$(1+1)$ dimensional Dirac equation with non Hermitian interaction

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Abstract

We study $(1+1)$ dimensional Dirac equation with non Hermitian interactions, but real energies. In particular, we analyze the pseudoscalar and scalar interactions in detail, illustrating our observations with some examples. We also show that the relevant hidden symmetry of the Dirac equation with such an interaction is pseudo supersymmetry.

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1 Introduction

Non Hermitian Hamiltonians have been studied widely primarily because of their intrinsic interest [1] and also for possible applications [2]. One interesting feature of some non Hermitian interactions, in particular the $\mathcal{PT}$ symmetric ones is that they have real eigenvalues and such potentials have been studied in detail within the framework of non relativistic quantum mechanics. Here our objective is to examine whether or not $\mathcal{PT}$ symmetric interactions are possible in non Hermitian Dirac equation [3]. In particular we shall consider $(1+1)$ dimensional Dirac equation and it will be shown that $\mathcal{PT}$ symmetric interactions can be accommodated within such a framework. It will also be shown that the underlying symmetry of such a system may be described by *pseudo supersymmetry*.

2 $(1+1)$ dimensional Dirac equation

The $(1+1)$-dimensional Dirac equation for a fermion of rest mass m (in units $\hbar = c = 1$) is given by [4, 5]

$$H\psi = E\psi \tag{1}$$

where

$$H = \alpha p + \beta m + \mathcal{V} \tag{2}$$

E is the energy of the fermion, p the momentum operator, and α , β are 2×2 matrices satisfying $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. A convenient choice of α , β may be given by the 2×2 Pauli matrices. The potential matrix $\mathcal{V}$ may be represented as [4]

$$\mathcal{V} = IV_t + \alpha V_e + \beta V_s + \beta\gamma^5 V_p \tag{3}$$

where I is the 2×2 unit matrix, V_t and V_e are the time and space components respectively, of the 2-vector potential, and V_s and V_p are the scalar and pseudoscalar terms.
It can be shown easily that the space component of the vector potential can be gauged away, as its only contribution is to change the spinors by a local phase factor. Furthermore, for non Hermitian V_t, $\mathcal{PT}$ invariance cannot be accommodated in the $(1+1)$ dimensional Dirac equation. Consequently, we choose a vanishing time component $V_t = 0$, and shall consider only the non Hermitian scalar and pseudoscalar interactions in (3), in further detail.

$\mathcal{PT}$ invariance of the $(1+1)$-dimensional Dirac equation

The operators $\mathcal{P}$ and $\mathcal{T}$ are defined by their action on the position and momentum operators x and p by [1] :

$$\left.\begin{array}{lllll} \mathcal{P} & : & x \to -x, & p \to -p & \\ \mathcal{T} & : & x \to x, & p \to -p, & i \to -i \end{array}\right\} \tag{4}$$

For the $(1+1)$ dimensional Dirac Hamiltonian with non Hermitian interactions, to be invariant under the combined action of $\mathcal{PT}$, i.e.,

$$H\ \mathcal{PT} = \mathcal{PT}\ H \tag{5}$$

one needs to put a restriction on the choice of α and β. Since p and m remain invariant under this transformation,

$$p\ \mathcal{PT} = \mathcal{PT}\ p\ , \qquad m\ \mathcal{PT} = \mathcal{PT}\ m \tag{6}$$

so should α and β :

$$\alpha\ \mathcal{PT} = \mathcal{PT}\alpha\ , \qquad \beta\ \mathcal{PT} = \mathcal{PT}\ \beta \tag{7}$$

Hence, to study the $(1+1)$ dimensional Dirac equation in the framework of $\mathcal{PT}$ symmetric quantum mechanics, only those non Hermitian interactions can be considered in (3), which behave under $\mathcal{PT}$ as follows : V_s should remain invariant while V_p should change sign, i.e.,

$$V_p^{\mathcal{PT}} = -V_p \ , \qquad V_s^{\mathcal{PT}} = V_s \tag{8}$$

2.1 Non Hermitian Pseudoscalar Interaction

First of all, we consider the interaction to be pseudoscalar.

