κ-Minkowski spacetime through exotic "oscillator"

Subir Ghosh [a], Probir Pal [b]

[a] *Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta 700108, India*
[b] *Physics Department, Uluberia College, Uluberia, Howrah 711315, India*

Abstract

We have proposed a generally covariant non-relativistic particle model that can represent the κ-Minkowski noncommutative spacetime. The idea is similar in spirit to the noncommutative particle coordinates in the lowest Landau level. Physically our model yields a novel type of dynamical system (termed here as exotic "oscillator"), that obeys a harmonic oscillator like equation of motion with a *frequency* that is proportional to the square root of *energy*. On the other hand, the phase diagram does not reveal a closed structure since there is a singularity in the momentum even though energy remains finite. The generally covariant form is related to a generalization of the Snyder algebra in a specific gauge and yields the κ-Minkowski spacetime after a redefinition of the variables. Symmetry considerations are also briefly discussed in the Hamiltonian formulation. Regarding continuous symmetry, the angular momentum acts properly as the generator of rotation. Interestingly, both the discrete symmetries, parity and time reversal, remain intact in the κ-Minkowski spacetime.

1. Introduction

Pointers from diverse areas in high energy physics indicate that one has to look beyond a *local* quantum field theoretic description in the formulation of quantum gravity. Very general considerations in black hole physics lead to the notion of a fuzzy or noncommutative (NC) spacetime which can avoid the paradoxes one faces in trying to localize a spacetime point within the Planck length [1]. This is also corroborated in the modified Heisenberg uncertainty principle that is obtained in string scattering results. The recent excitement in NC spacetime physics is generated from the seminal work of Seiberg and Witten [2] who explicitely demonstrated the emergence of NC manifold in certain low energy limit of open strings moving in the background of a two form gauge field. In this instance, the NC spacetime is expressed by the Poisson bracket algebra (to be interpreted as commutators in the quantum analogue),

$$\{x^{\mu}, x^{\nu}\} = \theta^{\mu\nu}, \tag{1}$$

E-mail address: subir_ghosh2@rediffmail.com (S. Ghosh).

where $\theta^{\mu\nu}$ is a c-number constant. Up till now this form of NC extension has been the popular one. However, notice that Lorentz invariance is manifestly violated in quantum field theories built on this spacetime. Somehow, it appears that the very idea of formulating field theories in this sort of spacetime, consistent with quantum gravity, gets defeated by this pathology!

In a parallel development, there have been intense activities in studying other forms of NC spacetimes, such as the Lie algebraic form [3] with structure constants $C_{\lambda}^{\mu\nu}$,

$$\{x^{\mu}, x^{\nu}\} = C_{\lambda}^{\mu\nu} x^{\lambda}. \tag{2}$$

It is important to note that the NC extension in (2) is *operatorial* [3] and do not jeopardize the Lorentz invariance in relativistic models, which is the case with (1) with constant $\theta^{\mu\nu}$. (For an introduction to this subject the readers are referred to [4].) Of particular importance in the above is a restricted class of spacetimes known as κ-Minkowski spacetime (or κ-spacetime in short), that is described by the algebra,

$$\{x_i, t\} = kx_i, \qquad \{x_i, x_j\} = \{t, t\} = 0. \tag{3}$$

In the above, x_i and t denote the space and time operators, respectively. Some of the important works in this topic that discusses, among other things, construction of a quantum field theory in κ-spacetime, are provided in [5–7]. Very interestingly, Amelino-Camelia [8] has proposed an alternative path to quantum gravity—"the doubly special relativity"—in which *two* observer independent parameters (the velocity of light and Planck's constant) are present. It has been shown [9] that κ-spacetime is a realization of the above. Furthermore, the mapping [9] between κ-spacetime and Snyder spacetime [10] (the first example of an NC spacetime), shows the inter-relation between these models and "two-time physics" [11], since the Snyder spacetime can be derived from two-time spaces in a particular gauge choice [12]. Our aim is to present a physically motivated realization of the κ-spacetime.

