Principal pivot transforms of some classes of matrices

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Abstract

In this paper, we study and characterize various classes of matrices that are defined based on principal pivot transforms. We show that matrices in these classes have nonnegative principal minors.

Keywords: Principal pivot transforms; Almost copositive matrix; Almost fully copositive matrix; Semi-monotone matrix; Q_0-matrix

1. Introduction

The concept of principal pivot transforms (PPTs) was introduced by Tucker [20]. PPTs play an important role in the study of linear complementarity theory. In the ensuing definitions, the symbol A will denote a real $n \times n$ matrix. The *principal pivot transform* (PPT) of A with respect to $\alpha \subseteq \{1, 2, \ldots, n\}$ is defined as the matrix given by

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$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. The PPT of LCP$(q, A)$ with respect to α (obtained by pivoting on $A_{\alpha\alpha}$) is given by LCP(q', M) where $q'_{\alpha} = -A_{\alpha\alpha}^{-1}q_{\alpha}$ and $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1}q_{\alpha}$.

Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details see [1, 19,2] in this connection. In what follows we first define some well-known matrix classes.

We say that A is *positive semidefinite* (PSD) if $x^tAx \geqslant 0$, $\forall\, x \in R^n$ and A is *positive definite* (PD) if $x^tAx > 0$, $\forall\, 0 \neq x \in R^n$. A is said to be a $P(P_0)$*-matrix* if all its principal minors are positive (nonnegative), and A is called a $N(N_0)$*-matrix* if all its principal minors are negative (nonpositive). A is called *copositive* (C_0) if $x^tAx \geqslant 0$, $\forall\, x \geqslant 0$. A is called *copositive* (PSD, PD) *of order k*, $0 \leqslant k \leqslant n$, if every principal submatrix of order k belongs to the class. A is said to be an E_0*-matrix* if for every $0 \neq y \geqslant 0$, $\exists$ an i such that $y_i > 0$ and $(Ay)_i \geqslant 0$. The class of such matrices is called semimonotone matrices. The *linear complementarity problem* is defined below.

Given $A \in R^{n\times n}$ and a vector $q \in R^n$, the problem of finding a solution $w \in R^n$ and $z \in R^n$ to the following system of linear equations and inequalities is the *linear complementarity problem* (LCP):

$$w - Az = q, \quad w \geqslant 0, \;\; z \geqslant 0, \tag{1.1}$$

$$w^tz = 0. \tag{1.2}$$

This problem is denoted as LCP(q, A). $A \in R^{n\times n}$ is called a Q*-matrix* if for every $q \in R^n$, LCP(q, A) has a solution. We say that a matrix A is a Q_0*-matrix* if for any $q \in R^n$, (1.1) has a solution implies that LCP(q, A) has a solution. A is said to be a *completely* $Q(Q_0)$*-matrix* if all its principal submatrices are $Q(Q_0)$ matrices. For details on this problem, see Cottle et al. [2] and Murty [12].

The notion of $N(N_0)$, $P(P_0)$, C_0, etc. is generalized to almost $N(N_0)$, almost $P(P_0)$, almost C_0 in [13,18,14–16]. We say that A is an *almost* $P_0(P)$*-matrix* if $\det A_{\alpha\alpha} \geqslant 0$ (>0) $\forall\, \alpha \subset \{1, 2, \ldots, n\}$ and $\det A < 0$. Similarly, A is called *an almost* $N_0(N)$*-matrix* if $\det A_{\alpha\alpha} \leqslant 0$ (<0) $\forall\, \alpha \subset \{1, 2, \ldots, n\}$ and $\det A > 0$. $A \in R^{n\times n}$ is said to be a *almost copositive* if it is copositive of order $n-1$ but not of order n. Almost copositive matrices are also called exact order matrices of order $(n-1)$ in Väliaho [17]. We say that a matrix is called *copositive of exact order k*, if it is copositive of order k but not of order $(k+1)$. For details see [16,17].