$$\mathcal{V}(x) = \beta\gamma^5 V_p(x) \tag{9}$$

A convenient choice of α, β may be [4]

$$\left.\begin{array}{lcl} \alpha & = & \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \\ \beta & = & \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \\ \beta\gamma^5 & = & \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{array}\right\} \tag{10}$$

so that the $(1+1)$ dimensional Dirac equation, with such an interaction term, reduces to

$$\{-i\sigma_1\partial_x + \sigma_3 m + \sigma_2 V_p(x)\}\,\psi(x) = E\psi \tag{11}$$

where $\psi(x)$ is a 2-component spinor :

$$\psi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \tag{12}$$

Writing (11) explicitly,

$$\begin{pmatrix} m & -i\partial_x - iV_p \\ -i\partial_x + iV_p & -m \end{pmatrix} \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \tag{13}$$

where ∂_x stands for $\dfrac{d}{dx}$.

Under the combined action of $\mathcal{PT}$, σ_2 changes sign while σ_1 and σ_3 remain unaltered. Thus the $(1+1)$ dimensional Dirac Hamiltonian remains invariant under $\mathcal{PT}$ if the pseudoscalar potential reverses its sign

$$V_p^*(-x) \;=\; -\,V_p(x) \tag{14}$$

Decomposing the Dirac equation into the upper and lower components of its spinor,

$$\left\{-i\frac{d}{dx} - iV_p\right\}\phi^{(2)} = (E-m)\,\phi^{(1)} \tag{15}$$

$$\left\{-i\frac{d}{dx} + iV_p\right\}\phi^{(1)} = (E+m)\,\phi^{(2)} \tag{16}$$

leads to the pair of equations

$$\mathcal{H}_i\phi^{(i)} = \left\{-\frac{d^2}{dx^2} + U_i(x)\right\}\phi^{(i)} = \varepsilon\phi^{(i)} \ , \qquad i = 1, 2 \tag{17}$$

where

$$\varepsilon = E^2 - m^2 \tag{18}$$

$$U_i = \left(V_p^2 \pm V_p'\right) \quad , \qquad i = 1, 2 \tag{19}$$

In conventional quantum mechanics, $\mathcal{H}_{1,2}$ are a supersymmetric (SUSY) pair of Hamiltonians, with $V_p(x)$ as the superpotential. The eigenfunctions of $\mathcal{H}_1$ ($\mathcal{H}_2$) are related to positive (negative) energies of the Dirac Hamiltonian H. Although $U_1(x)$ and $U_2(x)$ are different, all eigenvalues, with the possible exception of the ground state, are shared by $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\varepsilon \geq 0$ [5]. Defining an intertwining operator L by

$$L = \left\{ -\frac{d}{dx} + V_p(x) \right\} \tag{20}$$

so that its adjoint is

$$L^\dagger = \left\{ \frac{d}{dx} + V_p(x) \right\} \tag{21}$$

the SUSY partner Hamiltonians can be expressed as

$$\mathcal{H}_1 = LL^\dagger \ , \qquad\qquad \mathcal{H}_2 = L^\dagger L \tag{22}$$

The intertwining operators L and $L^\dagger$ generate the supercharges Q and $Q^\dagger$ given by

$$Q = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} \quad , \qquad\qquad Q^\dagger \ = \ \begin{pmatrix} 0 & 0 \\ L^\dagger & 0 \end{pmatrix} \tag{23}$$

which, in turn, obey the following closed algebra :

$$\left\{Q, Q^\dagger\right\} = H^2 \ , \qquad\qquad \left[Q, H^2\right] = \left[Q^\dagger, H^2\right] = 0 \tag{24}$$

We try to establish a similar hidden symmetry relationship between the potentials $U_i(x)$ when the pair of Hamiltonians $\mathcal{H}_1$, $\mathcal{H}_2$ are not Hermitian, rather they are pseudo Hermitian with respect to a linear, invertible, positive definite Hermitian operator η, i.e., [6],

$$\mathcal{H}_i^\dagger = \eta \mathcal{H}_i \eta^{-1} \quad , \qquad\qquad i = 1, 2 \tag{25}$$

