On almost type classes of matrices with Q-property§

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In this article, we introduce a new matrix class almost $\bar{N}$ (a subclass of almost N_0-matrices which are obtained as a limit of a sequence of almost N-matrices) and obtain a sufficient condition for this class to hold Q-property. We produce a counter example to show that an almost $\bar{N} \cap Q$-matrix need not be a R_0-matrix. We also introduce another two new limiting matrix classes, namely $\bar{N}$ of exact order 2, $\bar{E}(d)$ for a positive vector d and prove sufficient conditions for these classes to satisfy Q-property. Murthy *et al*. [Murthy, G.S.R., Parthasarathy, T. and Ravindran, G., 1993, A copositive Q-matrix which is not R_0. *Mathematical Programming*, **61**, 131–135.] showed that Pang's conjecture ($E_0 \cap Q \subset R_0$) is not true even when E_0 is replaced by C_0. We show that Pang's conjecture is true if E_0 is replaced by almost C_0. Finally, we present a game theoretic proof of necessary and sufficient conditions of an almost P_0-matrix satisfying Q-property.

Keywords: Almost $\bar{N}$-matrix; $\bar{N}$-matrix of exact order 2; $\bar{E}(d)$-matrix; Almost C_0-matrix; Q-property; Pang's conjecture

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1. Introduction

The notion of an *almost type* class was introduced by Väliaho for the first time where he defined and thoroughly investigated the class of almost copositive matrices [1,2] and showed that such matrices are of crucial importance in deriving criteria for copositivity. Olech *et al*. [3] introduced the class of almost N-matrices, namely the class of matrices whose determinant is positive and all proper principal minors are negative. Pye [4] studied the class of *almost* P_0-matrices of order n whose determinant is negative and all proper principal minors are nonnegative.

Suppose a class of square matrices of order n is defined by specifying a property $\mathcal{Y}$ which is satisfied by all proper principal submatrices but the property does not hold for the matrix A. We then say that $A \in R^{n\times n}$ is an *almost $\mathcal{Y}$-matrix* and the class of such matrices is the *almost $\mathcal{Y}$-class*. For example, for an almost N_0 matrix, all proper principal submatrices are N_0, but $A \notin N_0$. The almost type classes are referred as exact order matrices of order 1 in Mohan *et al.* [5].

We will now describe *the linear complementarity problem*, which is stated as follows: Given a real square matrix A of order n and a vector $q \in R^n$, the linear complementarity problem is to find $w \in R^n$ and $z \in R^n$ such that

$$w - Az = q, \qquad w \geq 0, \; z \geq 0 \tag{1.1}$$

$$w^t z = 0 \tag{1.2}$$

This problem is denoted as LCP(q, A). It is well studied in the literature on Mathematical Programming and arises in a number of applications in Operations Research, Mathematical Economics and Engineering. For recent books on this problem, see Cottle *et al.* [6] and Murty [7]. In what follows we first define some well known matrix classes.

A matrix $A \in R^{n\times n}$ is called a *Q-matrix* or is said to satisfy *Q-property* if for every $q \in R^n$, LCP(q, A) has a solution. We say that a matrix A is a *Q_0-matrix* (or a matrix satisfying *Q_0-property*), if for any $q \in R^n$ (1.1) has a solution that implies that LCP(q, A) has a solution. A is said to be an *R-matrix* (a subclass of Q-matrix introduced by Karamardian), if for all $t \geq 0$, LCP(te, A) has only the trivial solution. A is said to be an *$E(d)$-matrix* if LCP(d, A) has only the trivial solution for $d > 0$. A is said to be an *R_0-matrix* if LCP$(0, A)$ has only the trivial solution. Any solution (w, z) is said to be *nondegenerate* if $w + z > 0$. The vector $q \in R^n$ is said to be *nondegenerate with respect to A*, if every solution of LCP(q, A) is nondegenerate. Given a matrix A and a vector q, we define the feasible set $F(q, A) = \{(w, z) \mid w = Az + q, w \geq 0, z \geq 0\}$ and the solution set $S(q, A) = \{(w, z) \mid w = Az + q, w^t z = 0, w \geq 0, z \geq 0\}$.

We say that A is *positive semidefinite* (PSD) if $x^t Ax \geq 0 \; \forall \, x \in R^n$ and A is *positive definite* (PD), if $x^t Ax > 0 \; \forall \, 0 \neq x \in R^n$. A is said to be a $P\,(P_0)$-matrix, if all its principal minors are positive (nonnegative). $A \in R^{n\times n}$ is said to be an $N(N_0)$-matrix, if all its principal minors are negative (nonpositive). A is called *copositive* (C_0) (*strictly copositive* (C)), if $x^t Ax \geq 0 \; \forall \, x \geq 0$ $(x^t Ax > 0 \; \forall \, 0 \neq x \geq 0)$. A is called *copositive* (*strictly copositive*, *copositive-plus*, *PSD*, *PD*) *of order k*, $0 \leq k \leq n$, if every principal submatrix of order k belongs to the class. $A \in R^{n\times n}$ is said to be an *E_0-matrix* (*or Semimonotone*), if for every $0 \neq y \geq 0$, $\exists$ an i such that $y_i > 0$ and $(Ay)_i \geq 0$. If A belongs to any one of the classes E_0, C_0, E, C, C_0^+ then so is (i) any principal submatrix of A, (ii) any matrix $\bar{A}$, which is obtained by a principal permutation of the rows and columns of A.

