitian quantum mechanics, it will be shown that once a non-Hermitian Schrödinger potential $V(x)$ is exactly solvable, one can construct an isospectral partner $\widetilde{V}_N(x)$ from (24),

$$\widetilde{V}_N(x) = V(x) - 2\frac{d^2}{dx^2}\ln W(u_1, u_2, \ldots, u_N), \tag{40}$$

where W stands for the usual symbol for the Wronskian of the functions $u_1, u_2, \ldots, u_N$, which are eigenfunctions of h_0, $h_0 u_i = \alpha_i u_i$. As before the functions $u_i(x)$ may be just formal eigenfunctions. Our aim will be to study the spectrum of the new Hamiltonian in detail, to investigate the nature of the potential and the eigenfunctions, and to determine the symmetry which connects the original Hamiltonian h_0 and the transformed one h_N. For this purpose, we look for two intertwining operators $A^{(N)}$ and $B^{(N)}$ such that

$$A^{(N)}h_0 = h_N A^{(N)}, \quad h_0 B^{(N)} = B^{(N)} h_N, \tag{41}$$

where h_0 and h_N are no longer self-adjoint operators $(h_{0,(N)} \neq h^{\dagger}_{0,(N)})$; on the contrary, to ensure the reality of the spectrum, they are η pseudo-Hermitian,

$$\eta h_{0,(N)} \eta^{-1} = h^{\dagger}_{0,(N)}, \tag{42}$$

where η is a linear, invertible, Hermitian operator. However, the choice of η is not unique. For $\mathcal{PT}$ invariant potentials, a simple representation of η may be given by the parity operator,

$$\eta = \mathcal{P}, \quad \mathcal{P}f(x) = f(-x). \tag{43}$$

It follows that for real potentials, (43) leads to $\eta=1$ so that $B^{(N)}=A^{(N)\dagger}$, thus reproducing the standard result of supersymmetry.

It follows from Eqs. (41) and (42) that the operators $A^{(N)}$ and $B^{(N)}$ are pseudo-adjoint:

$$B^{(N)} = A^{(N)\#} = \eta^{-1} A^{(N)\dagger} \eta. \tag{44}$$

Considering first order Darboux transformation between every two juxtaposed Hamiltonians $h_0, h_1, \ldots, h_N$, each pair intertwined by first order operators L_k $(k=1,2,\ldots,N)$

$$h_k L_k = L_k h_{k-1}, \quad k = 1,2,3,\ldots,N, \tag{45}$$

$$L^{\#}_k h_k = h_{k-1} L^{\#}_k, \quad k = 1,2,3,\ldots,N, \tag{46}$$

where

$$L^{\#}_k = \eta^{-1} L_k \eta \tag{47}$$

then, analogous to the Hermitian case, the final Hamiltonian h_N is found to be related to the initial (or starting) Hamiltonian h_0 through

$$h_N = L_N L_{N-1} \cdots L_2 L_1 \; h_0 \; L^{\#}_1 L^{\#}_2 \cdots L^{\#}_N \tag{48}$$

so that the operator $A^{(N)}$ can be represented as a product of the N first order Darboux transformations

$$A^{(N)} = L_N L_{N-1} \cdots L_2 L_1 \tag{49}$$

with its pseudo-adjoint

$$B^{(N)} = A^{(N)\#} = \eta^{-1} L_1 L_2 \cdots L_N \eta = L^{\#}_1 \cdots L^{\#}_{N-1} L^{\#}_N. \tag{50}$$

It is worth mentioning here that in contrast to Hermitian quantum mechanics, all the intermediate Hamiltonians h_k are physically acceptable as their associated potentials contain no extra singularities which are not present in the initial potential $V(x)$. This is essentially because the associated eigenfunctions do not have nodes on the real line, and they are normalizable in the sense of Eq. (7).

