On $\mathbf{Q}$ and $\mathbf{R}_0$ properties of a quadratic representation in linear complementarity problems over the second-order cone

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Abstract

This paper studies the linear complementarity problem LCP(M, q) over the second-order (Lorentz or ice-cream) cone denoted by Λ^n_+, where M is a $n \times n$ real square matrix and $q \in R^n$. This problem is denoted as SOLCP(M, q). The study of second-order cone programming problems and hence an independent study of SOLCP is motivated by a number of applications. Though the second-order cone is a special case of the cone of squares (symmetric cone) in a Euclidean Jordan algebra, the geometry of its faces is much simpler and hence an independent study of LCP over Λ^n_+ may yield interesting results. In this paper we characterize the $\mathbf{R}_0$-property ($x \in \Lambda^n_+$, $M(x) \in \Lambda^n_+$ and $\langle x, M(x)\rangle = 0 \Rightarrow x = 0$) of a quadratic representation $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$ of Λ^n for $a, x \in \Lambda^n$ where '$\circ$' is a Jordan product and show that the $\mathbf{R}_0$-property of P_a is equivalent to stating that SOLCP(P_a, q) has a solution for all $q \in \Lambda^n$.

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1. Introduction

In a n-dimensional real vector space R^n with the usual inner product, the *second-order cone* denoted by Λ^n_+ is defined as $\Lambda^n_+ := \{x \in R^n : \|\bar{x}\| \leqslant x_0\}$ where $x = (x_0, \bar{x})^{\mathrm{T}}$ is indexed from zero. Given a $n \times n$ real square matrix M and a vector q in R^n the *second-order linear complementarity problem* (SOLCP(M, q)) is to find a vector x in Λ^n_+ such that $Mx + q$ is in Λ^n_+ and $\langle x, Mx + q\rangle = x^{\mathrm{T}}(Mx + q) = 0$. Note that the second-order cone is a closed convex cone which is self-dual in nature, that is, $\Lambda^n_+ = \{x \in R^n : \langle x, y\rangle \geqslant 0\, \forall y \in \Lambda^n_+\}$. It is well known that the second-order cone is the cone of squares (see Section 2) of its associated Euclidean Jordan algebra, see [4]. In this regard the complementarity problem SOLCP is a special case of the more general linear complementarity problem studied in the setting of a Euclidean Jordan algebra, see [6,9]. However, the important feature which makes the SOLCP interesting and draws a special attention is the nature of the faces of the second-order cone. Unlike the cone of symmetric positive semidefinite matrices, which is also studied in the setting of a Euclidean Jordan algebra, the only nontrivial faces of Λ^n_+ are its extreme rays and its only nonpolyhedral face is the cone Λ^n_+ itself, see [5].

Second-order cone programming and complementarity problems have been subjects of some recent studies. Pang et al. have studied the stability of solutions to semidefinite and second-order cone complementarity problems in [11]. Study of smoothing functions for second-order cone complementarity problems to develop noninterior continuation methods has been made by Fukushima, Luo, and Tseng in [5]. One can see the same paper and the references therein for various applications of second-order cone complementarity problems. For a comprehensive exposition to various applications and algorithmic aspects of second-order cone programming problems, the reader is advised to refer to Alizadeh and Golfarb, [1].

Loewy and Schneider [8] have studied the closed convex cone of matrices which leave Λ^n_+ invariant, denoted by $\Pi(\Lambda^n_+)$, and characterized the extreme rays of $\Pi(\Lambda^n_+)$. Our focus in this article is on the SOLCP(M, q) where the matrix $M \in \Pi(\Lambda^n_+)$. Our study is motivated by a result proved by Murty [10] in the context of a LCP over R^n_+ with a nonnegative square matrix. It states that LCP(M, q) is solvable for all $q \in R^n$ if and only if the diagonal entries of M are positive, where M is a $n \times n$ nonnegative matrix, that is, $M(R^n_+) \subseteq R^n_+$. Though we do not have a complete generalization of Murty's result to a second-order cone, in this article we shall show that for a *quadratic representation* P_a of Λ^n for $a \in \Lambda^n$, defined as $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$, see [4], SOLCP$(P_a, q)$ has a solution for all $q \in \Lambda^n$ if and only if a or $-a$ lies in the interior of Λ^n_+. An important feature that is useful in showing the above equivalence is the property of the faces of the second-order cone. Note that the quadratic representation $P_a \in \Pi(\Lambda^n_+)$ for all $a \in R^n$ and $P_a(\Lambda^n_+) = \Lambda^n_+$ when a is invertible. The quadratic representation plays a fundamental role in the study of Euclidean Jordan algebras. On the space of real symmetric matrices the quadratic representation is seen to be the map $X \to AXA$ where A is a real symmetric matrix. The solvability of semidefinite linear complementarity problem SDLCP(L, Q) with

$L(X) = AXA$, where A is real symmetric, has been characterized in terms of A being positive or negative definite in [12,13].