The notion of $N_0\,(N)$, $P_0\,(P)$, $C_0\,(C)$ etc. are further generalized to almost $N_0\,(N)$, $P_0\,(P)$, $C_0\,(C)$ in [1–5]. We say that A is an *almost P_0 (P)-matrix* if $\det A_{\alpha\alpha} \geq 0 \; (>0)$ $\forall \, \alpha \subset \{1, 2, \ldots, n\}$ and $\det A < 0$. Similarly, A is called *an almost $N_0\,(N)$-matrix* if $\det A_{\alpha\alpha} \leq 0 \; (<0) \; \forall \, \alpha \subset \{1, 2, \ldots, n\}$ and $\det A > 0$. A matrix $A \in R^{n\times n}$ is said to be an *$N_0\,(N)$-matrix of exact order k* $(1 \leq k \leq n)$, if every principal submatrix of order $(n - k)$ is an $N_0\,(N)$-matrix and every principal minor of order r, $(n - k) < r \leq n$ is positive. N-matrices of exact order 1 and 2 are studied in detail by Mohan *et al.* [5]. $A \in R^{n\times n}$ is said to be an *almost copositive*, if it is copositive of order $n-1$ but not of order n. A copositive matrix $A \in R^{n\times n}$ is said to be an *almost copositive-plus*, if it is copositive-plus of order $(n-1)$ but not of order n. $A \in R^{n\times n}$ is said to be *copositive* of exact order 2, if it is copositive of order $n-2$ but not of order n and $(n-1)$. Similarly, a copositive matrix $A \in R^{n\times n}$ is said to be *strictly copositive* (*copositive-plus*) of exact order 2, if it is strictly copositive (copositive-plus) of order $(n-2)$ but not of order n and $(n-1)$.

In linear complementarity theory, much of the research is devoted to find out constructive characterizations of matrices satisfying Q-property. The class of matrices due to Saigal [8] for which LCP$(0, A)$ has a unique solution and LCP(q, A) has odd number of solutions for some nondegenerate q with respect to A is a large class satisfying Q-property. The almost P_0 and almost C_0 classes satisfying Q-property are in R_0.

In section 2, some notations, definitions and some well known results in linear complementarity and matrix games are presented, which will be used in the sequel. In section 3, we introduce almost $\bar{N}$-matrix (a new subclass of almost N_0-matrices which are obtained as a limit of a sequence of almost N-matrices) and obtain a sufficient condition for almost $\bar{N}$ class with positive value to hold Q-property. We give a counter example to show that an almost $\bar{N} \cap Q$-matrix need not be an R_0-matrix. In section 4, we consider a generalization of almost $\bar{N}$-matrix, namely, $\bar{N}$-matrix of exact order 2 and extend the results proved for almost $\bar{N}$ class to this class. In section 5, we introduce another new class called $\bar{E}(d)$ and show that $\bar{E}(d) \cap R_0$ belongs to class Q. Finally, in section 6, we show that Pang's conjecture is true if E_0 is replaced by almost C_0. We also consider almost P_0-matrices and give a game theoretic proof of necessary and sufficient conditions for this class to hold Q-property.

2. Preliminaries

We consider matrices and vectors with real entries. For any matrix $A \in R^{n\times n}$, a_{ij} denotes its ith row and jth column entry. $A_{\cdot j}$ denotes the jth column and $A_{i\cdot}$, the ith row of A. If A is a matrix of order n, $\alpha \subseteq \{1, 2, \ldots, n\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$ then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β respectively. For any set α, $|\alpha|$ denotes its cardinality. For any set $\beta \subseteq \{1, 2, \ldots, n\}$, $\bar{\beta}$ denotes its complement in $\{1, 2, \ldots, n\}$. Any vector $x \in R^n$ is a column vector unless specified otherwise and x^t denotes the transpose of x.

The *principal pivot transform* (PPT) of A with respect to $\alpha \subseteq \{1, 2, \ldots, n\}$ is defined as the matrix given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details see [6].

2.1. *Matrix games*

The linear complementarity problem and the matrix game have some important connections. Many of the results of LCP can be stated in terms of the value of a matrix game. In this connection, Kaplansky's result [9] on matrix games is useful for deriving certain results. See also [10]. A matrix game may be stated as follows:

There are two players, player I and player II, and each player has a finite number of actions (called *pure strategies*). Let player I have m pure strategies and player II, n pure strategies. Suppose player I chooses to play a pure strategy i $(i = 1, 2, \dots, m)$ and player II chooses a pure strategy j $(j = 1, 2, \dots, n)$ simultaneously. Then player I pays player II an amount a_{ij} (which may be positive, negative or zero). Since player II's gain is player I's loss, the game is said to be zero-sum. A *mixed strategy* for player I is a probability vector $x \in R^m$ whose ith component x_i represents the probability of choosing pure strategy i where $x_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, a mixed strategy for player II is a probability vector $y \in R^n$.