Motivated by the class of exact order copositive matrices considered by Väliaho [16], Mohan et al. [8] studied exact order $N(P)$ matrices. Given a matrix $A \in R^{n\times n}$, let $B_i \in R^{(n-1)\times(n-1)}$, $i = 1, 2, \ldots, n$ denote the principal submatrices of A, obtained by deleting the ith row and ith column of A. Note that if A is of exact order k then B_i, $1 \leqslant i \leqslant n$ are the matrices of exact order $(k-1)$. We say that a matrix A

is called an $N(P)$-matrix of exact order k, $1 \leqslant k \leqslant n$, if every principal submatrix of order $(n-k)$ is an N-matrix (P-matrix) and if every principal minor of order r, $n-k < r \leqslant n$ is positive (negative). A is called a matrix of exact order k if it is a P-matrix or a N-matrix of exact order k. Note that an $N(P)$-matrix is an $N(P)$-matrix of exact order 0 and an almost $N(P)$-matrix is an $N(P)$-matrix of exact order one. An N-matrix of exact order 1 is of first category if both A and A^{-1} have at least one positive entry, otherwise it is N-matrix of exact order one of second category. A P-matrix of exact order 1 is of first category if A^{-1} has a positive entry otherwise it is said to be of second category. We say that a matrix A ($A \nleq 0$) of exact order 2 is of first category if there exists at most one index k ($1 \leqslant k \leqslant n$) such that the $(n-1) \times (n-1)$ exact order 1 principal submatrix B_k is nonpositive and every $(n-1) \times (n-1)$ principal submatrix B_i which is $\nleq 0$, $1 \leqslant i \leqslant n$ is exact order 1 of the *first category*. We say that it is of the second category, if all B_i are of the second category. For further details see [8].

Tucker [20] proved that if the diagonal entries for every PPT of A are positive, then A is a P-matrix. However if the diagonal entries for every PPT of A is nonnegative, then A need not be a P_0-matrix. Cottle and Stone [3] introduced the notion of a fully semimonotone matrix (E_0^{f}) by requiring that every PPT of such a matrix is a semimonotone matrix. For the class E_0^{f}, if $q \in R^n$ is in the interior of a full complementary cone (a complementary cone is *full* with respect to $\alpha \subseteq \{1, 2, \ldots, n\}$, if $\det A_{\alpha\alpha} \neq 0$) then LCP$(q, A)$ has a unique solution. This is a geometric characterization of E_0^{f} class. A is called *fully copositive* (C_0^{f}) if every legitimate PPT of A is C_0. By a *legitimate principal pivot transform* we mean the PPT obtained from A by performing a principal pivot on its nonsingular principal submatrices. For further details on the class of fully copositive matrices see [7,9,10]. If A belongs to any one of the class E_0, C_0, E_0^{f} or C_0^{f}, then so is (i) any principal submatrix of A and (ii) any matrix $\bar{A}$ obtained by a principal permutation of the rows and columns of A. If $A \in Q(Q_0)$ then every PPT of A is $Q(Q_0)$. Note that $P \subseteq P_0 \subseteq E_0^{\mathrm{f}} \subseteq E_0$ and $C_0^{\mathrm{f}} \subseteq E_0^{\mathrm{f}}$. We introduce two new classes of matrices based on principal pivot transforms. One of the new classes has the property that its PPTs are either C_0 or almost C_0 with at least one PPT almost C_0, and the other class has the property that its PPTs are either E_0 or almost C_0 with at least one PPT almost C_0.

In Section 2, some notations, definitions and a few well-known results in linear complementarity and matrix games are presented that will be used in the next section. In Section 3, we present some results on the class for which PPTs are either in C_0 (E_0) or almost C_0 with at least one PPT almost C_0. The almost classes studied in this paper have algorithmic significance and if these classes are also in Q_0 then these classes are processable by Lemke's algorithm. For a description of Lemke's algorithm see the book by Cottle et al. [2]. For many results we present proofs which use some terminology from matrix games. Finally in Section 4, we consider the problem of characterizing a class of matrices whose member possess at least one PPT that is a Z-matrix.

2. Preliminaries

For any set $\beta \subseteq \{1, 2, \ldots, n\}$, $\bar{\beta}$ denotes its complement in $\{1, 2, \ldots, n\}$. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^t denotes the row transpose of x. For any matrix $A \in R^{n\times n}$, a_{ij} denotes its ith row and jth column entry. $A_{\cdot j}$ denotes the jth column and $A_{i\cdot}$, the ith row of A. If A is a matrix of order $n, \emptyset \neq \alpha \subseteq \{1, 2, \ldots, n\}$ and $\emptyset \neq \beta \subseteq \{1, 2, \ldots, n\}$ then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β, respectively. For any set α, $|\alpha|$ denotes its cardinality. We state some game theoretic results due to Von Neumann and Morgenstern [18] which are needed in the sequel. See also [6]. A two person zero-sum matrix game may be stated as follows:

Suppose player I chooses an integer i ($i = 1, 2, \ldots, m$) and player II chooses an integer j ($j = 1, 2, \ldots, n$) simultaneously. Then player I pays player II an amount a_{ij} (which may be positive, negative or zero). Since player II's gain is player I's loss, the game is said to be zero-sum. A *strategy* for player I is a probability vector $x \in R^m$ whose ith component x_i represents the probability of choosing an integer i where $x_i \geqslant 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m x_i = 1$. Similarly, a strategy for player II is a probability vector $y \in R^n$.