Thus the initial and the transformed Hamiltonians h_0 and h_N are related by *nonlinear pseudosupersymmetry*. The super-Hamiltonian of this system consists of the pseudosupersymmetric pair of Hamiltonians h_0 and h_N as

$$H_N = \begin{pmatrix} h_0 & 0 \\ 0 & h_N \end{pmatrix}. \tag{51}$$

The supercharges generating this form of pseudosupersymmetry are constructed in the following way:

$$Q_N = \begin{pmatrix} 0 & A^{(N)} \\ 0 & 0 \end{pmatrix}, \quad Q_N^{\#} = \eta^{-1} Q_N^{\dagger} \eta = \begin{pmatrix} 0 & 0 \\ B^{(N)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A^{(N)\#} & 0 \end{pmatrix} \tag{52}$$

so that the supercharge Q_N and its pseudoadjoint $Q_N^{\dagger}$ of standard Hermitian quantum mechanics are replaced by Q_N and its pseudoadjoint $Q_N^{\#}$ for non-Hermitian Hamiltonians. Obviously, Q_N and $Q_N^{\#}$ are nilpotent

$$\{Q_N, Q_N\} = \{Q_N^{\#}, Q_N^{\#}\} = 0 \tag{53}$$

and satisfy the following closed algebra:

$$[Q_N, H_N] = [Q_N^{\#}, H_N] = 0, \tag{54}$$

$$\mathcal{H}_N \equiv \{Q_N^{\#}, Q_N\} = \begin{pmatrix} A^{(N)\#}A^{(N)} & 0 \\ 0 & A^{(N)}A^{(N)\#} \end{pmatrix} = \prod_{k=1}^{N} (H_N - \alpha_k \mathcal{I}), \tag{55}$$

i.e.,

$$A^{(N)\#}A^{(N)} = \prod_{k=1}^{N} (h_0 - \alpha_k), \tag{56}$$

$$A^{(N)}A^{(N)\#} = \prod_{k=1}^{N} (h_N - \alpha_k), \tag{57}$$

and $\mathcal{I}$ is the 2×2 unit matrix. Evidently, if $\psi_i(x)$ is an eigenfunction of h_0 with energy eigenvalue E_i, then $\tilde{\psi}_i(x) = A^{(N)}\psi_i(x)$ is an eigenfunction of h_N with the same energy E_i. However, for $i = 1, 2, \ldots, N$,

$$\tilde{\psi}_i(x) \,\alpha\, \frac{\psi_i(x)}{W(\psi_1, \psi_2, \ldots, \psi_N)}. \tag{58}$$

Clearly, the eigenfunctions $\tilde{\psi}_i(x)$ $(i=1,2,\ldots N)$ of h_N corresponding to the eigenvalues E_i $(i = 1, 2, \ldots, N)$ grow asymptotically, and so cannot be included in the set of solutions of h_N. Consequently, E_i $(i=1,2,\ldots,N)$ are excluded from the spectrum of h_N.

Next we note two interesting results which are in contrast to the Hermitian case.

(1) For $\tilde{V}_N(x)$ to be free of singularities, the Wronskian $W(\psi_1, \psi_2, \ldots, \psi_N) = W_{\psi_1, \psi_2, \ldots, \psi_N}(x)$ must be nodeless. In the case of Hermitian potentials, this is guaranteed only when ψ_i, $i = 1, 2, \ldots, N$ represent N consecutive eigenfunctions. However, in the case of generic non-Hermitian potentials, the eigenfunctions $\psi_n(x)$ $(n=0,1,2,\ldots)$ have no nodes on the real line. Consequently, the Wronskian is free of real singularities for any value of $i, j, k, \ldots$, and thus can be used to generate a wider class of isospectral Hamiltonians.

(2) The intermediate Hamiltonians are also physically acceptable, as the corresponding potentials are free of singularities, for the same reason as given above. For example, the first intertwining gives

$$V_1(x) = V(x) - 2\frac{d^2}{dx^2}\ln\, \psi_i(x) \tag{59}$$

which is well defined. However, this may not always be true for Hermitian potentials due to the presence of additional singularities in $V_1(x)$, which are not present in $V(x)$.

For the sake of simplicity, in the present work we shall restrict ourselves to second order nonlinear pseudosupersymmetry. Thus if an intertwining operator $A = L_2 L_1$ is constructed from the two first order Darboux transformation operators L_1 and L_2, given by

$$L_1 = -\partial_x + (\ln u_i)', \quad L_2 = -\partial_x + (\ln v)', \quad v = L_1 u_j, \tag{60}$$

where u_i and u_j are any two eigenfunctions of the non-Hermitian Hamiltonian h_0, then the transformed isospectral Hamiltonian,

$$h_2 = -\frac{d^2}{\mathrm{d}x^2} + \widetilde{V}_{i,j}(x) \tag{61}$$

has eigenfunctions

$$\widetilde{\psi}_n(x) = \frac{W(\psi_i,\psi_j,\psi_n)}{W(\psi_i,\psi_j)} = -E_n\psi_n + E_i\psi_i\frac{W(\psi_n,\psi_j)}{W(\psi_i,\psi_j)} + E_j\psi_j\frac{W(\psi_i,\psi_n)}{W(\psi_i,\psi_j)}, \tag{62}$$

where

$$\widetilde{V}_{i,j}(x) = V(x) - 2\frac{d^2}{\mathrm{d}x^2}\ln W(u_i,u_j). \tag{63}$$