From von Neumann's fundamental minimax theorem, we know that there exist mixed strategies x^*, y^* and a real number v such that

$$\sum_{i=1}^{m} x_i^* a_{ij} \leq v, \qquad \forall\, j = 1, 2, \dots, n.$$

$$\sum_{j=1}^{n} y_j^* a_{ij} \geq v, \qquad \forall\, i = 1, 2, \dots, m.$$

The mixed strategies (x^*, y^*) with $x^* \in R^m$ and $y^* \in R^n$ are said to be *optimal strategies* for player I and player II respectively and v is called *minimax value of game*. We write $v(A)$ to denote the value of the game corresponding to A. In the game described above, player I is the minimizer and player II is the maximizer. A mixed strategy is *completely mixed* if $x > 0$. The value of the game $v(A)$ is *positive* (*nonnegative*), if there exists a $0 \neq x \geq 0$ such that $Ax > 0$ $(Ax \geq 0)$. Similarly, $v(A)$ is *negative* (*nonpositive*) if there exists a $0 \neq y \geq 0$ such that $A^t y < 0$ $(A^t y \leq 0)$.

Remark 2.1 It is easy to show that $A \in Q$ if and only if $A \in Q_0$ with $v(A) > 0$. Also $A \in E_0$ if and only if $v(A_{\alpha\alpha}) \geq 0$ for all $\alpha \subseteq \{1, 2, \dots, n\}$. See [11].

We make use of the following result on the class R_0 due to Murty [12] and Saigal [8].

THEOREM 2.1 *If $A \in R_0$ and* LCP(q, A) *has odd number of solutions for a nondegenerate q, then $A \in Q$.*

The following results were proved by Väliaho [2] for symmetric almost copositive matrices. However, it is easy to see that these results hold for nonsymmetric almost copositive matrices as well.

THEOREM 2.2 *Let $A \in R^{n\times n}$ be almost copositive. Then A is PSD of order $n - 1$, and A is PD of order $n - 2$.*

THEOREM 2.3 *Suppose A is almost strictly copositive. Then A is PSD and PD of order $n - 1$.*

THEOREM 2.4 *Suppose that a copositive matrix A is almost copositive-plus. Then it is strictly copositive of exact order 2.*

The following result on semimonotone matrices is due to Pang [13].

THEOREM 2.5 ([13]) *Suppose $A \in E_0 \cap Q$. Then the system $Ax = 0$, $x > 0$ has no solution.*

The inconsistency of the above system is equivalent to the fact that any nonzero solution to LCP$(0, A)$ must have some zero components. Further, every nontrivial solution of LCP$(0, A)$ has at least two nonzero coordinates.

The following results will be used in the sequel.

THEOREM 2.6 ([14]) *Suppose $A \in R^{n\times n}$ is an almost P_0-matrix. Let $B = A^{-1}$. Then there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ such that $B_{\alpha\alpha} \leq 0$, $B_{\bar{\alpha}\bar{\alpha}} \leq 0, B_{\bar{\alpha}\alpha} \geq 0$ and $B_{\alpha\bar{\alpha}} \geq 0$.*

THEOREM 2.7 ([14, p. 1271]) *Suppose $A \in Q\ (Q_0)$. Assume that $A_{i\cdot} \geq 0$ for some $i \in \{1, 2, \ldots, \}$. Then $A_{\alpha\alpha} \in Q\ (Q_0)$, where $\alpha = \{1, 2, \ldots, n\} \backslash \{i\}$.*

THEOREM 2.8 ([8, p. 45]) *A sufficient condition for LCP(q, A) to have even number of solutions for all q for which each solution is nondegenerate is that there exists a vector $z > 0$ such that $z^t A < 0$.*

THEOREM 2.9 ([11, p. 195]) *Let $A \in R^{n\times n}$ be a E_0-matrix with $n \geq 3$. Suppose any one of the following conditions holds:*

(i) *Every principal submatrix of order $n-1$ is an R_0-matrix.*
(ii) *Every principal submatrix of order less than or equal to $n-2$ is an R_0-matrix.*

Then A is a Q-matrix if and only if A is an R_0-matrix.

Definition 2.1 Suppose $A \in R_0$ and q is nondegenerate with respect to A. For any $(w, z) \in S(q, A)$, define the index of z, $\text{ind}\,(A, q, z) = \text{sgn}\ \det(A_{\alpha\alpha}) = \frac{\det(A_{\alpha\alpha})}{|\det(A_{\alpha\alpha})|}$ where $\alpha = \{i\colon\ z_i \neq 0\}$. For an R_0-matrix A, the number $\sum_{z\in S(q,A)} \text{ind}\,(A, q, z)$ is the same for all vectors q such that LCP(q, A) has a finite number of solutions. This number is called the degree of the matrix A and we write $\deg(\text{A}) = \sum_{z\in S(q,A)} \text{sgn}\ \det(A_{\alpha\alpha})$. For further details on degree theory see Ref. [6, Chapter 6].

3. Almost $\bar{N}$-matrices

The class of $\bar{N}$-matrices was introduced by Mohan and Sridhar in [15]. The class of almost N-matrices is studied in [5]. We introduce here a new matrix class almost $\bar{N}$, which is a subclass of the almost N_0-matrices. See also Ref. [3, p. 119].