An altogether different form [13] of NC phase space is induced by spin degrees of freedom $S^{\mu\nu}$,

$$\{x^{\mu}, x^{\nu}\} = S^{\mu\nu}, \tag{4}$$

where once again the noncommutativity is operatorial and the model is Lorentz invariant.

Now we come to the motivation of our work. In a non-relativistic setup, NC space, originating from the lowest Landau level projection of charged particles moving in a plane under the influence of a uniform, perpendicular (and strong) magnetic field [1], has become the prototype of a simple physical system (qualitatively) describing considerably more complex and abstract phenomena, in this case open strings moving in the presence of a background two form gauge field [2] mentioned before. Under certain low energy limits, the mechanism by which NC manifolds emerge in the string boundaries on the branes, is similar to the way NC particle coordinates appear in the Landau problem. This sort of intuitive picture, if present, is very useful and appealing. The NC space (or spacetime) one is talking about here is of the form

$$\{x^{\mu}, x^{\nu}\} = \theta^{\mu\nu}, \qquad |\theta| \sim |B^{-1}|, \tag{5}$$

where $\theta^{\mu\nu}$ is constant and the strength is proportional to the inverse of magnetic field B. Note that in the classical set up the commutators are interpreted as Poisson brackets (or Dirac brackets).

In the phase space form of noncommutativity also [13] there is a physical picture concerning spinning particle models [14] that induces the NC spacetime. However, such an intuitive analogue is lacking for understanding the Lie algebraic form of NC [3]. Our present work is aimed at throwing some light in this area.

In this Letter we are going to put forward a non-relativistic particle model that has an underlying phase space algebra isomorphic to the κ-Minkowski one (3). Hamiltonian constraint analysis [15] reveals a novel dynamical system (termed here as exotic "oscillator"): *it has the square root of energy as its frequency*. This sort of feature is curiously reminiscent of the quantum particle whose frequency is proportional to its energy. Phase diagram analysis yields further surprises, to be elaborated later.

However, demonstrating that the model truly represents the NC κ-spacetime is not straightforward, the main hurdle being the identification of the time operator.[1] This requires a generalization of our model to a generally covariant one [16]. The gauge invari-

[1] There is a version of κ-spacetime [7] which has only NC space coordinates. We intend to study this model more closely in our framework since the NC time complications will be absent here. We wish to thank the referee for pointing this out.

ance (due to the symmetry under reparametrization of the evolution parameter) allows us to choose a gauge condition that fixes the time operator according to our requirement. This way of exploiting a non-standard gauge condition to induce NC coordinates has been used in [17] in constant spacetime noncommutativity (5).

2. Mechanical model for κ-spacetime

We start by considering a canonical phase space with the non-zero Poisson brackets,

$$\{X_i, P_j\} = \delta_{ij}, \qquad \{\eta, \pi\} = 1. \tag{6}$$

The sets (X_i, P_j) and (η, π) are decoupled. (We do not distinguish between upper and lower indices in the non-relativistic setup.) Let us posit the following set of second class constraints (SCC) [15]

$$\chi_1 \equiv \pi, \qquad \chi_2 \equiv \eta - k(\vec{P}\,.\,\vec{X}). \tag{7}$$

SCCs require the usage of Dirac brackets (DB) defined by

$$\{A, B\}_{\mathrm{DB}} = \{A, B\} - \{A, \chi_i\}\{\chi_i, \chi_j\}^{-1}\{\chi_j, B\}, \tag{8}$$

such that DB between an SCC and *any* operator vanishes. In the present case, the simple constraint Poisson bracket matrix and its inverse are, respectively,

$$\{\chi_i, \chi_j\} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

$$\{\chi_i, \chi_j\}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

The non-vanishing DBs are derived below:

$$\{X_i, \eta\} = kX_i, \qquad \{P_i, \eta\} = -kP_i,$$
$$\{X_i, P_j\} = \delta_{ij}. \tag{11}$$

Since we will always deal with DBs the subscript DB is dropped.