We shall begin with a brief survey of some Jordan algebraic properties of the second-order cone in Section 2. In Section 3 we present our main results and discuss some open problems.

2. Second-order cone and its Jordan algebra

A *Euclidean Jordan algebra* V is a finite dimensional vector space over the field of real numbers R equipped with an inner product $\langle x, y\rangle$ and a bilinear map $(x, y) \to x \circ y$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ where $x^2 = x \circ x$, and
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$,

for all $x, y, z \in V$. The product $x \circ y$ is called a Jordan product. A Euclidean Jordan algebra V has an identity element, if there exists a (unique) element $e \in V$ such that $x \circ e = e \circ x = x$ for all $x \in V$. The cone of squares in V is defined as $K := \{x \circ x : x \in V\}$, which is a symmetric cone [4], that is, K is a self-dual closed convex cone such that for any two elements $x, y \in \text{int}\, K$, there exists an invertible linear transformation $\Theta : V \to V$ such that $\Theta(K) = K$ and $\Theta(x) = y$.

For $x \in V$ let d be the smallest positive integer such that the set $\{e, x, x^2, \ldots, x^d\}$ is linearly dependent. Then d is called the *degree* of x. The *rank* of V is defined as the largest degree of any $x \in V$. An element $c \in V$ is an *idempotent* if $c^2 = c$. It is *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set $\{f_1, f_2, \ldots, f_n\}$ of primitive idempotents in V is a *Jordan frame* if

$$f_i \circ f_j = 0, \ \ i \neq j \quad \text{and} \quad \sum_{i=1}^{n} f_i = e.$$

Theorem 1 (Spectral theorem). *Let V be a Euclidean Jordan algebra with rank n. Then for every $x \in V$, there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

$$x = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n.$$

The numbers λ_i (with their multiplicities) are uniquely determined by x and are called the eigenvalues of x.

With the above decomposition we shall define *determinant* of $x \in \Lambda^n$ as follows:

$$\det(x) := \lambda_1 \lambda_2 \cdots \lambda_n.$$

x is said to be invertible if $\det(x) \neq 0$, in which case the inverse of x is defined as $x^{-1} := \lambda_1^{-1} f_1 + \cdots + \lambda_n^{-1} f_n$.

For a complete treatment on Euclidean Jordan algebra, one can refer to the book [4] by Faraut and Koranyi. Brief summaries can also be found in [14,15].

In this paper we shall confine our attention to the case when the space is R^n whose elements $x = (x_0, \bar{x})^{\mathrm{T}}$ are indexed from zero, equipped with the usual inner product and the Jordan product defined as

$$x \circ y = (\langle x, y \rangle, x_0\bar{y} + y_0\bar{x})^{\mathrm{T}}.$$

Then R^n is a Euclidean Jordan algebra, denoted by Λ^n, with the cone of squares as second-order cone which is seen to be $\Lambda^n_+ := \{x \in R^n : \|\bar{x}\| \leqslant x_0\}$. The interior of Λ^n_+ is the cone given by $\operatorname{int} \Lambda^n_+ = \{x \in R^n : \|\bar{x}\| < x_0\}$, see [1]. The identity element in this algebra is given by $e = (1, 0, \ldots, 0)^{\mathrm{T}}$. Also the spectral decomposition of any x with $\bar{x} \neq 0$ is given by $x = \lambda_1 f_1 + \lambda_2 f_2$ with

$$\lambda_1 := x_0 + \|\bar{x}\|, \ \lambda_2 := x_0 - \|\bar{x}\|,$$
$$f_1 := \frac{1}{2}(1, \bar{x}/\|\bar{x}\|)^{\mathrm{T}}, \quad \text{and} \quad f_2 := \frac{1}{2}(1, -\bar{x}/\|\bar{x}\|)^{\mathrm{T}},$$

where $\{f_1, f_2\}$ constitutes a Jordan frame. From the above decomposition $\det(x) = x_0^2 - \|\bar{x}\|^2$. The rank of Λ^n is always 2 and it can be shown that all Jordan frames are of the above form. Also $x \in \Lambda^n_+$ ($\operatorname{int} \Lambda^n_+$) if and only if both λ_1 and λ_2 are nonnegative (positive), see [1,5].

A linear transformation $L_x : \Lambda^n \to \Lambda^n$ for $x \in \Lambda^n$ is defined as $L_x(y) := x \circ y$ for all $y \in \Lambda^n$. We say that two elements x, y of a Euclidean Jordan algebra V *operator commute* if $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in V$, which is equivalent to stating that $L_x L_y = L_y L_x$. Specializing Lemma X.2.2 in [4] to the space Λ^n, we have the following characterization of operator commutativity. Also see [1] where an independent proof is provided.

Proposition 1. *Two elements x, y of Λ^n operator commute iff there is a Jordan frame $\{f_1, f_2\}$ such that $x = \lambda_1 f_1 + \lambda_2 f_2$ and $y = \beta_1 f_1 + \beta_2 f_2$ for some real numbers λ_1, λ_2, β_1, and β_2.*

In view of the above proposition it is easy to see that vectors x and y in Λ^n operator commute iff either $\bar{y}$ is a multiple of $\bar{x}$ or $\bar{x}$ is a multiple of $\bar{y}$.

