Response to "Comment on 'Exact solutions of cylindrical and spherical dust ion acoustic waves'" [Phys. Plasmas 12, 054701 (2005)]

Biswajit Sahu and Rajkumar Roychoudhury
Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700108, India

It is shown that the similarity solution of cylindrical Korteweg–de Vries equation given by Tian and Gao in their Comment on our paper can be obtained by the group analysis method using the same generators given in the paper by Sahu and Roychoudhury.
[DOI: 10.1063/1.1885478]

Recently Sahu and Roychoudhury[1] obtained some exact solutions of cylindrical and spherical dust ion acoustic waves,[2–5] using the Lie group analysis.[6] In their Comment Tian and Gao[7] presented a brief review on the various solutions of the cylindrical Korteweg–de Vries (CKdV) equation existing in the literature on CKdV. We have discussed a particular solution based on the group analysis method taking only one of the four generators which exist for CKdV. However each of the generators would give a solution, some trivial some nontrivial. However it can be shown that all the solutions are but particular cases of the general solution which can be obtained if one takes a linear combination of all the generators. The similarity solutions given by Tian and Gao are just particular cases of this general solution. Regarding the solution given in Eq. (3) of Tian and Gao's Comment,[7] one sees that though the solution is of mathematical interest, a closed form of this solution cannot be obtained as it involves evaluation of indefinite integral over product of Bessel functions which it self is a formidable task. However let us consider the most general solution obtained from the group analysis.

Let us take

$$X = aX_2 + bX_3 + cX_4 + d_0X_1, \tag{1}$$

where

$$X_1 = \frac{\partial}{\partial x},$$

$$X_2 = 3 + \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u},$$

$$X_3 = 2\sqrt{t}\frac{\partial}{\partial u} + \frac{1}{\sqrt{t}}\frac{\partial}{\partial u},$$

$$X_4 = 2x\sqrt{t}\frac{\partial}{\partial x} + 4t\sqrt{t}\frac{\partial}{\partial t} + \left(\frac{x}{\sqrt{t}} - 4u\sqrt{t}\right)\frac{\partial}{\partial u}. \tag{2}$$

Using the property of Lie algebra one can show that X is also generator of the group. $F(x,t)$ is an invariant of the CKdV equation iff $XF=0$. Using Eqs. (1) and (2) we have

$$(3at + 4ct\sqrt{t})\frac{\partial F}{\partial t} + (ax + 2b\sqrt{t} + 2cx\sqrt{t} + d_0)\frac{\partial F}{\partial t} + \left[\frac{b}{\sqrt{t}} - 2au + c\left(\frac{x}{\sqrt{t}} - 4u\sqrt{t}\right)\right]\frac{\partial F}{\partial u} = 0. \tag{3}$$

The characteristic equations for Eq. (3) are

$$\frac{dt}{3at + 4ct\sqrt{t}} = \frac{dx}{ax + 2b\sqrt{t} + 2cx\sqrt{t} + d_0} = \frac{du}{-2au + \frac{b}{\sqrt{t}} + c\left(\frac{x}{\sqrt{t}} - 4a\sqrt{t}\right)}. \tag{4}$$

From the first two terms in Eq. (4) one gets

$$(ax + 2b\sqrt{t} + 2cx\sqrt{t} + d_0) - (3at + 4ct\sqrt{t})dx = 0. \tag{5}$$

If we take $c=0$, Eq. (5) can be written as

$$-3a\left(tdx - \frac{x}{3}dt\right) + dt(2b\sqrt{t} + d_0) = 0. \tag{6}$$

Equation (6) can easily be integrated to give

$$axt^{-1/3} - 4bt^{1/6} + d_0t^{-1/3} = \bar{c}, \tag{7}$$

where $\bar{c}$ is the integration constant. Identifying $a=(6A)^{-1/3}$, $b=-c_2/4$, $d_0=c_1$, and $\bar{c}=c_0-3\bar{c}/A^2$, Eq. (7) can be written as

$$(6At)^{-1/3} + c_1t^{-1/3} + c_2t^{1/6} + \frac{3c}{A^2} = c_0. \tag{8}$$

From the first and last terms in Eq. (4) we have

$$\left(-2au + \frac{b}{\sqrt{t}}\right)dt - 3atdu = 0. \tag{9}$$

Integration of Eq. (9) gives

$$u = \frac{c'}{3a}t^{-2/3} + \frac{6}{3a}bt^{-1/2}, \tag{10}$$

c' being the integration constant. Using Eq. (8), Eq. (10) can be written as

$$u=(6At)^{-1/3}c''+\frac{x}{18t}+\frac{(6A)^{1/3}}{36}(2c_1t^{-1}-c_2t^{-1/2}), \tag{11}$$

c'' being an arbitrary constant.

From Eqs. (8) and (11), following the usual procedure adopted in solving partial differantial equation, the general solution can be written as

$$u=(6At)^{-1/3}\omega[z(x,t)]+\frac{x}{18t}+\frac{(6A)^{1/3}}{36}(2c_1t^{-1}-c_2t^{-1/2}), \tag{12}$$

which is the solution given in Eq. (4) by Tian and Gao. To obtain their second solution put $a=0$ in Eq. (4) to obtain

$$\frac{dt}{4ct\sqrt{t}}=\frac{dx}{2b\sqrt{t}+2cx\sqrt{t}+d_0}=\frac{du}{\frac{b}{\sqrt{t}}+\frac{cx}{\sqrt{t}}}. \tag{13}$$

From the first two terms in Eq. (13) we get the integral

$$cxt^{-1/2}+bt^{-1/2}-\frac{d_0}{t}=\bar{c}, \tag{14}$$

which is essentially the expression for $z(x,t)$ for the second solution of Tian and Gao.

Solving the first and the last equations in Eq. (13) the exact solution of Tian and Gao as given in Eq. (6) can be obtained. Solution of the ordinary differential equation satisfied by $W(z)$ for $C=0$ [Eq. (7) of Tian and Gao] is given in Ref. 1.

[1] B. Sahu and R. Roychoudhury, Phys. Plasmas **10**, 4162 (2003).
[2] N. S. Zakharov and U. P. Korobeinikov, J. Appl. Math. Mech. **44**, 668 (1980).
[3] R. Hirota, Phys. Lett. **71A**, 293 (1979).
[4] A. A. Mamun and P. K. Shukla, Phys. Lett. A **290**, 173 (2001).
[5] A. A. Mamun and P. K. Shukla, Phys. Plasmas **9**, 1468 (2002).
[6] *CRC Handbook of Lie Group Analysis of Differential Equations*, edited by N. H. Ibragimov (CRC, Ann Arbor, 1993), Vol. 1, pp. 192–194.
[7] B. Tian and Y.-T. Gao, Phys. Plasmas **12**, 054701 (2005).