We now construct the following Lagrangian that has the same SCC structure as in (7)

$$L = \frac{m}{2}\dot{\vec{X}}^2 - 2cmk\eta(\vec{X}\,.\,\dot{\vec{X}}) + c\eta^2 + 2mc^2k^2\eta^2\vec{X}^2. \tag{12}$$

m denotes the mass of the non-relativistic particle and c and k are two other constant parameters.

Reexpressed in the form

$$L = \frac{m}{2}\dot{\vec{X}}^2 + \left(2mc^2k^2\eta^2 + mck\dot{\eta}\right)\vec{X}^2 + c\eta^2,$$

one can think of the model as that of a generalized form of oscillator whose spring coupling is not constant and depends on X_i itself through η. In fact classically η can be eliminated by solving the Gaussian to yield a complicated non-linear model. Instead we prefer to work with this polynomial form with the extra variable η. As we shall see later, the model describes a novel dynamical system.

The conjugate momenta in (12) are defined by

$$P_i = m\dot{X}_i - c\eta^2 - 2cmk\eta X_i, \quad \pi = 0. \tag{13}$$

The primary constraint is

$$\chi_1 \equiv \pi \approx 0. \tag{14}$$

Time persistence of χ_1 generates the secondary constraint

$$\chi_2 \equiv \dot{\chi}_1 = \{\chi_1, H\} \to \chi_2 \equiv \eta - k(\vec{P}\,.\,\vec{X}) \approx 0. \tag{15}$$

These are the same as the constraints we started with at the beginning in (7). Obviously identical DBs as in (11) will be reproduced. The Hamiltonian

$$H = \frac{\vec{P}^2}{2m} + 2ck\eta(\vec{P}\,.\,\vec{X}) - c\eta^2 \tag{16}$$

in the reduced phase space simplifies to the exotic "oscillator"

$$H = \frac{\vec{P}^2}{2m} + ck^2(\vec{P}\,.\,\vec{X})^2. \tag{17}$$

It is worthwhile to emphasize the fact the model proposed here for simulating κ-spacetime has considerably more structure (in the form of additional variables η and π) than the analogous model for constant noncommutativity [1]. This is expected on the grounds that the Lie algebraic form of NC algebra is non-linear and operatorial in nature. We also encounter [18] similar complexities in analyzing a Lie algebraic space–space NC algebra.

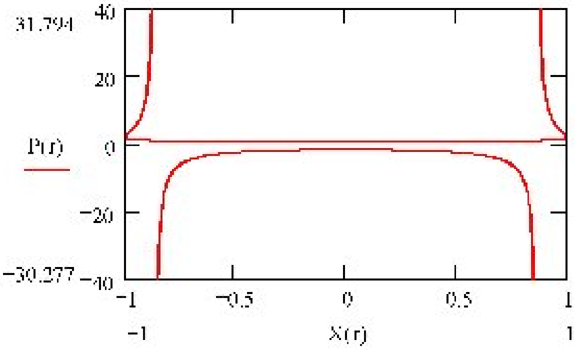

Fig. 1. The exotic "oscillator" phase diagram for $E = 1$ and $a = 0.5$.

3. The exotic "oscillator"

The relevant Hamiltonian equations of motion are

$$\dot{X}_i = \{X_i, H\} = \frac{P_i}{m} + 2ck^2(\vec{P}\,.\,\vec{X})X_i,$$
$$\dot{P}_i = -2ck^2(\vec{P}\,.\,\vec{X})P_i. \tag{18}$$

A further iteration in time derivative generates the following exotic "oscillator" dynamics

$$\ddot{X}_i = -w^2 X_i \tag{19}$$

with $c = -b^2$ and the frequency w identified as

$$w = \pm 2bk\sqrt{H}. \tag{20}$$

Note the novel characteristic of dispersion where the frequency is a function of the Hamiltonian or energy. This is curiously reminiscent of the quantum mechanical dispersion $w \sim$ energy. This is one of the interesting results of the present analysis.[2]

From the above analysis, the exotic "oscillator" interpretation seems to be straightforward, since for a particular value of the energy (which is a conserved quantity), the "oscillator" will have a definite frequency given by (20). However, a phase diagram of our model (see Figs. 1 and 2) will reveal that the above conclusion is *not* fully correct.