From Von Neumann's fundamental minimax theorem we know that there exist strategies x^*, y^* and a real number v such that

$$\sum_{i=1}^{m} x_i^* a_{ij} \leqslant v, \quad \forall\, j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} y_j^* a_{ij} \geqslant v, \quad \forall\, i = 1, 2, \ldots, m.$$

The strategies (x^*, y^*) with $x^* \in R^m$ and $y^* \in R^n$ are said to be *optimal strategies* for player I and player II respectively and v is called *minimax value of game*. We write $v(A)$ to denote the value of the game corresponding to A. In the game described above, player I is the minimizer and player II is the maximizer. The value of the game $v(A)$ is *positive* (*nonnegative*) if there exists a $0 \neq x \geqslant 0$ such that $Ax > 0$ ($Ax \geqslant 0$). Similarly, $v(A)$ is *negative* (*nonpositive*) if there exists a $0 \neq y \geqslant 0$ such that $A^t y < 0$ ($A^t y \leqslant 0$).

Theorem 2.1. *Let $M \in R^{n\times n}$ be a PPT of a given matrix $A \in R^{n\times n}$. Then $v(A) > 0$ if and only if $v(M) > 0$.*

Proof. It is enough to show that $v(A) > 0 \Rightarrow v(M) > 0$. Let $v(A) > 0$. Then there exists a $z > 0$ such that $Az > 0$.

$$\text{Let } \begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix}.$$

Premultiplying by $\begin{bmatrix} -A_{\alpha\alpha} & 0 \\ -A_{\bar{\alpha}\alpha} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}^{-1} = \begin{bmatrix} -A_{\alpha\alpha}^{-1} & 0 \\ -A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and rewriting we get

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix},$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Since $\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} > 0$ and $\begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} > 0$, it follows that $v(M) > 0$. □

If A is a Q-matrix then $v(A) > 0$ [11]. Since any PPT M of a Q-matrix is again a Q-matrix, it follows that for any Q-matrix $v(M) > 0$ in all of its PPTs M. However it is easy to prove that for any matrix A with $v(A) > 0$, $A \in Q$ if and only if $A \in Q_0$.

The following result was proved by Väliaho [16] for symmetric almost copositive matrices. However this holds for nonsymmetric almost copositive matrices as well.

Theorem 2.2. *Let $A \in R^{n\times n}$ be almost copositive. Then A is PSD of order $n-1$, and A is PD of order $n-2$.*

Theorem 2.3 [9]. *Assume $A \in R^{n\times n}$ is nonnegative, where $n \geqslant 2$. Then $A \in Q_0$ if and only if for every $i \in \{1, 2, \ldots, n\}$ $A_{i\cdot} \neq 0 \Rightarrow a_{ii} > 0$.*

Theorem 2.4 [9]. *Suppose $A \in Q_0$. Assume that for some $i \in \{1, 2, \ldots, n\}$, $A_{i\cdot} \geqslant 0$. Then $A_{\alpha\alpha} \in Q_0$ for $\alpha = \{1, 2, \ldots, n\}\backslash\{i\}$.*

Theorem 2.5 [11]. *Let $A \in R^{n\times n}$. The following statements are equivalent:*

(i) $A \in E_0$.
(ii) *The LCP(q, A) has a unique solution for every $q > 0$.*
(iii) $v(A_{\alpha\alpha}) \geqslant 0$ *for every index set* $\alpha \subseteq \{1, 2, \ldots, n\}$.
(iv) $v(A^{t}_{\alpha\alpha}) \geqslant 0$ *for every index set* $\alpha \subseteq \{1, 2, \ldots, n\}$.
(v) $A^{t} \in E_0$.

Theorem 2.6 [9]. *Suppose $A \in R^{n\times n}$ $(n \geqslant 3)$ is a nonsingular N_0-matrix. Then there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ satisfying*

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where $A_{\alpha\alpha} \leqslant 0$, $A_{\bar{\alpha}\bar{\alpha}} \leqslant 0$, $A_{\bar{\alpha}\alpha} \geqslant 0$ and $A_{\alpha\bar{\alpha}} \geqslant 0$.