Definition 2.2 A matrix $A \in R^{n\times n}$ is said to be an *almost $\bar{N}$-matrix* if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are almost N-matrices such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Example 3.1 Let

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

Note that A is an almost N_0-matrix. It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence of almost N-matrices

$$A^{(k)} = \begin{bmatrix} -1 & 2 & 2 \\ 1/k & -1/k & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

which converges to A as $k \to \infty$.

Remark 3.1 It is well known that for P_0 (almost P_0)-matrices, by perturbing the diagonal entries alone one can get a sequence of P (almost P)-matrices that converges to P_0 (almost P_0). However, this is not true for N_0 (almost N_0)-matrices. One of the reasons is that an N (almost N)-matrix needs to have all its entries nonzero. In the above example, we can see that even though the matrix $A \in$ almost N_0 but it cannot be obtained as a limit point of almost N-matrix by perturbing the diagonal. However, we show in the above example that $A \in$ almost $\bar{N}$.

The following example shows that an almost N_0-matrix need not be an almost $\bar{N}$-matrix.

Example 3.2 Let

$$A = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}.$$

Here A is an almost N_0-matrix. However, it is easy to verify that A is not an almost $\bar{N}$-matrix since we cannot get A as a limit point of a sequence of almost N-matrices.

Now we consider almost N_0-matrices and ask the following question. Suppose $A \in$ almost N_0. Then is it true that (i) $A \in Q$ implies $A \in R_0$ (ii) $A \in R_0$ implies $A \in Q$?

In the sequel, we partially settle the above questions. The following example demonstrates that $A \in$ almost $N_0 \cap Q$, but $A \notin R_0$.

Example 3.3 Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

It is easy to check that $A \in$ almost N_0. Now taking a PPT with respect to $\alpha = \{1, 3\}$ we get

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Now $A \in Q$ since M, a PPT of $A \in Q$ (see [11, p. 193]). However, $(0,1,0,0)$ solves LCP$(0,A)$. Hence $A \notin R_0$.

The following example due to Olech *et al*. [3, p. 120] shows that an almost N_0-matrix, even with value positive, need not be a Q-matrix or an R_0-matrix.

Example 3.4 Let

$$A = \begin{bmatrix} -2 & -2 & -2 & 2 \\ -2 & -1 & -3 & 3 \\ -2 & -3 & -1 & 3 \\ 2 & 3 & 3 & 0 \end{bmatrix} \quad q = \begin{bmatrix} -1001 \\ -500 \\ -500 \\ 500 \end{bmatrix}.$$

It is easy to check that $A \in$ almost N_0 but $A \notin Q$ even though $v(A)$ is positive. Furthermore $A \notin R_0$.

However, if $A \in$ almost $\bar{N} \cap R_0$ and $v(A) > 0$, then we show that $A \in Q$.

In the statement of some theorems that follow, we assume that $n \geq 4$, to make use of the sign pattern stated in the following lemma.

LEMMA 3.1 *Suppose $A \in R^{n\times n}$ is an almost $\bar{N}$-matrix of order $n \geq 4$. Then there exists a nonempty subset α of $\{1,2,\ldots,n\}$ such that A can be written in the partitioned form as (if necessary, after a principal rearrangement of its rows and columns)*

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Proof This follows from Remark 3.1 in [5, p. 623] and from the definition of almost $\bar{N}$-matrices. ■

Remark 3.2 In the proof of the sign pattern in Lemma 3.1, we assume $n \geq 4$ since the lemma requires that all the principal minors of order 3 or less are negative.

THEOREM 3.1 *Suppose $A \in E_0 \cap$ almost $\bar{N}$ $(n \geq 4)$. Then there exists a principal rearrangement*

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A where $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices and $B_{\bar{\alpha}\alpha}$, $B_{\alpha\bar{\alpha}}$ are nonnegative matrices.

Proof The proof follows from Lemma 3.1 and the properties of E_0-matrix. ■

The following example shows that almost $\bar{N} \cap E_0$ is nonempty.

Example 3.5 Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Here A is an $E_0 \cap N_0$-matrix. It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence

$$A^{(k)} = \begin{bmatrix} -1/k & -1 & 2/k & 2 \\ -1/k & -1/k & 1 & 2/k \\ 4/k & 1 & -1/k & -1 \\ 1 & 2/k & -1/k & -1/k \end{bmatrix}$$

of almost N-matrices which converges to A as $k \to \infty$.