The quadratic representation of Λ^n, denoted by P_a for $a \in \Lambda^n$, is the matrix

$$P_a := 2L_a^2 - L_{a^2} = 2aa^{\mathrm{T}} - \det(a)\mathscr{J}_n,$$

where $\mathscr{J}_n$ is the $n \times n$ matrix defined as

$$\mathscr{J}_n := \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$

Observation 1. *For $a \in \Lambda^n$, $P_a \in \Pi(\Lambda^n_+)$.*

Proof. Case 1: When $\det(a) \neq 0$, $P_a(\Lambda_+^n) = \Lambda_+^n$ and $P_a(\text{int}\,\Lambda_+^n) = \text{int}\,\Lambda_+^n$, see [1].

Case 2: When $\det(a) = 0$, $a_0^2 = \|\bar{a}\|^2$ and $P_a(x) = 2a^{\mathrm{T}}xa$ for $x \in \Lambda_+^n$. If $a_0 = \|\bar{a}\|$, then $a \in \Lambda_+^n$ and $2a^{\mathrm{T}}xa_0 - 2|a^{\mathrm{T}}x|a_0 = 0$, because $a^{\mathrm{T}}x \geqslant 0$. Again if $a_0 = -\|\bar{a}\|$, then $2a^{\mathrm{T}}xa_0 + 2|a^{\mathrm{T}}x|a_0 = 0$, because $a^{\mathrm{T}}x \leqslant 0$. $\square$

Below we shall state some of the important properties of a quadratic representation.

Proposition 2 [1]. *Let α be a real number and $x \in \Lambda^n$. For $a \in \Lambda^n$ with the spectral decomposition $a = \lambda_1 f_1 + \lambda_2 f_2$ with $\lambda_1 \neq \lambda_2$, we have the following properties*:

(i) $\lambda_1 = a_0 + \|\bar{a}\|$ *and* $\lambda_2 = a_0 - \|\bar{a}\|$ *are the eigenvalues of L_a, each with multiplicity one and corresponding eigenvectors f_1 and f_2, respectively. Also a_0 is an eigenvalue of L_a with multiplicity $n-2$.*
(ii) $\lambda_1^2 = (a_0 + \|\bar{a}\|)^2$ *and* $\lambda_2^2 = (a_0 - \|\bar{a}\|)^2$ *are eigenvalues of P_a, each with multiplicity one and corresponding eigenvectors f_1 and f_2, respectively. Also* $\det(a) = a_0^2 - \|\bar{a}\|^2$ *is an eigenvalue of P_a with multiplicity $n-2$.*
(iii) *P_a is an invertible matrix iff a is invertible.*
(iv) $P_{\alpha a} = \alpha^2 P_a$.
(v) $P_{P_a(x)} = P_a P_x P_a$.
(vi) $\det P_a(x) = \det^2(a)\det(x)$.
(vii) *If a is invertible, then* $P_a(a^{-1}) = a$ *and* $P_{a^{-1}} = P_a^{-1}$.

Proposition 3 [5,6]. *For $x, y \in \Lambda^n$, the following conditions are equivalent*:

(i) $x \in \Lambda_+^n$, $y \in \Lambda_+^n$, *and* $\langle x, y\rangle = 0$.
(ii) $x \in \Lambda_+^n$, $y \in \Lambda_+^n$, *and* $x \circ y = 0$.

In each case, elements x and y operator commute.

Definition 1 [2].
(a) A subset F of a closed convex cone C in Λ^n is called a *face* of C, denoted by $F \lhd C$, if F is a convex cone and

$$x \in C,\quad y - x \in C \text{ and } y \in F \quad\Rightarrow\quad x \in F.$$

(b) The *smallest face* of a closed convex cone C containing a point $x \in C$ is defined as

$$\Phi(x) := \cap\{F : F \lhd C, x \in F\}.$$

(c) For a self-dual closed convex cone C the *complementary face* of the face F of C is defined as

$$F^{\triangle} := \{x \in C : \langle x, y\rangle = 0\ \forall y \in F\}.$$

Theorem 2 [2]. *Let $F \lhd C$ and $x \in C$. Then F is the smallest face of C containing x if and only if x lies in the relative interior of F.*

In view of the above discussions the second-order cone Λ^n_+ can also be represented by $\Lambda^n_+ := \{x \in \Lambda^n : x^{\mathrm{T}} \mathscr{J}_n x \geqslant 0,\ x_0 \geqslant 0\}$ and any $x \in \Lambda^n$ has $\det(x) = x^{\mathrm{T}} \mathscr{J}_n x$. The elements on the boundary of Λ^n_+ are exactly those for which $x_0 = \|\bar{x}\|$. Below we shall state some relations on the boundary structure of the cone Λ^n_+, which will be useful in this paper.