The mother Hamiltonian $\mathcal{H}_2$ is constructed from the anticommutator by

$$\mathcal{H}_2 = \{Q_2^\#, Q_2\} = \begin{pmatrix} A^\# A & 0 \\ 0 & AA^\# \end{pmatrix} = \begin{pmatrix} (h_0-\alpha_1\mathcal{I})(h_0-\alpha_2\mathcal{I}) & 0 \\ 0 & (h_2-\alpha_1\mathcal{I})(h_2-\alpha_2\mathcal{I}) \end{pmatrix}, \tag{64}$$

where $\mathcal{I}$ is 2×2 unit matrix and H_2 is given by (51).

In the following sections we shall investigate this formalism further with the help of explicit examples.

IV. $\mathcal{PT}$ SYMMETRIC OSCILLATOR

In this section we shall apply our formalism to the well-known example of the $\mathcal{PT}$ symmetric oscillator[18]

$$V(x) = (x-i\epsilon)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x-i\epsilon)^2} \tag{65}$$

with eigenfunctions

$$\psi_n(x) = e^{-(1/2)(x-i\epsilon)^2}(x-i\epsilon)^{-q\alpha+1/2}L_n^{-q\alpha}((x-i\epsilon)^2) \tag{66}$$

and eigenvalues

$$E_n = 4n - 2q\alpha + 2, \quad n = 0,1,2,\ldots, \tag{67}$$

where $q = \pm 1$ is called the quasiparity.

In this study we shall restrict ourselves to $N=2$ only. If one performs Darboux transformations with two eigenfunctions $\psi_i(x)$ and $\psi_j(x)$ of the potential $V(x)$, corresponding to energies E_i and E_j (i and j need not be consecutive), then the intertwining operators take the form

$$L_1 = -\frac{d}{\mathrm{d}x} + \frac{\psi_i'}{\psi_i}, \quad L_1^\# = \frac{d}{\mathrm{d}x} + \frac{\psi_i'}{\psi_i}, \tag{68}$$

$$L_2 = -\frac{d}{dx} + \frac{W'_{i,j}}{W_{i,j}} - \frac{\psi'_i}{\psi_i}, \quad L_2^{\#} = \frac{d}{dx} + \frac{W'_{i,j}}{W_{i,j}} - \frac{\psi'_i}{\psi_i}, \tag{69}$$

where $W_{i,j}$ is the usual Wronskian given by

$$W_{i,j} = W(\psi_i, \psi_j) = \psi_i(x)\psi'_j(x) - \psi'_i(x)\psi_j(x) \tag{70}$$

and η has been taken as in (43). Replacing the intertwining operators $A^{(2)}$ (and $B^{(2)}$) by A (and B) for simplicity, we obtain

$$A = L_2 L_1 = \frac{d^2}{dx^2} + \beta_{i,j}\frac{d}{dx} - \beta_{i,j}\frac{\psi'_i}{\psi_i} - \frac{\psi''_i}{\psi_i}, \tag{71}$$

$$B = A^{\#} = \eta^{-1} A^{\dagger} \eta = \frac{d^2}{dx^2} - \beta_{i,j}\frac{d}{dx} - \beta_{i,j}\frac{\psi'_i}{\psi_i} - \frac{\psi''_i}{\psi_i} - \beta'_{i,j}, \tag{72}$$

where

$$\beta_{i,j} = -\frac{W'_{i,j}}{W_{i,j}} \tag{73}$$

and η is simply the parity operator $\mathcal{P}$ for $\mathcal{PT}$ symmetric potentials. The new exactly solvable non-Hermitian potential, which is isospectral to the $\mathcal{PT}$ symmetric oscillator in (65), is obtained from

$$\tilde{V}_{i,j}(x) = V(x) - 2\frac{d^2}{dx^2}\ln W_{i,j} \tag{74}$$

with solutions

$$\tilde{\psi}_k(x) = A\psi_k(x). \tag{75}$$

Thus for each set (i,j), one obtains two sets of $\tilde{V}_{i,j}(x)$ because of the presence of quasiparity q. Obviously, $\tilde{\psi}_k(x)=0$ for $k=i,j$. Thus the new potential so constructed in (74) above, has all the eigenenergies of the original $\mathcal{PT}$ symmetric oscillator except for the levels i,j, which are missing from the spectrum of (74).