THEOREM 3.2 *Suppose $A \in R^{n\times n}$ is an almost $\bar{N} \cap Q_0 \cap E_0$-matrix with $n \geq 4$. Then there exists a principal rearrangement B of A such that all the leading principal submatrices of B are Q_0-matrices.*

Proof The proof of this theorem follows from Theorem 3.1 and Theorem 2.7. ■

THEOREM 3.3 *Let $A \in$ almost $\bar{N} \cap R^{n\times n}, n \geq 4$ with $v(A) > 0$. Then $A \in Q$ if $A \in R_0$.*

Proof Let $A \in$ almost $\bar{N} \cap R_0$. Then by Lemma 3.1, there exists $\emptyset \neq \alpha \subseteq \{1, 2, \ldots, n\}$,

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0,\ A_{\bar{\alpha}\bar{\alpha}} \leq 0,\ A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Now consider $A_{\alpha\alpha}$. Suppose $A_{\alpha\alpha}$ contains a nonnegative column vector. Then clearly LCP$(0, A)$ has a nontrivial solution which contradicts our hypothesis that $A \in R_0$. Hence every column of $A_{\alpha\alpha}$ should have at least one negative entry. Hence $\exists$ an $x \in R^{|\alpha|}$, $x > 0$, such that $x^t A_{\alpha\alpha} < 0$. It now follows from Theorem 2.8 that for any $q_\alpha > 0$, where q_α is nondegenerate with respect to $A_{\alpha\alpha}$, LCP$(q_\alpha, A_{\alpha\alpha})$ has r solutions ($r \geq 2$ and even). Similarly, LCP$(q_{\bar{\alpha}}, A_{\bar{\alpha}\bar{\alpha}})$ has s solutions ($s \geq 2$ and even) for any $q_{\bar{\alpha}} > 0$, where $q_{\bar{\alpha}}$ is nondegenerate with respect to $A_{\bar{\alpha}\bar{\alpha}}$. Now suppose (w^i_α, z^i_α) is a solution for LCP$(q_\alpha, A_{\alpha\alpha})$. Note that

$$w = \begin{bmatrix} w^i_\alpha \\ q_{\bar{\alpha}} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z^i_\alpha \\ 0 \end{bmatrix}$$

solves LCP(q, A). Similarly, associated with every solution $(w^i_{\bar{\alpha}}, z^i_{\bar{\alpha}})$ we can construct a solution LCP(q, A). Thus LCP(q, A) has $(r + s - 1)$ solutions accounting for only once the solution $w = q, z = 0$. Thus there are an odd number $(r + s - 1 \geq 3)$ of solutions to LCP(q, A) with all solutions nondegenerate. We shall show that $(r + s - 1) \leq 3$ and hence there are only three solutions to LCP(q, A). Let $S(q, A)$ be the set of solutions to LCP(q, A). Since q is nondegenerate with respect to A, this is a finite set [12, p. 85]. Suppose $(\bar{w}, \bar{z})$ is a nondegenerate solution to LCP(q, A). Then $(\bar{w}, \bar{z}) \in S(q, A)$. Now since A is a limit point of almost N-matrices $\{A^{(k)}\}$, we note that the complementary basis corresponding $(\bar{w}, \bar{z})$ will also yield a solution to LCP$(q, A^{(k)})$ for all k sufficiently large. From Theorem 3.2 [5, p. 625], which asserts that there are exactly three solutions for LCP$(q, A^{(k)})$, for any nondegenerate $q\ (> 0)$ with respect to $A^{(k)}$, we obtain $(r + s - 1) \leq |S(\alpha, A)| \leq |S(q, A^{(k)})| = 3$.

But $(r+s-1) \geq 3$. Hence LCP(q, A) has exactly three solutions for any nondegenerate $q(>0)$ with respect to A. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 2.1 that $A \in Q$. ■

COROLLARY 3.1 *Suppose $A \in$ almost $\bar{N} \cap R_0$ with $v(A) > 0$. Then $|\deg(A)| = odd$.*

Proof This follows from the fact that LCP(q, A) has three solutions for any nondegenerate q (>0) with respect to A and $A \in R_0$.

However, the converse of Theorem 3.3 is not true. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

in example 3.3 which is also a Q-matrix. Note that $A \in$ almost $\bar{N}$, since we can get A as a limit point of the sequence of almost N-matrices

$$A^{(k)} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1/k^2 & -1/k & -1/k \\ 1 & -1/k & -1/k & -1 \\ 1 & -1/k & -1 & -1/k \end{bmatrix}$$

which converges to A as $k \to \infty$. However, $A \notin R_0$.

The converse of the statement is not true for $n < 4$ is illustrated in the following example.

Example 3.6 Consider the matrix

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence

$$A^{(k)} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1/k \end{bmatrix}$$

of almost N-matrices which converges to A as $k \to \infty$.

We show that $A \in Q$, by showing that its $A^{-1} \in Q$. Now look at

$$A^{-1} = \begin{bmatrix} -1/6 & 1/6 & 1/2 \\ 1/6 & -1/6 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}.$$

Suppose that $q_1 \geq q_2$ in LCP(q, A) where

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

It is easy to see that $A_{23} \in Q$ and $A_{13} \in Q$. Since $A_{23} \in Q$, there exists a solution

$$\left(\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}, \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} \right)$$

to LCP$(q_\alpha, A_{\alpha\alpha})$ where $\alpha = \{2, 3\}$. Now define

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix}$$

where $w_1 = w_2 + q_1 - q_2 + \frac{1}{3} z_2$. It is easy to check that (w, z) is a solution to LCP(q, A).

If $q_1 < q_2$, then we can get a solution to LCP(q, A) using a solution to LCP$(q_\alpha, A_{\alpha\alpha})$ where $\alpha = \{1, 3\}$. Now define

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_1 \\ 0 \\ z_3 \end{bmatrix}$$

where $w_2 = w_1 + q_2 - q_1 + \frac{1}{3} z_1$. It is easy to check that (w, z) is a solution to LCP(q, A). Since q is arbitrary, it follows that $A \in Q$. However, $A \notin R_0$.