Theorem 2.7 [9]. *Suppose $A \in R^{n\times n}$ $(n \geqslant 3)$ is a nonsingular $E_0 \cap N_0$-matrix. Then there exists a principal rearrangement*

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \emptyset$, $\alpha \neq \{1, 2, \ldots, n\}$, $B_{\bar{\alpha}\alpha} \geqslant 0$, $B_{\alpha\bar{\alpha}} \geqslant 0$ *and* $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ *are strict upper triangular nonpositive matrices.*

It is easy to observe the following.

Theorem 2.8. *Assume* $A \in R^{n\times n}$ $(n \geqslant 3)$ *is a* $E_0 \cap N_0 \cap Q_0$*-matrix. Then there exist a principal rearrangement B of A such that all the leading principal submatrices of B are* Q_0*-matrices.*

Proof. By Theorem 2.7, there exists a principal rearrangement

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \emptyset$, $\alpha \neq \{1, 2, \ldots, n\}$, $B_{\bar{\alpha}\alpha} \geqslant 0$, $B_{\alpha\bar{\alpha}} \geqslant 0$ and $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are strict upper triangular nonpositive matrices. It is easy to conclude from the structure of B that $B_n \geqslant 0$. Note that $B \in Q_0$, since B is a principal rearrangement of A. Therefore by Theorem 2.4, $B_{\beta\beta} \in Q_0$ where $\beta = \{1, 2, \ldots, n\}\backslash\{n\}$. Similarly, we can show that the other leading principal submatrices of B are Q_0. □

3. Some PPT based matrix classes and its subclasses

Cottle and Stone [3] introduced a class called fully semimonotone matrices (E_0^{f}) for which every legitimate PPT is a semimonotone matrix. Stone conjectured that a fully semimonotone Q_0-matrix has nonnegative principal minors. Various subclasses such as E_0^{f}, C_0^{f} were studied earlier in [3,9,10,7]. In this section, we consider some more classes, defined using principal pivot transforms. One of these classes has the property that its PPTs are either C_0 or almost C_0 with at least one PPT almost C_0. The other class considered in this paper has the property that its PPTs are either E_0 or almost C_0 with at least one PPT almost C_0. Note that an almost C_0-matrix is not necessarily E_0. We show that if this class also belongs to Q_0, then it is in E_0^{f} by showing this class is in P_0.

Definition 3.1. A is said to be an *almost fully copositive* (almost C_0^{f})-matrix if its PPTs are either C_0 or almost C_0 and there exists at least one PPT M of A for some $\alpha \subset \{1, 2, \ldots, n\}$ that is almost C_0.

Example 3.1. The following matrix A is almost fully copositive:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Theorem 3.1. *If $A \in R^{n\times n} \cap$ almost $C_0^{\mathrm{f}} \cap Q_0$ $(n \geqslant 3)$, then A is a P_0-matrix.*

Proof. Suppose M is a PPT of A so that $M \in$ almost C_0. By Theorem 2.2, all the principal submatrices of order $(n-1)$ of M are PSD. Now to show $M \in P_0$ it is enough to show that $\det M \geqslant 0$. Suppose $\det M < 0$. Then M is an almost P_0-matrix. Therefore $M^{-1} \in N_0$ and by Theorem 2.6 there exists a nonempty subset $\alpha \subseteq \{1, 2, \ldots, n\}$ satisfying

$$M^{-1}_{\alpha\alpha} \leqslant 0,\ M^{-1}_{\bar{\alpha}\bar{\alpha}} \leqslant 0, M^{-1}_{\alpha\bar{\alpha}} \geqslant 0 \text{ and } M^{-1}_{\bar{\alpha}\alpha} \geqslant 0. \tag{3.1}$$

But M^{-1} is a PPT of M and by definition of almost C_0^{f}, $M^{-1} \in$ almost C_0 or $M^{-1} \in C_0$. We consider the following cases:

Case (i). $M^{-1} \in$ almost C_0. Note that by Theorem 2.2, the principal submatrices of order $(n-2)$ are PD. Therefore the diagonal entries of M^{-1} are positive. But $M^{-1} \in N_0$ and hence contradicts (3.1). Therefore $\det(M) \geqslant 0$ and $M \in P_0$. Since M is a PPT of A it follows that of $A \in P_0$.