(we note here that for $\mathcal{PT}$ symmetric potentials U_i, η may simply be taken as the parity operator $\mathcal{P}$). One can write the partner Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ in terms of two intertwining differential operators L and M,

$$L = \left\{ \frac{d}{dx} + V_p(x) \right\} \tag{26}$$

$$M = \left\{ -\frac{d}{dx} + V_p(x) \right\} \tag{27}$$

as

$$\mathcal{H}_1 = LM \ , \qquad\qquad \mathcal{H}_2 = ML \tag{28}$$

so that

$$\mathcal{H}_2 M = M \mathcal{H}_1 \quad , \qquad\qquad L\mathcal{H}_2 = \mathcal{H}_1 L \tag{29}$$

Evidently, if $\phi^{(1)}$ is an eigenfunction of $\mathcal{H}_1$ with energy eigenvalue ε , i.e.,

$$\mathcal{H}_1 \phi^{(1)} = \varepsilon \phi^{(1)} \tag{30}$$

then

$$\phi^{(2)} = \frac{i}{E + m} M \phi^{(1)} \tag{31}$$

is an eigenfunction of $\mathcal{H}_2$ with the same eigenvalue ε, except for the lowest state :

$$\mathcal{H}_2\phi^{(2)} = \frac{i}{E+m}\mathcal{H}_2 M\phi^{(1)} = \frac{i}{E+m}(ML)M\phi^{(1)} = \frac{i}{E+m}M(\mathcal{H}_1\phi^{(1)}) = \varepsilon\left(\frac{i}{E+m}M\phi^{(1)}\right) = \varepsilon\phi^{(2)} \tag{32}$$

Thus L and M intertwine the Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ in such a way that M maps the eigenfunctions of $\mathcal{H}_1$ to those of $\mathcal{H}_2$, and L does its converse. It is worth noting here that L and M are no longer mutually adjoint operators ($L \neq M^\dagger$). On the contrary, they are mutually pseudo adjoint [6, 7]

$$M \;=\; L^{\#} \;=\; \eta^{-1}L^\dagger\eta \;, \qquad\qquad L \;=\; M^{\#} \;=\; \eta^{-1}M^\dagger\eta \tag{33}$$

Though $\mathcal{H}_1$ and $\mathcal{H}_2$ being given by (22) are still isospectral, with the possible exception of the ground state, it can be shown by straightforward algebra that (24) no longer holds in such a situation. Instead, it is replaced by

$$H^2 = \{Q, Q^{\#}\} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix} = \begin{pmatrix} LL^{\#} & 0 \\ 0 & L^{\#}L \end{pmatrix} \;, \qquad [Q, H^2] = [Q^{\#}, H^2] = 0 \tag{34}$$

where the pseudo supercharges Q and $Q^{\#}$ are generated from the intertwining operators L and $L^{\#}$ as

$$Q = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} \;, \qquad Q^{\#} = \eta^{-1}Q^\dagger\eta \;=\; \begin{pmatrix} 0 & 0 \\ L^{\#} & 0 \end{pmatrix} \tag{35}$$

Thus, in terms of its components, the pair of Hamiltonians in $(1+1)$ dimensional Dirac equation with non Hermitian pseudoscalar interaction, possesses a hidden pseudo supersymmetry.

We shall illustrate these results by a couple of explicit examples which have been $\mathcal{PT}$ symmetrized in two different ways, viz., the $\mathcal{PT}$ invariant Scarf II potential [7, 8] and the $\mathcal{PT}$ symmetric oscillator [9].

Explicit Examples :

(i) $\mathcal{PT}$ invariant Scarf II potential

We consider the following non Hermitian form of $V_p(x)$

$$V_p(x) = (p+q)\,tanh\;x - i\,(p-q)\,sech\;x \tag{36}$$

Evidently,

$$V_p^*(-x) = -V_p(x) \tag{37}$$

The pseudo supersymmetric partners $U_i(x)$ reduce to the form

$$\begin{aligned} U_1(x) &= \left\{V_p^2(x) + V_p'(x)\right\} \\ &= \left[-\left\{2(p^2+q^2) - (p+q)\right\} sech^2 x - i(p-q)\left\{2(p+q)-1\right\} sech\;x\;\;tanh\;x + (p+q)^2\right] \end{aligned} \tag{38}$$