For the simplicity of calculations we shall now construct and examine some potentials using low values of i and j in further detail.

A. New potential for $i=1$, $j=2$

Applying the above formalism with the two eigenstates $\psi_1(x)$ and $\psi_2(x)$ of the potential in (65), the Wronskian is found to be

$$W(\psi_1, \psi_2) = c_{12} e^{-(x-i\epsilon)^2}(x-i\epsilon)^{2-2q\alpha} g, \tag{76}$$

where c_{12} is some real constant and

$$g = (1-q\alpha)(2-q\alpha) - 2(1-q\alpha)(x-i\epsilon)^2 + (x-i\epsilon)^4. \tag{77}$$

The intertwining operators A and $A^{\#}$ are obtained from $A=L_2L_1$, $A^{\#}=L_1^{\#}L_2^{\#}$, where

$$L_1 = -\frac{d}{dx} + \frac{\psi'_1}{\psi_1} = -\frac{d}{dx} - (x-i\epsilon) + \frac{-q\alpha+\frac{1}{2}}{(x-i\epsilon)} - \frac{2(x-i\epsilon)}{1-q\alpha-(x-i\epsilon)^2}, \tag{78}$$

$$L_2 = -\frac{d}{\mathrm{d}x} + \frac{d}{\mathrm{d}x}\ln(L_1\psi_2) = -\frac{d}{\mathrm{d}x} - (x - i\epsilon) + \frac{-q\alpha + \frac{3}{2}}{(x - i\epsilon)} + \frac{2(x - i\epsilon)}{1 - q\alpha - (x - i\epsilon)^2} + \frac{g'}{g}, \tag{79}$$

so that

$$A = \frac{d^2}{\mathrm{d}x^2} - \left\{ -2(x - i\epsilon) + \frac{2(1 - q\alpha)}{(x - i\epsilon)} + \frac{g'}{g} \right\}\frac{d}{\mathrm{d}x} + (x - i\epsilon)^2 + \frac{(-q\alpha + \frac{1}{2})(-q\alpha + \frac{5}{2})}{(x - i\epsilon)^2} + 2q\alpha - 1$$
$$+ \left\{ -(x - i\epsilon) + \frac{-q\alpha + \frac{1}{2}}{(x - i\epsilon)} - \frac{2(x - i\epsilon)}{1 - q\alpha - (x - 1\epsilon)^2} \right\}\frac{g'}{g}, \tag{80}$$

$$A^{\#} = \frac{d^2}{\mathrm{d}x^2} + \left\{ -2(x - i\epsilon) + \frac{2(1 - q\alpha)}{(x - i\epsilon)} + \frac{g'}{g} \right\}\frac{d}{\mathrm{d}x} + (x - i\epsilon)^2 + \frac{(-q\alpha + \frac{3}{2})(-q\alpha - \frac{1}{2})}{(x - i\epsilon)^2} + 2q\alpha - 3$$
$$+ \left\{ -(x - i\epsilon) + \frac{-q\alpha + \frac{1}{2}}{(x - 1\epsilon)} - \frac{2(x - i\epsilon)}{1 - q\alpha - (x - i\epsilon)^2} \right\}\frac{g'}{g} + \frac{g''}{g} - \left(\frac{g'}{g}\right)^2. \tag{81}$$