$$\begin{aligned}
&x \in \Lambda^n_+ \cup (-\Lambda^n_+) \;\Rightarrow\; \mathscr{J}_n x \in \Lambda^n_+ \cup (-\Lambda^n_+),\\
&x^{\mathrm{T}} \mathscr{J}_n x > 0 \;\Rightarrow\; x \in \operatorname{int} \Lambda^n_+ \cup \operatorname{int}(-\Lambda^n_+),\\
&x^{\mathrm{T}} \mathscr{J}_n x \geqslant 0 \;\Rightarrow\; x \in \Lambda^n_+ \cup (-\Lambda^n_+),\\
&x^{\mathrm{T}} \mathscr{J}_n x = 0 \;\Rightarrow\; x \in \operatorname{bd} \Lambda^n_+ \cup \operatorname{bd}(-\Lambda^n_+),\\
&x^{\mathrm{T}} \mathscr{J}_n x < 0 \;\Rightarrow\; x \notin \Lambda^n_+ \cup (-\Lambda^n_+).
\end{aligned}$$

Definition 2 [4]. A linear transformation $\Psi : \Lambda^n \to \Lambda^n$ is said to be an *automorphism* of Λ^n if Ψ is invertible and $\Psi(x \circ y) = \Psi(x) \circ \Psi(y)$ for all $x, y \in \Lambda^n$. The set of all automorphisms of Λ^n is denoted by $\mathrm{Aut}(\Lambda^n)$.

One of the important properties of automorphisms of Λ^n, which we shall make use of in this paper, is that for any two Jordan frames $\{e_1, e_2\}$ and $\{f_1, f_2\}$ in Λ^n there exists an automorphism Ψ such that $\Psi f_1 = e_1$ and $\Psi f_2 = e_2$, see [4]. Also any automorphism Ψ of Λ^n can be written as

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

where U is an $(n-1) \times (n-1)$ orthogonal matrix, see [8].

Definition 3

(a) A matrix $M : \Lambda^n \to \Lambda^n$ has the $\mathbf{R}_0$-property if

$$x \in \Lambda^n_+,\ \ M(x) \in \Lambda^n_+,\ \ \langle x, M(x) \rangle = 0 \quad\Rightarrow\quad x = 0.$$

(b) M is said to have the $\mathbf{Q}$-property if SOLCP(M, q) has a solution for all q in Λ^n.

There is a nice geometric interpretation of $\mathbf{Q}$-property in terms of the complementary cones associated with a matrix M which are defined as follows. Given a matrix $M : \Lambda^n \to \Lambda^n$ a *complementary cone* of M corresponding to the face F of Λ^n_+ is defined as

$$K_F := \left\{ y - L(x) : x \in F,\ y \in F^{\triangle} \right\}.$$

The SOLCP(M, q) has a solution if and only if $q \in K_F$ for some face F, see [9]. Thus M has the $\mathbf{Q}$-property if and only if the union of all complementary cones is the whole space Λ^n. The above interpretation is motivated by the study of complementary cones in LCP over R^n_+, see [3]. But due to the nonpolyhedral nature of the second-order cone, complementary cones in SOLCP need not be closed.

Example 1. Let $M : \Lambda^3 \to \Lambda^3$ be a matrix defined as

$$M(x) = \begin{pmatrix} x_0 + x_1 \\ 0 \\ x_2 \end{pmatrix}.$$

Note that

$$M \begin{pmatrix} \frac{1}{2}(\epsilon + 1/\epsilon) \\ \frac{1}{2}(\epsilon - 1/\epsilon) \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{as } \epsilon \to 0.$$

However, there exists no $x \in \Lambda^3_+$ such that

$$M(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the complementary cone of M corresponding to the face Λ^3_+ is not closed.

However, it can be shown (in a more general setting) that the $\mathbf{R}_0$-property of a matrix M provides a sufficient condition for the closedness of complementary cones, see [9].

Fix a canonical Jordan frame $\{e_1, e_2\}$ where

$$e_1 := (1/2, 1/2, 0)^{\mathrm{T}} \quad \text{and} \quad e_2 := (1/2, -1/2, 0)^{\mathrm{T}}.$$

In case of a second-order cone with a fixed Jordan frame $\{e_1, e_2\}$ consider the following subspaces of Λ^n. (Similar subspaces have been considered in Theorem IV 2.1 in [4].)

$$V_{11} = \{x \in \Lambda^n : x \circ e_1 = x\} = \{\lambda e_1 : \lambda \in R\},$$
$$V_{22} = \{x \in \Lambda^n : x \circ e_2 = x\} = \{\beta e_2 : \beta \in R\}, \quad \text{and}$$
$$V_{12} = \left\{x \in \Lambda^n : x \circ e_1 = \frac{1}{2}x = x \circ e_2\right\} = \{x \in \Lambda^n : x_0 = x_1 = 0\}.$$

Thus given an $x \in \Lambda^n$ we can write

$$x = (x_0 + x_1)e_1 + (x_0 - x_1)e_2 + (0, 0, x_2, \ldots, x_{n-1})^{\mathrm{T}}.$$

We shall designate $(x_0 + x_1)$ and $(x_0 - x_1)$ as the diagonal entries of a vector x with respect to the Jordan frame $\{e_1, e_2\}$.