In the figures we have considered a simplified version of the Hamiltonian (17), in the "oscillator" regime, in one space dimension and all the parameters are taken to be unity

$$H \equiv E = P^2(1 - X^2). \tag{21}$$

The phase diagram for $E = 1$ is drawn in Fig. 1. In Fig. 2 phase diagrams are drawn for three values of energy $E = 1, 3, 5$ and they are compared with the harmonic oscillator Hamiltonian, for the same set of energy values $E_{ho} = 1, 3, 5$

$$H \equiv E_{ho} = p^2 + q^2. \tag{22}$$

In (21) and (22), we use the parametric representations, respectively,

$$X = \cos(r), \qquad P = \frac{\sqrt{E}}{\sin(r)}, \tag{23}$$
$$X = \sqrt{E}\cos(r), \qquad P = \sqrt{E}\sin(r). \tag{24}$$

It is evident that in (23) there is a singularity at $r = 0$. Actually in the figures we have plotted

$$X = \cos(r), \qquad P = \frac{\sqrt{E}}{a + \sin(r)}$$

with $a = 0.5$ in Fig. 1 and $a = 1.5$ in Fig. 2. The asymmetries in the figures are due to the choice of the value of a. Indeed, in this model the momentum can diverge even though the energy remains finite. It will be useful to construct a variant of our model with the

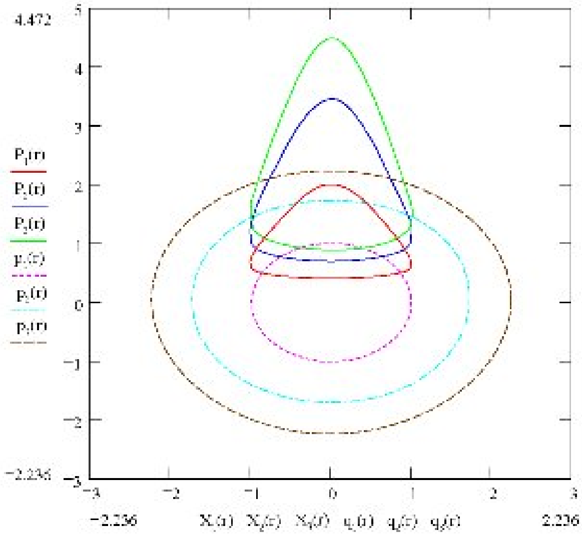

Fig. 2. Phase diagrams of exotic "oscillator" are compared with those of harmonic oscillator for energies $E = 1, 3, 5$ and $a = 1.5$.

[2] One might be tempted to think that tuning the exponent in the $(\vec{P}\,.\,\vec{X})$ term in (17), the quantum particle dispersion can be obtained. However, this is not the case as we show in Appendix A. As a curiosity, the quantum dispersion *will* be obtained if the Hamiltonian is proportional to $(H)^{3/2}$ with H of the form of (17).

a-damping in-built. Clearly one has to be more careful in interpreting w containing the Hamiltonian explicitly as frequency. It is clear from the Lagrangian in (12) that the model is qualitatively different from a harmonic oscillator. An intuitive physical understanding of this behavior of our exotic "oscillator", with the apparently simple looking dynamics as depicted in (19), is possible in the Lagrangian version.