Applying Eq. (74), the new potential isospectral to the one in (65) except for the states corresponding to $\psi_1(x)$ and $\psi_2(x)$, comes out as

$$\widetilde{V}_{1,2}(x) = (x - i\epsilon)^2 + \frac{(-q\alpha + \frac{3}{2})(-q\alpha + \frac{5}{2})}{(x - i\epsilon)^2} - 2\frac{g''}{g} + 2\left(\frac{g'}{g}\right)^2 + 4 \tag{82}$$

which has solutions

$$\widetilde{\psi}_n(x) = -E_{n+2}\psi_{n+2} + E_1\psi_1\frac{W(\psi_{n+2},\psi_2)}{W(\psi_1,\psi_2)} + E_2\psi_2\frac{W(\psi_1,\psi_{n+2})}{W(\psi_1,\psi_2)} \tag{83}$$

with energy eigenvalues

$$\widetilde{E}_n = E_{n+2} = 4n + 10 - 2q\alpha, \quad n = 1,2,3,\ldots . \tag{84}$$

The ground state is given by

$$\widetilde{\psi}_0(x) = e^{-(1/2)(x - i\epsilon)^2}(x - i\epsilon)^{-q\alpha+1/2}\left\{ B_1 + \frac{B_2(x - i\epsilon)^2}{g} \right\} \tag{85}$$

with eigenvalue

$$\widetilde{E}_0 = E_0 = 2 - 2q\alpha, \tag{86}$$

where B_1 and B_2 are some x, ϵ independent constants. Thus the energies E_1 and E_2 of $V(x)$ are absent in the spectrum of $\widetilde{V}_{1,2}(x)$. It can be verified that the eigenfunctions $\widetilde{\psi}$ are also $\mathcal{PT}$ invariant, and can be normalized using (7). Furthermore, the supercharges Q_2 and $Q_2^{\#}$, generated from the operators A and $A^{\#}$, satisfy the following algebra :

$$\mathcal{H}_2 = \{Q_2, Q_2^{\#}\} = H_2^2 - 4(4 - q\alpha)H_2 + (6 - 2q\alpha)(10 - 2q\alpha), \tag{87}$$

where H_2 is given by (51). The intermediate potential given by

$$V_1(x) = V(x) - 2\frac{d^2}{dx^2}\ln \psi_1(x) = (x-i\epsilon)^2 + \frac{(-q\alpha+\frac{1}{2})(-q\alpha+\frac{3}{2})}{(x-i\epsilon)^2} + \frac{12}{1-q\alpha-(x-i\epsilon)^2} - \frac{8(1-q\alpha)}{\{1-q\alpha-(x-i\epsilon)^2\}^2} + 4 \tag{88}$$

does not have any singularity on the real line, and hence is physically acceptable as well. By arguments similar to those given above, its ground state eigenfunction is given by

$$\phi_0 = \frac{1}{1-q\alpha-(x-i\epsilon)^2} e^{-(1/2)(x-i\epsilon)^2}(x-i\epsilon)^{-q\alpha+(3/2)} \tag{89}$$

with energy

$$e_0 = E_0 = 2 - 2q\alpha \tag{90}$$

and the excited states

$$\phi_n = \frac{W(\psi_{n+1},\psi_1)}{\psi_1} \tag{91}$$

with corresponding energies

$$e_n = E_{n+1} = 4n + 6 - 2q\alpha, \quad n = 1,2,3,\ldots . \tag{92}$$

It is easy to observe that applying (4), both the intermediate and the final potentials (as well as their eigenfunctions) satisfy (3), and hence are $\mathcal{PT}$ invariant, having real spectra.

B. New potentials for $i=0$, $j=2$

In a similar manner, the expressions for the different quantities are obtained as follows:

$$W(\psi_0,\psi_2) = c_{02} e^{-(x-i\epsilon)^2}(x-i\epsilon)^{-2q\alpha+2}\left\{\frac{(x-i\epsilon)^2}{2-q\alpha} - 1\right\} \tag{93}$$

with c_{02} some real constant

$$L_1 = -\frac{d}{dx} - (x-i\epsilon) + \frac{(-q\alpha+\frac{1}{2})}{(x-i\epsilon)}, \tag{94}$$

$$L_2 = -\frac{d}{dx^2} - (x-i\epsilon) + \frac{(-q\alpha+\frac{3}{2})}{(x-i\epsilon)} + \frac{2(x-i\epsilon)}{[(x-i\epsilon)^2-(2-q\alpha)]}, \tag{95}$$