4. A generalization of almost $\bar{N}$-matrix

Mohan *et al.* [5] introduced the N-matrix of exact order 2 as a generalization of almost N-matrix studied by Olech *et al.* [3]. In this section, we introduce a new class of matrix as a generalization of almost $\bar{N}$-matrix introduced in the earlier section. This class originates from the limit of a sequence of N-matrices of exact order 2.

Definition 4.1 A matrix $A \in R^{n \times n}$ is said to be an *$\bar{N}$-matrix of exact order* 2 if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are N-matrices of exact order 2 such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Example 4.1 Let

$$A = \begin{bmatrix} 0 & -90 & -80 & -70 & 0 \\ -90 & -2 & -2 & -2 & 2 \\ -70 & -2 & -1 & -3 & 3 \\ -50 & -2 & -3 & -0.8 & 3 \\ 0 & 2 & 3 & 3 & 0 \end{bmatrix}.$$

Here A is an N_0-matrix of exact order 2.

Also A is an $\bar{N}$-matrix of exact order 2 since we can get A as a limit point of the sequence of N-matrices of exact order 2

$$A^{(k)} = \begin{bmatrix} -1/k^2 & -90 & -80 & -70 & 1/k \\ -90 & -2 & -2 & -2 & 2 \\ -70 & -2 & -1 & -3 & 3 \\ -50 & -2 & -3 & -0.8 & 3 \\ 1/k & 2 & 3 & 3 & -1/k^2 \end{bmatrix}$$

which converges to A as $k \to \infty$.

The following example shows that $\bar{N}$-matrix of exact order 2 is a proper subclass of N_0-matrix of exact order 2.

Example 4.2 Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly A is an N_0-matrix of exact order 2. However, A is not an $\bar{N}$-matrix of exact order 2 since we cannot get A as a limit of a sequence of N-matrices of exact order 2.

Remark 4.1 It is easy to see that Lemma 3.1 holds for $\bar{N}$-matrices of exact order 2 for $n \geq 5$.

THEOREM 4.1 *Let $A \in \bar{N} \cap R^{n\times n}, n \geq 5$ of exact order 2 with $v(A) > 0$. Suppose there exists at most one nonpositive principal submatrix of order $n-1$ and the values of proper principal submatrices of order ($\geq$2) which contains at least one positive entry are positive. Then $A \in Q$ if $A \in R_0$.*

Proof

Case 1 Suppose there is a nonpositive principal submatrix of order $(n-1)$. We may assume, without loss of generality that $A_{\alpha\alpha} \leq 0$ where $\alpha = \{2, \ldots, n\}$. Since $A \in \bar{N} \cap R^{n\times n}, n \geq 5$ of exact order 2 with $v(A) > 0$ and $A \in R_0$ the sign pattern of A can be written as

$$A = \begin{bmatrix} - & \oplus & \oplus & \ldots & \oplus \\ + & & & & \\ + & & & A_{\alpha\alpha} & \\ \vdots & & & & \\ + & & & & \end{bmatrix}$$

where the sign symbol $\oplus$ denotes a nonnegative real number. Choose a $q > 0$ which is nondegenerate with respect to A and the partitioned form of q is $q = [q_1, q_\alpha]^t$ where $|\alpha| = (n-1)$. By repeating a similar argument as in Theorem 3.3 we can show that LCP$(q_\alpha, A_{\alpha\alpha})$ has r solutions ($r \geq 2$ and even). Similarly, LCP(q_1, a_{11}) has two solutions. Thus there are an odd number ($r + 1 \geq 3$) of solutions to LCP(q, A) with all solutions nondegenerate.

Now we show that for this q ($q > 0$), LCP(q, A) has no other solution. Suppose $(\hat{w}, \hat{z})$ is another solution distinct from the odd number of solutions listed above.

Let $\beta = \{i : \hat{z}_i > 0\}$. Since $(\hat{w}, \hat{z})$ is different from the solutions listed above, it follows that the index $1 \in \beta$ and $\beta \cap \{2, \ldots, n\} \neq \phi$. Note that all $A_{\beta\beta}$ contains at least one positive entry. So, by assumption $v(A_{\beta\beta}) > 0$.

Now $\hat{w} - A\hat{z} = q$, leads to $A_{\beta\beta}\hat{z} < 0$ which contradicts our assumption $v(A_{\beta\beta}) > 0$.

Thus LCP(q, A) has an odd number of solutions. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 2.1 that $A \in Q$.

Case II Suppose there is no nonpositive principal submatrix of order $(n-1)$. Then by Remark 4.1, there exists a $\emptyset \neq \alpha \subseteq \{1, 2, \ldots, n\}$ such that A can be written in the partitioned form as

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0$, $A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Now consider $A_{\alpha\alpha}$. We proceed as in Theorem 3.3. Thus there are an odd number (≥ 3) of solutions to LCP(q, A) with all solutions nondegenerate. As before (Case I) we can show that there are no other solution. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 2.1 that $A \in Q$.