A Lagrangian framework is better suited to get the physical picture corresponding to the exotic "oscillator". For the one-dimensional model (21), exploiting the first order formalism, we get the Lagrangian as

$$L \equiv P\dot{X} - H = \frac{1}{2}\frac{\dot{X}^2}{(1-cX^2)}, \tag{25}$$

where we have eliminated P using the equation of motion. The impression is that of a "free" particle with an *effective* mass. The singularity of this effective mass leads to the momentum blowup.[3]

The higher-dimensional action is more involved:

$$L = P_i\dot{X}_i - H = \frac{1}{2}\left[(\dot{X}_i)^2 + c\frac{(X_i\dot{X}_i)^2}{1-cX_i^2}\right]. \tag{27}$$

For a single space dimension, (27) reduces to (25)

$$L = \frac{1}{2}\frac{(\dot{X}_i)^2}{(a+\sqrt{1-cX_i^2})^2}. \tag{28}$$

Expressions similar to (27) have appeared in [12]. It will be very interesting if these models are related to known physical systems. We will comment on this possibility at the end.

4. Generally covariant framework

Let us now come to the main topic: κ-spacetime. From the DBs (11) it is evident that our aim is to identify the degree of freedom η as time. So far in this formulation, $\eta(t)$ is a (configuration space) degree of freedom just as $X_i(t)$ and their evolution is dictated by the Hamiltonian in the conventional way. Hence further work is to be done if η is to identified as time. Quite obviously, in κ-spacetime time is an operator since it has non-trivial commutation relations. In our classical scenario this will be reflected in the non-zero Dirac brackets concerning η.

Hence in order to identify η as the time operator, the natural way to proceed is to generalize the model to a *generally covariant* one [16], which has more freedom since the evolution is dependent on another parameter τ and "time" is still not fixed or identified. In this formulation one works in an extended phase space with one extra canonical pair $\{X_0(\tau), P_0(\tau)\} = 1$ and all the dynamical variables are functions of the parameter τ. The system is elevated to a local gauge theory where the gauge symmetry is invariance under reparametrizations of τ. Therefore one has the freedom of choosing a gauge condition in order to lift the above invariance and this choice in effect can fix the time operator. Conventionally $X_0(\tau)$ plays the role of the time operator and normal Hamiltonian mechanics is recovered in the gauge $X_0(\tau) = \tau$. In the present context our aim is to fix the gauge so that the variable η (introduced above) becomes the time operator.

We follow [16] and re-express the action S of our model in the following generally covariant form

$$S = \int d\tau\,[P_i\dot{X}_i + \pi\dot{\eta} + P_0\dot{X}_0 - H_E]. \tag{29}$$

The extended Hamiltonian

$$H_E = u_0\phi_0 + u_1\chi_1 + u_2\chi_2, \tag{30}$$

has become a linear combination of constrains only and weakly vanishes. In (30) ϕ_0 represents the FCC inducing τ-diffeomorphism

$$\phi_0 \equiv P_0 + H \approx 0, \tag{31}$$

and χ_1 and χ_2 are the SCC pair introduced at the beginning in (7) and u_is denote multipliers. Note that in this form we have reverted back to completely canonical (X_i, P_j) and (η, π) phase space. However, for convenience, we will exploit the partially reduced phase space where the SCCs χ_1 and χ_2 are strongly zero with the phase space algebra given in (11). As mentioned before, conventional dynamics, as obtained in (17), (18) is recovered in the gauge $\phi_1 \equiv X_0(\tau) - \tau \approx 0$ which constitutes the SCC pair together with ϕ_0.

Now comes the most important part of our work. Since we are interested in a particle model that gener-

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[3] The ad hoc momentum cut off a, introduced to get a closed phase space diagram is in-built in the following Lagrangian

$$L = \frac{1}{2}\frac{(\dot{X}_i)^2}{(a+\sqrt{1-cX_i^2})^2}. \tag{26}$$

ates the κ-spacetime, we instead choose a gauge

$$\phi_1 \equiv X_0 - k(\vec{P}\,.\,\vec{X}). \tag{32}$$

The reason for this choice is the following. Remember that we are working in a truncated phase space where the SCC $\chi_2 \equiv \eta - kX_iP_i$ strongly vanishes and thus η is already identified with kX_iP_i. So in the above gauge (32) the time variable X_0 becomes related to η.