$$A = \frac{d^2}{dx^2} + 2\left\{(x-i\epsilon) + \frac{(q\alpha-1)}{(x-i\epsilon)} - \frac{(x-i\epsilon)}{(x-i\epsilon)^2-(2-q\alpha)}\right\}\frac{d}{dx} + (x-i\epsilon)^2 + \frac{(-q\alpha+\frac{1}{2})(-q\alpha+\frac{5}{2})}{(x-i\epsilon)^2} - \frac{3}{(x-i\epsilon)^2-(2-q\alpha)} + 2q\alpha - 3, \tag{96}$$

$$A^{\#} = -\frac{d}{dx^2} - 2\left\{(x-i\epsilon) + \frac{(q\alpha-1)}{(x-i\epsilon)} + \frac{(x-i\epsilon)}{(x-i\epsilon)^2-(2-q\alpha)}\right\}\frac{d}{dx} + (x-i\epsilon)^2 + \frac{(-q\alpha-\frac{1}{2})(-q\alpha+\frac{3}{2})}{(x-i\epsilon)^2} - \frac{5}{\{(x-i\epsilon)^2-(2-q\alpha)\}} - \frac{4(-q\alpha+2)}{\{(x-i\epsilon)^2-(2-q\alpha)\}^2} + 2q\alpha - 5. \tag{97}$$

The new potential

$$\widetilde{V}_{0,2}(x) = (x-i\epsilon)^2 + \frac{\sigma(\sigma-1)}{(x-i\epsilon)^2} + \frac{4}{(x-i\epsilon)^2-(2-q\alpha)} + \frac{8(2-q\alpha)}{\{(x-i\epsilon)^2-(2-q\alpha)\}^2} + 4, \tag{98}$$

where

$$\sigma = -q\alpha + \tfrac{5}{2} \tag{99}$$

is totally different from the initial potential of the $\mathcal{PT}$ symmetric oscillator, yet shares the same spectrum except for the states $n=0$, 2 of the original potential, which are missing in the partner.

The ground state wave function of the Hamiltonian in (98) is given by

$$\widetilde{\psi}_0(x) = \left\{ A_1(x-i\epsilon)^2 + A_2 + \frac{A_3}{(x-i\epsilon)^2-(2-q\alpha)} \right\} e^{-(1/2)(x-i\epsilon)^2}(x-i\epsilon)^{-q\alpha+(1/2)} \tag{100}$$

with ground state energy

$$\widetilde{E}_0 = E_1 = 6 - 2q\alpha, \tag{101}$$

where A_1, A_2, A_3 are x-independent constants, while the excited states are obtained from (62)

$$\widetilde{\psi}_n = A\psi_{n+2} = -E_{n+2}\psi_{n+2} + E_0\psi_0 \frac{W(\psi_{n+2},\psi_2)}{W(\psi_0,\psi_2)} + E_2\psi_2 \frac{W(\psi_0,\psi_{n+2})}{W(\psi_0,\psi_2)} \tag{102}$$

with energies

$$\widetilde{E}_n = E_{n+2} = 4n + 10 - 2q\alpha, \quad n = 1,2,\ldots . \tag{103}$$

It can also be verified that eigenfunctions $\widetilde{\psi}_n(x)$ have correct asymptotic behavior and are also $\mathcal{PT}$ invariant. Consequently, they also satisfy Eq. (7). The intermediate potential is given by

$$V_1(x) = (x-i\epsilon)^2 + \frac{(-q\alpha+\frac{1}{2})(-q\alpha+\frac{3}{2})}{(x-i\epsilon)^2} + 2 \tag{104}$$

which is also physically acceptable. By arguments similar to those given above, its ground state eigenfunction is given by

$$\phi_0 = e^{-(1/2)(x-i\epsilon)^2}(x-i\epsilon)^{-q\alpha+(3/2)} \tag{105}$$

with energy

$$e_0 = 6 - 2q\alpha \tag{106}$$

and excited states

$$\phi_n = \frac{W(\psi_{n+1},\psi_0)}{\psi_0} \tag{107}$$

with energies

$$e_n = E_{n+1} = 4n + 6 - 2q\alpha, \quad n = 1,2,3,\ldots . \tag{108}$$

Once again, both the intermediate and the final potentials (as well as their eigenfunctions) are $\mathcal{PT}$ invariant, having real spectra.

The supercharges Q_2 and $Q_2^\dagger$ generated from the intertwining operators A and $A^\#$ can be shown to satisfy the following algebra:

$$\mathcal{H}_2 = \{Q_2, Q_2^\#\} = H_2^2 - 4(3-q\alpha)H_2 + (2-q\alpha)(10-2q\alpha) \tag{109}$$

where H_2 is given by (51).