Observation 2. *For $a \in \Lambda^n_+$ we have the following relationship among its entries. $a \in \Lambda^n_+(\operatorname{int} \Lambda^n_+)$ if and only if $(a_0 + a_1) \geqslant 0$ (> 0), $(a_0 - a_1) \geqslant 0$ (> 0), and $(a_0 + a_1)(a_0 - a_1) - \|\tilde{a}\|^2 \geqslant 0$ (> 0) where $\tilde{a} = (a_2, \ldots, a_{n-1})^{\mathrm{T}}$.*

Similar to the notion of a diagonal matrix we introduce the notion of a diagonal vector $d \in \Lambda^n$ with respect to a canonical Jordan frame $\{e_1, e_2\}$ as

$$d = \lambda_1 e_1 + \lambda_2 e_2, \quad \lambda_1, \lambda_2 \in R.$$

3. Equivalence of Q and R_0-property of a quadratic representation

Proposition 4. *Let $M \in \Pi(\Lambda^n_+)$. Then M has the $\mathbf{R}_0$-property if and only if $\langle x, M(x)\rangle > 0$ for all $0 \neq x \in \Lambda^n_+$.*

The proof of the above proposition follows easily from the definitions.

Remark 1. For a linear complementarity problem LCP(M, q) over R^n_+ with $M(R^n_+) \subseteq R^n_+$, we have LCP(M, q) is solvable for all $q \in R^n$ iff LCP$(M, 0)$ has a unique solution zero, which is also equivalent to stating that M is strictly copositve, see [3].

Theorem 3. *Let $M \in \Pi(\Lambda^2_+)$. Then M has the $\mathbf{Q}$-property if and only if M has the $\mathbf{R}_0$-property.*

Proof. Observing the fact that $\{e_1, e_2\}$ is the only Jordan frame of Λ^n_+ unique up to permutation, $\mathbf{R}_0$-property of the above matrix M is equivalent to the property that $\langle e_1, M(e_1)\rangle > 0$ and $\langle e_2, M(e_2)\rangle > 0$. Now suppose without loss of generality that $\langle e_1, M(e_1)\rangle = 0$. We shall show that M does not have the $\mathbf{Q}$-property. Take $q = e_2 - e_1 = (0, -1, 0)^{\mathrm{T}}$. Let $x \in \Lambda^n_+$ be a solution to SOLCP(M, q). Then we have

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad M(x) + q = \beta_1 e_1 + \beta_2 e_2$$

such that $\lambda_1\beta_1 = 0$ and $\lambda_2\beta_2 = 0$. Since $M(x) \in \Lambda^n_+$ and $x \neq 0$, $\lambda_2 = \beta_1 = 0$. Thus $\langle e_1, M(x) + q\rangle = \langle e_1, \lambda_1 M(e_1) + e_2 - e_1\rangle = -\langle e_1, e_1\rangle < 0$, which contradicts the fact that $M(x) + q \in \Lambda^n_+$. The 'if part' is apparent from Karamardian's theorem [7] on the solvability of the complementarity problem. □

Loewy and Schneider [8] proved the following result which characterizes the extreme matrices of the closed convex cone of square matrices which leave the cone Λ^n_+ invariant.

Theorem 4 [8]. *Let M be an $n \times n$ real matrix, with $n \geqslant 3$. Then M is n extreme matrix of $\Pi(\Lambda^n_+)$ (generates a 1-dimensional face of $\Pi(\Lambda^n_+)$) if and only if either $M(\Lambda^n_+) = \Lambda^n_+$ or $M = uv^{\mathrm{T}}$ for $u, v \in \mathrm{bd}\,\Lambda^n_+$.*

Remark 2. Any $M : \Lambda^n \to \Lambda^n$ satisfying $M(\Lambda^n_+) = \Lambda^n_+$ can be written as $M = P_a\Psi$, where $a \in \mathrm{int}\,\Lambda^n_+$ and $\Psi \in \mathrm{Aut}(\Lambda^n)$, see page 56 of [4].