Remark 4.2 Now since A is a limit of a sequence $\{A^{(k)}\}$ of N-matrices of exact order 2, we note that the complementary basis corresponding to a solution will also yield a solution to LCP($q, A^{(k)}$) for all k sufficiently large. Hence there are exactly five solutions [5, p. 634] for LCP($q, A^{(k)}$) for any nondegenerate $q(>0)$ with respect to $A^{(k)}$. Therefore $3 \leq |S(q, A)| \leq |S(q, A^{(k)})| = 5$.

5. $\bar{E}(d)$-matrices

Garcia [16] introduced the class of matrices $E(d)$ which is dependent only on d as a generalization of E_0. For a given $d > 0$, the class $E(d)$ is the class of matrices for which LCP(d, A) has a unique solution $w = d$, $z = 0$. Now we ask the following question.

Is $E(d)$ closed for a given $d > 0$?

The answer is no and it is illustrated in the following example.

Example 5.1 Consider the following matrix

$$A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad A^{(k)} = \begin{bmatrix} -2 & 3 \\ -3 - \frac{1}{k} & 4 \end{bmatrix}.$$

It is easy to see that for

$$d = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

LCP$(d, A^{(k)})$ has a unique solution $w = d,\ z = 0$ but LCP(d, A) has two solutions. Thus we have a sequence $\{A^{(k)}\}$ of matrices where $A^{(k)} \in \mathrm{E}(d)$ and as $k \to \infty,\ a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. However, $A \notin E(d)$. Thus the class $E(d)$ is not closed.

We now introduce a new matrix class, similar to almost $\bar{N}$-matrix, containing a matrix A if it is the limit of a sequence $\{A^{(k)}\}$ of matrices where $A^{(k)} \in E(d)$ for a given positive vector d. We call this class as $\bar{E}(d)$. Note that the matrix A in the above example belongs to $\bar{E}(d)$.

Definition 5.1 For a given positive vector $d \in R^n$, a matrix $A \in R^{n\times n}$ is said to be an $\bar{E}(d)$-matrix if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are in $E(d)$ such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Although $E(d)$ is not closed, so that LCP(d, A) may have more than one solution, we have the following theorem.

THEOREM 5.1 *Suppose $A \in \bar{E}(d) \cap R_0$ for a given positive vector $d \in R^n$. Then $A \in Q$.*

Proof Since $A \in \bar{E}(d)$ $\exists$ a sequence $\{A^{(k)}\}$ of matrices such that $A^{(k)} \in E(d)$ and $A^{(k)} \to A$. Note that d is nondegenerate with respect to $A^{(k)}$ for all k and $d > 0$. Suppose d is degenerate with respect to A. Since the set $\{q \mid q$ is degenerate with respect to A$\}$ has dimension $\leq (n-1)$, it follows that we can find a $\epsilon > 0$ and $d^* \in N_\varepsilon(d)$ where $N_\varepsilon(d)$ is the ϵ-neighborhood of d such that d^* is nondegenerate with respect to A and also $A^{(k)}$ for all k. Now let $S(d^*, A) = \{(w, z) \mid (w, z)$ is a solution to LCP$(d^*, A)\}$. Note that $S(d^*, A) \neq \emptyset$, since $(d^*, 0) \in S(d^*, A)$ and also $S(d^*, A)$ is finite since d^* is nondegenerate with respect to A [12, p. 85].

Let $\epsilon > 0$ be given. Suppose $(w^*, z^*) \in S(d^*, A)$. Thus for k large enough $S(d^*, A^{(k)}) \cap N_\epsilon(w^*, z^*) \neq \emptyset$ where $N_\epsilon(w^*, z^*)$ is the ϵ-neighborhood of (w^*, z^*). To see this, let B be the complementary basis submatrix of $(I, -A)$ induced by (w^*, z^*) and let $B^{(k)}$ be the corresponding complementary basis submatrix of $(I, -A^{(k)})$. Note that $B^{(k)}$ is arbitrarily close to B for large k and hence $(B^{(k)})^{-1}d^*$ can be made arbitrarily close to $B^{-1}d^*$ and in particular $(B^{(k)})^{-1}d^* > 0$. Therefore the corresponding solution (w^k, z^k) of LCP$(d^*, A^{(k)}) \in N_\epsilon(w^*, z^*)$.

Thus every solution of LCP(d^*, A) corresponds to a distinct solution of LCP$(d^*, A^{(k)})$ for k sufficiently large. Hence $\emptyset \neq |S(d^*, A)| \leq |S(d^*, A^{(k)})| = 1$, since LCP$(d^*, A^{(k)})$ has a unique solution by our choice of d^*. Therefore LCP(d^*, A) has a unique nondegenerate solution. Using Theorem 2.1, it follows that $A \in Q$.

COROLLARY 5.1 *Suppose $A \in \bar{E}(d) \cap R_0$. Then* $\deg(A) = 1$.