$$\begin{aligned} U_2(x) &= \left\{V_p^2(x) - V_p'(x)\right\} \\ &= \left[-\left\{2(p^2+q^2) + (p+q)\right\} sech^2 x - i(p-q)\left\{2(p+q)+1\right\} sech\;x\;\;tanh\;x + (p+q)^2\right] \end{aligned} \tag{39}$$

and thus are $\mathcal{PT}$ invariant :

$$U_i^*(-x) = U_i(x) \tag{40}$$

In this particular case both the partners belong to the class of Scarf II potential. The energy eigenvalues and eigenfunctions of this exactly solvable model are well known [7, 8]. $\mathcal{H}_1$ and $\mathcal{H}_2$ have respective bound state energies

$$\left.\begin{array}{rcll} \varepsilon_n^{(1)} & = & \left\{2\,(p+q)\,n - n^2\right\}\ , & n = 0, 1, 2, \cdots \\ \varepsilon_n^{(2)} & = & \left\{2\,(p+q)\,n - n^2\right\}\ , & n = 1, 2, \cdots \end{array}\right\} \tag{41}$$

with corresponding eigenfunctions

$$\begin{aligned} \phi_n^{(1)}(x) &= \frac{\Gamma\left(n - 2p + \frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2} - 2p\right)}\, z^{-p}\, (z^*)^{-q}\, P_n^{-2p-\frac{1}{2},\,-2q-\frac{1}{2}}(i\ sinh\ x)\ , \qquad n = 0, 1, 2, \cdots \\ \phi_n^{(2)}(x) &= \frac{i}{E+m}\ M\ \phi_n^{(1)}(x)\ , \qquad n = 1, 2, \cdots \end{aligned} \tag{42}$$

where $z = \dfrac{1 - i\ sinh\ x}{2}$ and $P_n^{\alpha,\beta}$ are the Jacobi polynomials [10]. Thus, using (18), the discrete spectrum for the Dirac Hamiltonian consists of a positive series

$$E_n^+ = +\sqrt{m^2 + \{2\,(p+q)\,n - n^2\}} \qquad n = 0, 1, 2, \cdots \tag{43}$$

and a negative series

$$E_n^- = -\,E_{n+1}^+ = -\sqrt{m^2 + \{2\,(p+q)\,(n+1) - (n+1)^2\}} \qquad n = 0, 1, 2, \cdots \tag{44}$$

(ii) $\mathcal{PT}$ symmetric oscillator

Next we move on to the pseudoscalar potential given by

$$V_p(x) = -z + \frac{\left(-q\alpha + \frac{1}{2}\right)}{z} \tag{45}$$

where

$$z = x - i\epsilon \tag{46}$$

The pseudo supersymmetric partners being given by

$$\begin{aligned} U_1(x) &= \left\{V_p^2(x) + V_p'(x)\right\} \\ &= z^2 + \frac{\alpha^2 - \frac{1}{4}}{z^2} + 2q\alpha - 2 \end{aligned} \tag{47}$$

$$\begin{aligned} U_2(x) &= \left\{V_p^2(x) - V_p'(x)\right\} \\ &= z^2 + \frac{\alpha^2 - 2q\alpha + \frac{3}{4}}{z^2} + 2q\alpha \end{aligned} \tag{48}$$

are once again $\mathcal{PT}$ symmetric

$$U_i^*(-x) = U_i(x) \tag{49}$$

and belong to the widely studied class of $\mathcal{PT}$ symmetric oscillator [9]. The pair of Hamiltonians $\mathcal{H}_{1,2}$ possesses real energies given by

$$\left.\begin{array}{rcll} \varepsilon_n^{(1)} & = & 4n\ , & n = 0, 1, 2, \cdots \\ \varepsilon_n^{(2)} & = & 4n\ , & n = 1, 2, \cdots \end{array}\right\} \tag{50}$$

with corresponding eigenfunctions (double set, owing to the quasi-parity $q = \pm 1$)