Case (ii). $M^{-1} \in C_0 \cap Q_0$. Since $M^{-1} \in N_0$ we must have $M^{-1}_{\alpha\alpha} = 0$, $M^{-1}_{\bar{\alpha}\bar{\alpha}} = 0$. Therefore

$$M^{-1} = \begin{bmatrix} 0 & M^{-1}_{\alpha\bar{\alpha}} \\ M^{-1}_{\bar{\alpha}\alpha} & 0 \end{bmatrix}.$$

But this contradicts that M^{-1} is a Q_0-matrix. See Theorem 2.3 Therefore $M \in P_0$. □

Now we consider the matrix class whose members have PPTs that are either E_0 or almost C_0 with at least one PPT that is almost C_0. The following example shows that this class is nonempty.

Example 3.2. Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

It is easy to verify that all its PPT are either E_0 or almost C_0. Also $A \in Q_0$.

Theorem 3.2. *Suppose $A \in R^{n\times n} \cap Q_0$ $(n \geqslant 3)$ and the PPTs of A are either E_0 or almost C_0 with at least one PPT almost C_0. Then $A \in P_0$.*

Proof. Suppose M be a PPT of A so that $M \in$ almost C_0. By Theorem 2.2, all the submatrices of order $n-1$ of M are PSD. Now to complete the proof, we need to show that $\det M \geqslant 0$. Suppose $\det M < 0$. Then M is an almost P_0-matrix. Therefore $M^{-1} \in N_0$ and by Theorem 2.6 there exists a nonempty subset $\alpha \subseteq \{1, 2, \ldots, n\}$ satisfying

$$M^{-1}_{\alpha\alpha} \leqslant 0, \quad M^{-1}_{\bar{\alpha}\bar{\alpha}} \leqslant 0, \quad M^{-1}_{\alpha\bar{\alpha}} \geqslant 0 \quad \text{and} \quad M^{-1}_{\bar{\alpha}\alpha} \geqslant 0. \tag{3.2}$$

But M^{-1} is a PPT of M and by definition $M^{-1} \in$ almost C_0 or $M^{-1} \in E_0$. We consider the following cases:

Case (i). $M^{-1} \in$ almost C_0. Then the diagonal entries of M^{-1} are positive. But $M^{-1} \in N_0$ and contradicts (3.2). Therefore $\det(M) \geqslant 0$ and $M \in P_0$. Since M is a PPT of A it follows that of $A \in P_0$.

Case (ii). $M^{-1} \in E_0 \cap Q_0$. Since $M^{-1} \in E_0 \cap N_0$ then by Theorem 2.7 there exists a principal rearrangement

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of M^{-1} such that $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices and $B_{\alpha\bar{\alpha}}$, $B_{\bar{\alpha}\alpha}$ are nonnegative matrices.

Take $\alpha = \{1, 2, \ldots, p\}$ and $\gamma = \{1, 2, \ldots, (p+1)\}$. Note that by Theorem 2.8, $B_{\gamma\gamma} \in Q_0$. Consider

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha(p+1)} & B_{\alpha\bar{\gamma}} \\ B_{(p+1)\alpha} & B_{(p+1)(p+1)} & B_{(p+1)\bar{\gamma}} \\ B_{\bar{\gamma}\alpha} & B_{\bar{\gamma}(p+1)} & B_{\bar{\gamma}\bar{\gamma}} \end{bmatrix}.$$

Note that

$$B_{\bar{\alpha}\bar{\alpha}} = \begin{bmatrix} B_{(p+1)(p+1)} & B_{(p+1)\bar{\gamma}} \\ B_{\bar{\gamma}(p+1)} & B_{\bar{\gamma}\bar{\gamma}} \end{bmatrix}$$

is a strict upper triangular matrix nonpositive matrix. Therefore $B_{(p+1)(p+1)} = 0$ and $B_{\bar{\gamma}(p+1)} = 0$.