Proposition 5. *Let $M \in \Pi(\Lambda^n_+)$. Then M has the* **Q***-property if and only if $\Theta M \Theta^{\mathrm{T}}$ has the* **Q***-property for all $\Theta : \Lambda^n \to \Lambda^n$ such that $\Theta(\Lambda^n_+) = \Lambda^n_+$.*

Proof. Take an arbitrary $q \in \Lambda^n$ and define $\tilde{q} = \Theta^{-1}q$. Then SOLCP$(M, \tilde{q})$ has a solution $\tilde{x}$ such that

$$\tilde{x} \in \Lambda^n_+,\ M(\tilde{x}) + \tilde{q} \in \Lambda^n_+ \quad \text{and} \quad \langle \tilde{x}, M(\tilde{x}) + \tilde{q}\rangle = 0.$$

Define $x = (\Theta^{-1})^{\mathrm{T}}\tilde{x}$. Since both Θ^{-1}, $\Theta^{\mathrm{T}} \in \Pi(\Lambda^n_+)$, we have

$$x \in \Lambda^n_+, \quad M(\Theta^{\mathrm{T}}x) + \Theta^{-1}q \in \Lambda^n_+ \quad \text{and} \quad \langle \Theta^{\mathrm{T}}x, M\Theta^{\mathrm{T}}x + \Theta^{-1}q\rangle = 0,$$

equivalently,

$$x \in \Lambda^n_+, \quad \Theta M \Theta^{\mathrm{T}}x + q \in \Lambda^n_+ \quad \text{and} \quad \langle x, \Theta M \Theta^{\mathrm{T}}x + q\rangle = 0,$$

which implies that $\Theta M \Theta^{\mathrm{T}}$ has the **Q**-property. □

Theorem 5. *Let $M = uv^{\mathrm{T}}$ for $u, v \in \operatorname{bd}\Lambda^n_+$. Then M does not have the* **Q***-property.*

Proof. First we shall show that if $M = e_i e_j^{\mathrm{T}}$, where $i, j \in \{1, 2\}$, then M does not have the **Q** property. When $M = e_1 e_2^{\mathrm{T}}$ or $M = e_1 e_1^{\mathrm{T}}$ we can take $q = (0, 1, 0)^{\mathrm{T}}$ and can show easily that SOLCP(M, q) does not have a solution. Again when $M = e_2 e_1^{\mathrm{T}}$ or $M = e_2 e_2^{\mathrm{T}}$ we can take $q = (0, -1, 0)^{\mathrm{T}}$ and can easily prove that SOLCP(M, q) is not solvable. In fact, in both the above cases SOLCP(M, q) is not even feasible. Now we consider the two cases.

Case 1: Suppose that u and v are linearly dependent. Then there exists an automorphism Ψ of Λ^n such that $\Psi u = e_1$. By Proposition 5, uv^{T} has the **Q**-property implies that $e_1 e_1^{\mathrm{T}}$ has the **Q**-property. Since we know that $e_1 e_1^{\mathrm{T}}$ is not **Q**, we have proved our claim.

Case 2: Suppose that u and v are linearly independent. Then by Lemma 3.7 in [8], there exists a $\Theta \in \Pi(\Lambda^n_+)$ with $\Theta(\Lambda^n_+) = \Lambda^n_+$ such that $\Theta u = e_1$ and $\Theta v = e_2$. Again by Proposition 5, uv^{T} has the **Q**-property implies that $e_1 e_2^{\mathrm{T}}$ has the **Q**-property. Since we know that $e_1 e_2^{\mathrm{T}}$ is not **Q**, uv^{T} does not have the **Q**-property. □

Theorem 6. *For a quadratic representation P_a of Λ^n, $n \geqslant 3$, we have the following equivalence:*

(i) $a \in \operatorname{int}\Lambda^n_+$ *or* $-a \in \operatorname{int}\Lambda^n_+$.
(ii) P_a *is positive definite.*
(iii) SOLCP(P_a, q) *has a unique solution for all* $q \in \Lambda^n$.
(iv) P_a *has the* $\mathbf{R}_0$*-property.*

Proof. The proof of (i) ⇒ (ii) follows from Theorem 3 in [1], since all the eigenvalues of P_a are positive. From (ii), P_a is positive definite implies that SOLCP(P_a, q) has atmost one solution for all $q \in \Lambda^n$. Also from Karamardian's theorem [7], P_a

has the **Q**-property. Thus (iii) follows. The proof of (iii) $\Rightarrow$ (iv) is obvious. To prove (iv) $\Rightarrow$ (i), let us suppose that neither $a \in \operatorname{int} \Lambda^n_+$ nor $-a \in \operatorname{int} \Lambda^n_+$. To complete the proof, it is sufficient to show that there exists some $z \in \operatorname{bd} \Lambda^n_+$ such that $\langle a, z\rangle = 0$. We consider the following two cases.

Case 1. When $a \in \operatorname{bd} \Lambda^n_+$ or $-a \in \operatorname{bd} \Lambda^n_+$ we can find a nonzero $z \in \operatorname{bd} \Lambda^n_+$ or $-z \in \operatorname{bd} \Lambda^n_+$ on a face complementary to which a or $-a$ lies.