To get the fully reduced phase space, we now compute the secondary set of DBs induced by the SCC pair (ϕ_0, ϕ_1) with

$$\{\phi_0, \phi_1\} = -\left(1 - \frac{k\vec{P}^2}{m}\right) \equiv -\alpha. \tag{33}$$

We must remember to use the first set of DBs in (11) as the existing bracket structure in the definition of DB in (8) in the analysis at hand. This leads to the following more involved final DB structure involving coordinate and momenta:

$$\{X_i, X_j\} = \frac{k}{m\alpha}(X_iP_j - X_jP_i),$$
$$\{X_i, P_j\} = \delta_{ij} + \frac{k}{m\alpha}P_iP_j,$$
$$\{P_i, P_j\} = 0. \tag{34}$$

The algebra (34) is a slightly more general form of the one proposed by Snyder [10] due to the scaling by α in the right-hand sight. The pure form of Snyder algebra [10] have appeared in [12] in a gauge fixed reduced two-time model. The algebra with X_0 turns out to be

$$\{X_i, X_0\} = \frac{kX_i}{\alpha}, \qquad \{P_i, X_0\} = -\frac{kP_i}{\alpha}. \tag{35}$$

Notice that the spacetime, as obtained in (34), (35), is not the κ-spacetime that we set out to generate. But this is rectified by introducing the following set of variables

$$x_i \equiv X_i - \frac{k}{m}(\vec{P}\,.\,\vec{X})P_i, \qquad p_i \equiv P_i \tag{36}$$

in terms of which we obtain the following DBs

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = 0. \tag{37}$$

Hence, we will obtain identical dynamics as in (19), (20) if in the general covariant framework we take the Hamiltonian

$$H = H = \frac{p^2}{2m} + ck^2(\vec{p}\,.\,\vec{x})^2. \tag{38}$$

This can be obtained from (17) by replacing the set (X_i, P_j) by (x_i, p_j) in the limit $\alpha \sim 1$.[4]

To get a representation of the time operator, we note that

$$\{x_i, X_0\} = \left\{X_i - \frac{k}{m}(\vec{P}\,.\,\vec{X})P_i, X_0\right\}$$
$$= \frac{k}{\alpha}\left(X_i + \frac{k}{m}(\vec{P}\,.\,\vec{X})P_i\right). \tag{39}$$

However, the correct DB for κ-spacetime is generated with the time variable

$$t \equiv k\alpha(\vec{P}\,.\,\vec{X}), \tag{40}$$

for which we obtain

$$\{x_i, t\} = kx_i. \tag{41}$$

The operator conjugate to the time is obtained below

$$\left\{t, \frac{1}{2\kappa}\ln \vec{P}^2\right\} = 1. \tag{42}$$

This constitutes our final result. We also note that the $k = 0$ limit that reduces κ-Minkowski to commutative spacetime is smooth everywhere.

5. Exotic "oscillator" in Snyder space

Because of the non-linearity involved in the Snyder algebra (34), probably one of the simplest but interesting mechanical model in Snyder space is the exotic "oscillator". Consider the Snyder algebra,

$$\{X_i, X_j\} = -\gamma(X_iP_j - X_jP_i),$$
$$\{X_i, P_j\} = \delta_{ij} - \gamma P_iP_j, \qquad \{P_i, P_j\} = 0 \tag{43}$$

(where for later convenience we have taken the NC κ-parameter to be $-\gamma$). For small γ one finds the following set of equations of motion

$$\ddot{X}_i = -2\gamma HX_i, \qquad \ddot{P}_i = -2\gamma HP_i \tag{44}$$

for the Hamiltonian

$$H = \frac{1}{2}X_iX_i + \frac{\gamma}{2}(X_iP_i)^2. \tag{45}$$

[4] Another derivation of the Hamiltonian operator is provided in Appendix B where the second stage DBs (induced by the pair (ϕ_0, ϕ_1)) are not needed.