Proof This follows from the uniqueness of the solution of LCP(q, A) for a nondegenerate $q = d^* > 0$ and $\det A_{\emptyset\emptyset} = 1$. ■

For implementation of Lemke's algorithm (see [6] for details), one needs a positive vector d, called the *covering vector*. The above proof uses the fact that if $A^{(k)} \to A$ with $A^{(k)} \in E(d)$ one can find a d^* by perturbing d slightly, to be used as covering vector for processing LCP(q, A) by Lemke's algorithm. In the above example one may take

$$d^* = \begin{bmatrix} 2 \\ 2.98 \end{bmatrix}.$$

It is easy to check that LCP(d^*, A) has a unique solution $w = d^*,\ z = 0$.

6. Almost C_0 and almost P_0-matrices

Väliaho [1,2] introduced symmetric almost C_0-matrices. The following example shows that an almost C_0-matrix need not be an E_0-matrix.

Example 6.1 Consider the following matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}.$$

It is easy to see that $A \in$ almost C_0 but $A \notin E_0$.

Pang [16] proved the following theorem.

THEOREM 6.1 *Suppose $A \in E_0$. If $A \in R_0$ then $A \in Q$.*

Pang conjectured that the converse must be true, i.e., $E_0 \cap Q \subset R_0$. However, this was disproved by Jeter and Pye [17]. Murthy *et al*. [18] showed that the conjecture is not true even if E_0-matrix is replaced by C_0-matrix. Here we show that if C_0-matrix is replaced by an almost C_0-matrix then Pang's conjecture is true. We present a game theoretic proof.

THEOREM 6.2 *Suppose $A \in$ almost C_0 with $n \geq 3$. If $A \in Q$ then $A \in R_0$.*

Proof $A \in Q$ implies $v(A) > 0$. Suppose $v(A_{\alpha\alpha}) < 0$ for $\alpha \subset \{1, 2, \ldots, n\}$. Then there exists a mixed strategy y such that $y_\alpha^t A_{\alpha\alpha} < 0$. Define $x \in R_+^n$ such that $y_\alpha = x_\alpha$ and $x_{\bar{\alpha}} = 0$. Hence $x^t Ax = y_\alpha^t A_{\alpha\alpha} y_\alpha < 0$ which contradicts that submatrices of order $n-1$ are PSD. Therefore $v(A_{\alpha\alpha}) \geq 0 \; \forall \; \alpha \subseteq \{1, 2, \ldots, n\}$. It follows from Remark 2.1 that $A \in E_0$. From Theorem 2.2, it follows that A is PD of order $(n-2)$. Hence every principal submatrix of order less than or equal to $n-2$ is an R_0-matrix. Since $A \in Q$, by Theorem 2.9 it follows that $A \in R_0$. ■

To prove the converse we need the additional assumption $v(A) > 0$.

THEOREM 6.3 *Suppose $A \in$ almost C_0 with $v(A) > 0$. If $A \in R_0$ then $A \in Q$.*

Proof Using a similar argument we can see that $A \in E_0$. Since $E_0 \cap R_0$-matrix is a Q_0-matrix with $v(A) > 0$. It follows from the Remark 2.1 that $A \in Q$. ■

THEOREM 6.4 *Suppose a copositive matrix A with $n \geq 3$ is almost copositive-plus. Then $A \in Q$ if and only if $A \in R_0$.*

Proof By Theorem 2.4, A is strictly copositive of exact order 2. So, by definition every principal submatrix of order less than or equal to $n-2$ is a strictly copostive matrix. It follows that every principal submatrix of order less than or equal to $n-2$ is an R_0-matrix. Since every C_0-matrix is an E_0-matrix. By Theorem 2.9, it follows that $A \in Q$ if and only if $A \in R_0$. ■

The following result was proved by Pye [4]. We present a game theoretic proof.

THEOREM 6.5 *Let A be a nonsingular almost $P_0 \cap R^{n\times n}$ matrix with $v(A) > 0$. Then the following statements hold.*

(i) *if $A \in R_0$ then $A \in Q$.*
(ii) *if $A \in Q$ then $A \in R$.*

Proof Note that $A \in E_0$ since $v(A) \geq 0 \; \forall \, \alpha \subseteq \{1, 2 \ldots, n\}$. Assume $A \in R_0$. Then $A \in Q_0$ with $v(A) > 0$. Hence $A \in Q$. Conversely assume that $A \in Q$. We show that $A \in R$. Suppose $A \notin R$. Let z be a nontrivial solution of LCP(te, A) where $t \geq 0$.

Case(a) $t > 0$. Let $\beta = \{i \mid z_i = 0\}$. Let $\alpha = \{1, 2, \ldots, n\} \setminus \beta$. Note that (w_α, z_α), $z_\alpha \neq 0$ is a solution of LCP$(te_\alpha, A_{\alpha\alpha})$. Hence $A_{\alpha\alpha} z_\alpha < 0$, $z_\alpha > 0$. Therefore $z_\alpha^t A_{\alpha\alpha}^t < 0$, $z_\alpha > 0$. But this implies $v(A_{\alpha\alpha}^t) < 0$. This contradicts that $A_{\alpha\alpha}^t$ is a P_0-matrix. Therefore LCP(te, A) where $t > 0$ has no nontrivial solution.
Case(b) $t = 0$. By Theorem 5[4, p. 441], it follows that if LCP$(0, A)$ has a nontrivial solution then $A \notin Q$. Hence $A \in R$. ■

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