We note that the potentials obtained in this section are unique in the sense that they do not have any counterpart in standard quantum mechanics (i.e., in the Hermitian case).

V. $\mathcal{PT}$ SYMMETRIC SCARF II POTENTIAL

We note that the generalized oscillator problem considered in the last section was made non-Hermitian by an imaginary displacement of the coordinate variable x. However, there are other methods of constructing non-Hermitian models. To see how the formalism described in Sec. III works with such models, in this section we shall study an example, viz., the $\mathcal{PT}$ symmetric non-Hermitian Scarf II potential, which has been $\mathcal{PT}$ symmetrized in a different way. This exactly solvable potential, given by

$$V(x) = -\lambda\,\mathrm{sech}^2 x - i\mu\,\mathrm{sech}\,x\tanh x, \quad \lambda > 0,\ \mu \neq 0 \tag{110}$$

has a discrete spectrum that admits both real as well as complex conjugate energies, depending on the relative strengths of its parameters λ and μ. For $|\mu| \leqslant \lambda + \frac{1}{4}$, the system possesses a real and discrete bound state spectrum, whereas for $|\mu| > \lambda + \frac{1}{4}$, the system exhibits spontaneous $\mathcal{PT}$ symmetry breaking, with complex conjugate pairs of energies. The normalized wave functions for this potential are well known, being given by[15,19]

$$\psi_n(x) = \frac{\Gamma(n-2p+\frac{1}{2})}{n!\Gamma(\frac{1}{2}-2p)} z^{-p}(z^*)^{-q} P_n^{-2p-1/2,-2q-1/2}(i\sinh x), \tag{111}$$

where $P_n^{\alpha,\beta}$ are the Jacobi polynomials,[20]

$$P_n^{\alpha,\beta}(i\sinh x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} F(-n, n+\alpha+\beta+1;\alpha+1;z) \tag{112}$$

and

$$z = \frac{1-i\sinh x}{2}, \tag{113}$$

$$p = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4}+\lambda+\mu} = -\frac{1}{4} \pm \frac{t}{2}, \tag{114}$$

$$q = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4}+\lambda-\mu} = -\frac{1}{4} \pm \frac{s}{2}. \tag{115}$$

However, for normalization of the wave functions, only the positive sign is allowed in p. The energy spectrum

$$E_n = -(n-p-q)^2, \quad n = 0,1,2,\ldots < \left(\frac{s+t-1}{2}\right) \tag{116}$$

is real and bound for $|\mu| \leqslant \lambda + \frac{1}{4}$, i.e., for real p and q, with two towers characterized by the two values of q.

If the formalism developed above is applied to this example for $N=2$, with states $\psi_0(x)$ and $\psi_2(x)$, then the Wronskian is calculated to be

$$W(\psi_0,\psi_2) = (1-i\sinh x)^{-2p}(1+i\sinh x)^{-2q}\cosh x\{-i(p-q)+(p+q-\tfrac{3}{2})\sinh x\} \tag{117}$$

and the intertwining operators A and $A^{\#}$ are given by

$$A = L_2 L_1, \quad A^{\#} = L_1^{\#} L_2^{\#}, \tag{118}$$

where L_1 and L_2 take the form

$$L_1 = -\frac{d}{dx} + \frac{\psi_0'}{\psi_0} = \frac{d}{dx} + i(p-q)\operatorname{sech} x - (p+q)\tanh x, \tag{119}$$

$$L_2 = -\frac{d}{dx} + \frac{W_{0,2}'}{W_{0,2}} - \frac{\psi_0'}{\psi_0} = \frac{d}{dx} + i(p-q)\operatorname{sech} x - (p+q)\tanh x + \frac{(-p-q+\frac{3}{2}) + i(p-q)\sinh x + (-2p-2q+3)\sinh^2 x}{i(p-q)\cosh x + (-p-q+\frac{3}{2})\sinh x \cosh x}. \tag{120}$$