The above equations are valid up to $O(\gamma)$. The choice of the sign of γ is tuned to get the dynamics in the exotic oscillator form. Notice the difference between the Hamiltonian (45) and the Hamiltonian (36) where $\{X_i, P_j\} = \delta_{ij}$ as in (11). Thus effectively the set $\{X_i, P_j\}$ is the canonical set $\{x_i, p_j\}$ in our notation. To $O(\gamma)$, one can recover the Hamiltonian (38) from (45) by exploiting the mapping (48) given below.

6. Symmetries

Symmetry principles are playing increasingly major roles in contemporary physics. The fate of conventional spacetime symmetries in the context of NC theories is an important issue since one is changing the underlying spacetime structure itself. Poincaré invariance in the canonically NC field theories is explicitly broken [19]. However, in κ-spacetime, there appears a *deformation* of Lorentz symmetry [9,20]. These issues are more pertinent where relativistic field theories are concerned. Hence we will restrict ourselves to the symmetries that are relevant for non-relativistic (Hamiltonian) quantum mechanics. We will find non-trivial differences from the results obtained in [21] where a spacetime constant (i.e., canonical) form of noncommutativity has been considered.

We start with the angular momentum operator $L_{ij} = X_i P_j - X_j P_i$ and find

$$\{L_{ij}, X_k\} = \delta_{ik} X_j - \delta_{jk} X_i,$$
$$\{L_{ij}, P_k\} = \delta_{ik} P_j - \delta_{jk} P_i. \tag{46}$$

With $l_{ij} = x_i p_j - x_j p_i$ this is isomorphic to the conventional algebra

$$\{l_{ij}, x_k\} = \delta_{ik} x_j - \delta_{jk} x_i,$$
$$\{l_{ij}, p_k\} = \delta_{ik} p_j - \delta_{jk} p_i. \tag{47}$$

It should be remembered that (X_i, P_j) obey the generalized algebra (34) whereas (x_i, p_j) obey the commutative spacetime algebra (37). This shows that Snyder phase space behaves canonically under rotations. In fact, exploiting the inverse mapping of (36)

$$X_i \equiv x_i + \frac{k}{m\alpha}(\vec{p}\,.\,\vec{x})p_i, \qquad P_i \equiv p_i, \tag{48}$$

it is easy to see that

$$L_{ij} = X_i P_j - X_j P_i = x_i p_j - x_j p_i \equiv l_{ij}. \tag{49}$$

Interestingly, we find the *time operator to be rotationally invariant*,

$$\{L_{ij}, X_0\} = 0, \qquad \{l_{ij}, t\} = 0. \tag{50}$$

Hence, unlike the case discussed in [21], L_{ij} can regarded as the generator of spatial rotations in κ-spacetime.

Let us now turn to the discrete symmetries of the quantum theory where we identify $\{,\} \Rightarrow -i[,]$ and $p_i = -i\frac{\partial}{\partial x_i}$. Considering parity transformations in κ-spacetime,

$$P: \quad t \to t, \quad x_i \to -x_i, \tag{51}$$

we find the NC commutation relations

$$[x_i, x_j] = 0, \qquad [x_i, t] = i\kappa x_i, \quad i \to i, \tag{52}$$

are preserved under parity, where P is a linear operator. At the same time, considering the time reversal operator T as an antilinear operator, we find that the transformations

$$T: \quad t \to -t, \qquad x_i \to x_i, \quad i \to -i, \tag{53}$$

preserve (52) as well. *Hence P and T symmetries remain intact in κ-spacetime*. For charge conjugation invariant models based on κ-spacetime algebra, CPT will remain a valid symmetry. Once again we note the crucial difference with canonical NC spacetime results [21].