Now look at the principal submatrix $B_{\gamma\gamma}$ of order $(p+1)$. We shall show that $B_{\alpha(p+1)} = 0$. Suppose $b_{i_0(p+1)} > 0$ for some $i_0 \in \alpha$. Since $b_{i_0(p+1)} > 0$ there exists a q_γ such that $q_{i_0} < 0$ and $q_i > 0$ for all $i \in \gamma$, $i \neq i_0$ and the set of feasible solution $F(q_\gamma, B_{\gamma\gamma})$ of LCP$(q_\gamma, B_{\gamma\gamma})$ is nonempty. Let $(w_\gamma, z_\gamma) \in F(q, A)$. Then $z_{p+1} > 0$. Now $B_{(p+1)\alpha} \geqslant 0$ implies $w_{p+1} > 0$ contradicts $B_{\gamma\gamma} \in Q_0$. Therefore $B_{\alpha(p+1)} = 0$. Hence B is singular. But this leads to a contradiction. Therefore $A \in P_0$. □

Remark 3.1. Note that Theorem 3.1 also follows from Theorem 3.2 However, in the proof of Theorem 3.1, we use different arguments that uses the structure of a C_0-matrix.

Theorem 3.3. *Let $A \in E_0^{\mathrm{f}}$ with one zero principal minor. Assume that $A \in Q_0 \backslash Q$. Then there exist a PPT M of A such that the following holds*: (i) $\operatorname{rank}(M) = n - 1$, (ii) $Mz = 0$ *and* $\pi^{\mathrm{t}} M = 0$ *for vectors* $z, \pi > 0$.

Proof. Assume $\det(A_{\alpha\alpha}) = 0$ for some $\alpha \subseteq \{1, 2, \ldots, \}$. Let M be a PPT of A with respect to a nonsingular principal submatrix, say $A_{\beta\beta}$ of A such that $\det(M) = 0$. Hence $\text{rank}(M) = n - 1$. Since $M \in E_0^{\text{f}}$, LCP(d, M) has a unique solution for $d > 0$. Note that $M \in Q_0 \backslash Q$ since M is a PPT of A. Thus there exist a $q \in R^n$ such that LCP(q, M) does not have a solution. Therefore Lemke's algorithm when applied LCP(q, M) terminates in a secondary ray. Since no proper principal minor of M is zero and $M \in E_0^{\text{f}}$, it follows that, we get a positive vector z such that $Mz = 0$. Now we show that there is a positive vector $\pi > 0$ such that $\pi^{\text{t}} M = 0$. Without loss of generality, assume that z and π are probability vectors. Note that $M^{\text{t}} \in E_0$. Therefore $\text{val}(M^{\text{t}}) \geqslant 0$ by Theorem 2.5 Let $0 \neq \pi \geqslant 0$ be the optimal strategy for M^{t}. Therefore $M^{\text{t}}\pi \geqslant 0$. Now since $z^{\text{t}} M^{\text{t}} = 0$, therefore $\text{val}(M^{\text{t}}) = 0$ which implies $M^{\text{t}}\pi = 0$. Since $\det(M^{\text{t}}) = 0$ and the principal minors are nonzero, it follows that there is a positive vector $\pi > 0$ such that $\pi^{\text{t}} M = 0$. □

The class Q_0 matrices identified in the above theorem is contained in the class of Q_0 matrices of order n and rank$(n - 1)$ with positive vectors d and π satisfying $Md = 0$ and $\pi^{\text{t}} M = 0$ mentioned in [4]. Note that the class is not contained in any well-known classes of Q_0 matrices such as those studied in Garcia [5]. Lemke's algorithm is not applicable for this class. However Algorithm-I of Eagambaram and Mohan [4] can be applied to solve this class. Finally, we conclude the paper by mentioning an open problem associated with PPTs in Section 4.

4. Characterization of matrices for which at least one PPT is a Z-matrix: an open problem

The principal pivot transform of a Z-matrix need not be a Z-matrix. However Väliaho's [15] observed that the inverse of a symmetric almost copositive matrix is a Z-matrix. Mohan et al. [8] considered a class of matrices of exact order 2 whose inverses belong to class Z and observed the following result.

Theorem 4.1. *Let $A \in R^{n\times n}$ $(n \geqslant 5)$ be a matrix of exact order 2. $A^{-1} \in Z$ if and only if $v(A) < 0$ and A is of second category with each $B_i \nleq 0$.*

For the class stated in the theorem the following result on algorithmic significance was also proved by Mohan et al. [8].

Theorem 4.2. *Let $A \in R^{n\times n}$ $(n \geqslant 5)$ be a matrix of exact order 2 of the second category with $B_i \nleq 0$ for $1 \leqslant i \leqslant n$. Then a solution to LCP$(q, A)$, if one exists, can be computed by obtaining a solution to LCP$(-A^{-1}q, A^{-1})$, in at most n steps.*

However the complete characterization of the class of matrices for which at least one PPT is a Z-matrix remains an interesting open problem.

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