Now using (40) the new potential is found to be

$$\widetilde{V}_{0,2}(x) = -\widetilde{\lambda}\operatorname{sech}^2 x - i\widetilde{\mu}\operatorname{sech} x \tanh x - 2\left(\frac{\sigma^2 \operatorname{sech}^2 x - i\rho\sigma \operatorname{sech} x \tanh x}{(\rho \operatorname{sech} x - i\sigma \tanh x)^2}\right), \tag{121}$$

where

$$\widetilde{\lambda} = \lambda - 4p - 4q + 2, \tag{122}$$

$$\widetilde{\mu} = \mu - 4p + 4q, \tag{123}$$

$$\lambda = 2(p^2 + q^2) + (p+q), \tag{124}$$

$$\mu = 2(p^2 - q^2) + (p-q), \tag{125}$$

$$\rho = p - q, \tag{126}$$

$$\sigma = -p - q + \tfrac{3}{2}. \tag{127}$$

Once again, the final potential $\widetilde{V}_{0,2}(x)$ is also $\mathcal{PT}$ invariant. The eigenfunctions are obtained from (62), with the ground state as

$$\widetilde{\psi}_0 = (E_0 - E_1)\psi_1 + (E_2 - E_0)\psi_2 \frac{P_1'}{P_2'} \tag{128}$$

and excited states

$$\widetilde{\psi}_n = (E_0 - E_{n+2})\psi_{n+2} + (E_2 - E_0)\psi_2 \frac{P_{n+2}'}{P_2'} \tag{129}$$

where P_n denotes the Jacobi polynomial $P_n^{-2p-1/2-2q-1/2}(i \sinh x)$ and P_n' denotes its derivative with respect to x. It can be shown that for $|\mu| \leqslant \lambda + \frac{1}{4}$, the wave functions $\widetilde{\psi}$ are also $\mathcal{PT}$ invariant, and can be normalized following (7). The new potential $\widetilde{V}_{0,2}(x)$ has real bound state spectrum given by

$$\widetilde{E}_0 = -(1-p-q)^2, \tag{130}$$

$$\widetilde{E}_n = -(n+2-p-q)^2, \quad n = 1,2,\ldots, < \left(\frac{s+t-5}{2}\right), \tag{131}$$

and the algebra satisfied by the supercharges turns out to be

$$\mathcal{H}_2 = \left\{Q_2, Q_2^{\#}\right\} = H_2^2 + (2-2p-2q)H_2 + (p+q)(p+q-2), \tag{132}$$

where H_2 is given by (51).

The intermediate potential takes the form

$$V_1(x) = -\tilde{v}_1 \operatorname{sech}^2 x - i\tilde{v}_2 \operatorname{sech} x \tanh x, \tag{133}$$

where

$$\tilde{v}_1 = \lambda - 2(p+q), \tag{134}$$

$$\tilde{v}_2 = \mu - 2(p-q) \tag{135}$$

with eigenfunctions

$$\phi_n = \frac{W(\psi_{n+1}, \psi_0)}{\psi_0} \tag{136}$$

and the corresponding energies

$$e_n = E_{n+1} = -(n+1-p-q)^2, \quad n = 0,1,\ldots, < \left(\frac{s+t-3}{2}\right). \tag{137}$$

Thus $V_1(x)$ and the corresponding wave functions (136) are also physically acceptable as well as $\mathcal{PT}$ invariant.

VI. CONCLUSIONS

In this paper we have suggested an application of higher order Darboux algorithm to non-Hermitian $\mathcal{PT}$ symmetric potentials. For the sake of definiteness the method has been applied to two specific potentials, namely, the generalized oscillator and the Scarf II potentials and a number of new potentials having nearly the same spectrum as the original ones have been obtained. It may be noted that in each of these cases, starting from a $\mathcal{PT}$ symmetric potential we have obtained new potentials which are again $\mathcal{PT}$ symmetric. In other words the higher order Darboux algorithm does not induce spontaneous $\mathcal{PT}$ symmetry breaking. Among the different cases considered here the one involving nonconsecutive levels deserves special mention. The potentials thus obtained have no Hermitian analogues. Also the intermediate potentials in all the cases are perfectly well behaved since the Darboux algorithm does not introduce any new singularity or break $\mathcal{PT}$ symmetry. Furthermore it has been shown that the symmetry underlying the original and the new potentials is a fusion of *nonlinear SUSY* and $\mathcal{PT}$ symmetry which we call *nonlinear pseudosupersymmetry*. Finally we note that analogous to the study of breaking N-fold supersymmetry,[21] it would be of interest to examine breaking of this new symmetry.

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