Case 2. When $a \notin \operatorname{bd} \Lambda^n_+$ and $-a \notin \operatorname{bd} \Lambda^n_+$, there exists an $x \in \operatorname{bd} \Lambda^n_+$ and $y \in \operatorname{bd} \Lambda^n_+$ such that $\langle a, x\rangle < 0$ and $\langle a, y\rangle > 0$. Since the inner product $\langle a, \cdot\rangle$ is continuous on the line segment joining x and y denoted as $[x, y]$, there exists a $\tilde{z} = \alpha x + (1-\alpha)y$ for $0 < \alpha < 1$ such that $\langle a, \tilde{z}\rangle = 0$. Since Λ^n_+ has faces of dimension 0, 1 and n only, either $\tilde{z}$ lies on a one dimensional face of Λ^n_+ in which case we are done otherwise $\tilde{z} \in \operatorname{int} \Lambda^n_+$. It means that $a^\perp \cap \operatorname{int} \Lambda^n_+$ is nonempty where $a^\perp$ is the orthogonal complement of the span of a in Λ^n. Note that $a^\perp$ is a $(n-1)$ dimensional subspace of Λ^n intersecting a full dimensional cone Λ^n_+ in its interior where $n \geqslant 3$. Hence from [5, Lemma 3.6] there exists an invertible linear transformation Γ on Λ^n such that $\Gamma(\Lambda^n_+) = \Lambda^n_+$ and $\Gamma(a^\perp) = \{x \in \Lambda^n : x_{n-1} = 0\}$. Thus there exists a nonzero $z \in a^\perp$ such that $\Gamma z = e_1$, which by [5, Lemma 3.3] lies on the boundary of Λ^n_+.

Now for the above chosen z we have $\langle z, P_a(z)\rangle = \langle z, (2aa^{\mathrm{T}} - \det(a)\mathscr{J}_n)z\rangle = 0$, which contradicts that P_a has the $\mathbf{R}_0$-property. $\square$

Lemma 1. *If the quadratic representation P_a has the* **Q**-*property then a is invertible.*

Proof. Taking $-q \in \operatorname{int} \Lambda^n_+$ let $x \in \Lambda^n_+$ be a solution to SOLCP(P_a, q). Since $P_a(x) + q \in \Lambda^n_+$ and $-q \in \operatorname{int} \Lambda^n_+$, $P_a(x) \in \operatorname{int} \Lambda^n_+$. From Proposition 2, $\det P_a(x) = \det^2(a)\det(x)$ and since $x \in \Lambda^n_+$, we have a is invertible. $\square$

The following lemma is analogous to Lemma 4.3.2 in [13].

Lemma 2. *Let $d = d_1e_1 + d_2e_2$ be a diagonal vector with nonzero entries. Let $|d| = |d_1|e_1 + |d_2|e_2$. Write $d = |d| \circ s$ where $s = \pm e_1 \pm e_2$. The coefficients of e_1 and e_2 in the expression for s are determined by the signs of d_1 and d_2, respectively. If P_d has the* **Q**-*property then P_s has the* **Q**-*property.*

Proof. Fix an arbitrary $q \in \Lambda^n$ and define $\tilde{q} := P_{\sqrt{|d|}}(q)$ where $\sqrt{|d|} = \sqrt{|d_1|}e_1 + \sqrt{|d_2|}e_2$. Let $x \in \Lambda^n_+$ be a solution to SOLCP$(P_d, \tilde{q})$. Then $z := P_d(x) + \tilde{q} \in \Lambda^n_+$ is such that $x \circ z = 0$. We have $z = P_{|d|\circ s}(x) + \tilde{q} = P_{|d|}P_s(x) + P_{\sqrt{|d|}}(q) = P_{\sqrt{|d|}}P_{\sqrt{|d|}}P_s(x) + P_{\sqrt{|d|}}(q)$, where the last equality follows from Proposition 2(v). Thus by Proposition 2(vii), $P_{\sqrt{|d|}^{-1}}(z) = P_{\sqrt{|d|}}P_s(x) + q = P_sP_{\sqrt{|d|}}(x) + q$. Define $y = P_{\sqrt{|d|}}(x)$ and $w = P_{\sqrt{|d|}^{-1}}(z)$. Then $y \in \Lambda^n_+$, $w = P_s(y) + q \in \Lambda^n_+$ and $\langle y, w\rangle = \left\langle x, P_{\sqrt{|d|}}P_{\sqrt{|d|}^{-1}}(z)\right\rangle = 0$. Hence $y \in \Lambda^n_+$ solves SOLCP(P_s, q). $\square$

Lemma 3. *Let s be a diagonal vector of the form* $s = \pm e_1 \pm e_2$ *and* $s \neq \pm e$. *Then* P_s *does not have the* **Q**-*property.*

Proof. Without loss of generality take $s = e_1 - e_2 = (0, 1, 0)^{\mathrm{T}}$. Then the matrix P_s will be

$$P_s = 2ss^{\mathrm{T}} - \det(s)J_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I \end{pmatrix}.$$

Consider a vector $q = (0, 0, -1, 0, \ldots, 0)$. If $y \in \Lambda^n$ is the solution of the SOLCP (P_s, q) then

$$y \in \Lambda^n_+, \ P_s(y) + q \in \Lambda^n_+ \quad \text{and} \quad \langle y, P_s(y) + q \rangle = 0.$$