$$\left.\begin{array}{rcll}\phi^{(1)}_{nq}(z) & = & N_{nq} e^{-\frac{z^2}{2}} z^{-q\alpha+\frac{1}{2}} L_n^{(-q\alpha)}\left(z^2\right) \ , & n = 0, 1, 2, \cdots \\ \phi^{(2)}_{nq}(z) & = & \dfrac{i}{E+m}\ M\ \phi^{(1)}_{nq}(x) = \dfrac{i}{E+m}\left\{\dfrac{1}{L_n^{(-q\alpha)}\left(z^2\right)}\dfrac{d}{dx}L_n^{(-q\alpha)}\left(z^2\right)\right\}\phi^{(1)}_{nq} \ , & n = 1, 2, \cdots\end{array}\right\} \tag{51}$$

where $L_n^{(\sigma)}$ are the associated Laguerre polynomials [10]. In a similar manner, the positive and negative series of the discrete spectrum possessed by the Dirac Hamiltonian turns out to be

$$\left.\begin{array}{rcl}E_n^+ & = & +\sqrt{m^2+4n} \\ E_n^- & = & -E_{n+1}^+ = -\sqrt{m^2+4(n+1)}\end{array}\right\}, \qquad n = 0, 1, 2, \cdots \tag{52}$$

2.2 Non Hermitian Scalar Interaction

Now we shall explore the possibility of a non Hermitian scalar interaction

$$\mathcal{V}(x) = \beta V_s(x) \tag{53}$$

However, since $\mathcal{PT}$ invariance is neither a necessary nor a sufficient condition for the reality of the spectrum, we can look for a $(1+1)$ dimensional Dirac equation with non $\mathcal{PT}$ symmetric, non Hermitian interaction, with real energies. For this purpose, we can suitably choose α, β as [5]

$$\alpha = \sigma_2 \ , \qquad \beta = \sigma_1 \tag{54}$$

so that the Dirac equation

$$\left\{\alpha p + \beta(m+V_s)\right\}\psi = E\psi \tag{55}$$

can be decoupled and reduced to the following pair of equations

$$\left.\begin{array}{rcl}\left\{-\dfrac{d}{dx}+m+V_s\right\}\phi^{(2)} & = & E\phi^{(1)} \\ \left\{\dfrac{d}{dx}+m+V_s\right\}\phi^{(1)} & = & E\phi^{(2)}\end{array}\right\} \tag{56}$$

giving

$$\mathcal{H}_i\phi^{(i)} = \left\{-\frac{d^2}{dx^2}+U_i(x)\right\}\phi^{(i)} = \varepsilon\phi^{(i)}, \qquad i = 1, 2 \tag{57}$$

with

$$U_i(x) = \left\{(m+V_s)^2 - m^2 \mp \frac{dV_s}{dx}\right\} \tag{58}$$

$$\varepsilon = E^2 - m^2 \tag{59}$$

As is obvious from the above equations, analogous to the non Hermitian pseudoscalar interaction dealt with in section 2.1 above, a non Hermitian scalar interaction can also be studied in the framework of pseudo supersymmetry for a mass dependent interaction

$$V_s = (W_s - m) \tag{60}$$

so that

$$\mathcal{H}_i \phi^{(i)} = \left\{ -\frac{d^2}{dx^2} + \widehat{U}_i(x) \right\} \phi^{(i)} = \widehat{\varepsilon} \phi^{(i)}, \qquad i = 1,2 \tag{61}$$

where

$$\widehat{U}_i(x) = \left(W_s^2 \mp W_s' \right) \quad , \qquad i = 1,2 \tag{62}$$

and

$$\widehat{\varepsilon} = E^2.$$

For example, one can choose a scalar interaction of the form

$$V_s = \lambda tanh\ x + ia - m \tag{63}$$

Evidently, only such energy dependent scalar interactions can be accommodated within the non Hermitian framework, which admit real energies.

3 Conclusion

To conclude, we have studied the $(1+1)$ dimensional solvable Dirac equation with non Hermitian scalar and pseudoscalar interactions, possessing real energies. As explicit examples of the non Hermitian pseudoscalar interaction, we have considered the Schrödinger equivalent of two $\mathcal{PT}$ invariant cases, viz., the $\mathcal{PT}$ symmetric Scarf II potential and the $\mathcal{PT}$ oscillator. Additionally, we observed that the relevant hidden symmetry of the $(1+1)$ dimensional Dirac equation with non Hermitian interaction is *pseudo supersymmetry*, which is in contrast to the supersymmetry of its conventional Hermitian counterpart.

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