7. Conclusion and future outlook

We have succeeded in presenting a non-relativistic particle model which reproduces the κ-spacetime. The spirit of our work is in analogy with (the lowest level projection of) Landau problem of charges moving in a plane in a perpendicular magnetic where the particle positions become effectively noncommutative with constant θ. In the process, we have found that physically our particle model yields a novel type of dynamics that appears to be "oscillator"-like with a frequency proportional to the square root of the energy. Surprisingly the motion is not truly periodic which is revealed in the study of the phase diagram. Subsequently a generalization of the model to a generally covariant one leads to a definition of time that gives the full κ-Minkowski algebra. Furthermore, we have shown how a generally covariant reformulation of the model describes the (generalized) Snyder spacetime in a particular gauge and eventually leads to the κ-Minkowski spacetime. Study of the continuous (rotational) and discrete (parity and time reversal) symmetries reveal that the κ-Minkowski spacetime is probably a better option than the constant spacetime noncommutativity, as studied in [21]. This is primarily because angular momentum is the correct generator for rotations and parity and time reversal symmetries are kept intact. Hence maintaining CPT-invariance will not pose any problem.

We note some points that are to be studied in future. In the exotic "oscillator" context, a physical interpretation of the open phase diagram is required. One can try to construct an extension of the model, with the characteristic features as we have noted, but having at the same time a closed periodic motion. It will be very interesting to quantize the model. Also it would be interesting to investigate the type of systems that can induce quantum particle like dispersion and to study the kind of spacetimes they represent. Similar analysis, as has been performed here, for the general Lie algebraic noncommutative spacetime is under study.

In the context of obtaining the κ-Minkowski spacetime from our model, one can exploit an alternative framework (see Banerjee et al. [17]) where the identification of the time operator might be more direct. There is the possibility that some familiar interacting model, in a *non-standard gauge*, will be equivalent to the particle model proposed here. As a more ambitious programme, taking a cue from the Landau problem—string analogy in the context of noncommutative spacetime, one can try to construct string models yielding κ-Minkowski spacetimes. Our exotic "oscillator" model can help in the construction of the latter. A positive indication in this direction is that in a relativistic generalization of the present model the de Sitter metric plays a pivotal role and it is indeed natural to extend the framework for strings moving in de Sitter background. These results will be reported elsewhere [22].

Acknowledgements

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Appendix A

For a more general form of the Hamiltonian, comprising of canonical (x_i, p_j) variables

$$\tilde{H} = \frac{(\vec{p})^2}{2m} + c(\vec{p}\,.\,\vec{x})^n \tag{A.1}$$

we obtain the following equation of motion

$$\ddot{x}_i = nc(\vec{p}\,.\,\vec{x})^{n-2}\left[(n-1)\frac{(\vec{p})^2}{m} + nc(\vec{p}\,.\,\vec{x})^n\right]x_i. \tag{A.2}$$

It is easy to see that only $n = 2$ reproduces the exotic oscillator.

Appendix B

From the weakly vanishing Hamiltonian

$$H = u\phi_0 \tag{B.1}$$

and the explicitly time (τ) dependent gauge condition $\phi_1 \equiv X_0 - k(\vec{P}\,.\,\vec{X}) - \tau$, time persistence of ϕ_1 determines the multiplier u in (B.1) in the following way:

$$\frac{d\phi_1}{d\tau} = \frac{\partial\phi_1}{\partial\tau} + \{\phi_1, \phi_0\} \to u = \frac{1}{\alpha}. \tag{B.2}$$

The equations of motion (modulo constraint) are

$$\dot{X}_i = \left\{X_i, \frac{\phi_0}{\alpha}\right\} = \frac{1}{\alpha}\left(\frac{P_i}{m} + 2ck^2(\vec{P}\,.\,\vec{X})X_i\right),$$
$$\dot{P}_i = -\frac{2ck^2}{\alpha}(\vec{P}\,.\,\vec{X})P_i. \tag{B.3}$$

For low velocity (or large mass) $\alpha \approx 1$ the dynamical equations of (18) are reproduced.

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