The complementary condition $\langle y, P_s(y) + q \rangle = 0$ gives us $y_0^2 + y_1^2 - y_2 = \sum_{i=2}^{n-1} y_i^2$. Also from $y \in \Lambda^n_+$ and $P_s(y) + q \in \Lambda^n_+$ we have

$$y_0^2 \geqslant y_1^2 + \cdots + y_{n-1}^2 \quad \text{and} \quad y_0^2 \geqslant y_1^2 + (1 + y_2)^2 + y_3^2 + \cdots + y_{n-1}^2.$$

Substituting the value of $\sum_{i=2}^{n-1} y_i^2$ in the above two inequalities we get $2y_1^2 \leqslant y_2$ and $2y_1^2 + 1 + y_2 \leqslant 0$, which are inconsistent. □

Theorem 7. *Let* P_a *be a quadratic representation of* Λ^n, $n \geqslant 3$. *Then* P_a *has the* **Q**-*property if and only if* P_a *has the* $\mathbf{R_0}$-*property.*

Proof. Since P_a has the **Q**-property we have by Lemma 1, a is invertible. By the spectral decomposition and Theorem IV.2.5 in [4] we can write $a = d_1 \Psi e_1 + d_2 \Psi e_2$, where $d_1, d_2 \neq 0$ and $\Psi \in \mathrm{Aut}(\Lambda^n)$. Since $\Psi P_a \Psi^{\mathrm{T}} = P_{\Psi a}$, by Proposition 5, P_a has the **Q**-property implies that $P_{\Psi^{\mathrm{T}} a}$ has the **Q**-property, which is equivalent to the statement that P_d has **Q**-property where $d := d_1 e_1 + d_2 e_2$. By Lemma 2, P_s has the **Q**-property where s is the vector corresponding to d as defined above. Since P_s has the **Q**-property if and only if $s = \pm e$, either $d_1 > 0$ and $d_2 > 0$ or $d_1 < 0$ and $d_2 < 0$. It means that either $a \in \mathrm{int}\, \Lambda^n_+$ or $-a \in \mathrm{int}\, \Lambda^n_+$, which by Theorem 6 proves that P_a has the $\mathbf{R}_0$-property. Conversely, $\mathbf{R}_0$ implies **Q** is evident from Karamardian's theorem [7]. □

The above result can also be extended to the linear complementarity problem over the direct product of second-order cones, which is defined as follows. Given a matrix M on R^n and $q \in R^n$ the linear complementarity problem over the cone K^n is the problem of finding an $x \in K^n$ such that $M(x) + q$ is in K^n and $\langle x, M(x) + q \rangle = 0$, where the cone K^n is defined as

$$K^n := \Lambda^{n_1}_+ \times \cdots \times \Lambda^{n_m}_+,$$

with $n = n_1 + \cdots + n_m$.

It should be noted that R^n is a Euclidean Jordan algebra with the usual inner product and the Jordan product defined as the componentwise Jordan product of the elements of Λ^{n_i}, $i = 1 \dots m$. Formally, for $x := (x_1, \dots, x_m)^{\mathrm{T}}$ and $y := (y_1, \dots, y_m)^{\mathrm{T}}$, where each component vector x_i is written in a row vector form, the Jordan product is defined as

$$x \circ y := (x_1 \circ y_1, \dots, x_m \circ y_m)^{\mathrm{T}}.$$

The cone of squares with respect to the above Jordan product is the cone K^n. Also the quadratic representation P_a for $a \in R^n$ is a block diagonal matrix of the form

$$P_a = \begin{pmatrix} P_{a_1} & 0 & \dots & 0 \\ 0 & P_{a_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & P_{a_m} \end{pmatrix},$$

where $a = (a_1, \dots, a_m)^{\mathrm{T}}$. For further information one can see the Refs. [1,4].

Theorem 8. *In the space R^n with the cone of squares K^n and the Jordan product defined above the following statements are equivalent*:

(i) *For $a \in R^n$, P_a has the $\mathbf{R}_0$-property, that is, $x \in K^n$, $P_a(x) \in K^n$ and $\langle x, P_a(x)\rangle = 0$ implies $x = 0$.*
(ii) *The linear complementarity problem* $\mathrm{LCP}(P_a, q)$ *over the cone K^n has the solution for all $q \in R^n$.*

Proof. The proof of the above theorem is apparent once we notice that P_a has the $\mathbf{R}_0$-property if and only if P_{a_i} has the $\mathbf{R}_0$-property for each component vector $a_i \in \Lambda^i$ for $i = 1, \dots, m$. $\square$

An open problem

In this paper we have shown that when the matrix $M \in R^{n \times n}$ is a quadratic representation, $\mathrm{SOLCP}(M, q)$ has a solution for all $q \in R^n$ if and only if $\mathrm{SOLCP}(M, 0)$ has a unique solution zero. However, it is not known whether the above equivalence holds for all matrices $M \in \Pi(\Lambda^n_+)$ for $n \geqslant 3$. This question remains open.

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