SYMMETRICAL UNEQUAL BLOCK ARRANGEMENTS

By K. KISHEN

Statistical Laboratory, Calcutta

INTRODUCTION

In agronomic tests involving a large number of varieties or treatments among which interactions do not exist or are of no particular interest, the need for the reduction of the size of the block for the efficient control of soil heterogeneity is insistent. As there are here no interactions available which may be completely or partially confounded with differences between block fertility, leading to the diminution of the size of the block, the orthodox procedure of confounding ceases to be applicable in these cases. In his quasi-factorial arrangements and symmetrical incomplete randomized blocks, Yates[7, 8] has succeeded in obtaining such special types of designs which, whilst very considerably reducing the size of block, permit of comparison between all possible pairs of varieties with unequal or equal precision. Only a limited number of such designs are, however; available, so that it is often necessary either to cut out a certain number of varieties or add "dummy" varieties to make the number of varieties to be tested equal to that for which a quasi-factorial or incomplete randomised block arrangement is available. There is obvious need, therefore, for the construction of designs in cases for which the above arrangements do not exist.

In a valuable contribution entitled "Partially Balanced Incomplete Block Designs", Bose and Nair[1] have investigated a more general class of these designs. But even this new class of designs does not suffice to overcome all the difficulties of the agricultural experimenter. It was, for instance, brought to the notice of the present writer that in certain hilly tracts growing coffee, tea or other crops, and other places where the land under cultivation is uneven, it is sometimes not agriculturally feasible to lay out blocks of equal size. It was, therefore, felt that designs with blocks of unequal size needed detailed investigation.

It is true that Yates's two-dimensional and three-dimensional quasi-factorial arrangements (in two and three unequal groups of sets respectively) provide such designs in cases where the number of varieties is factorisable as the product of two and three integers respectively. However, besides being not available for numbers which are primes, these designs are only partially balanced arrangements and include sometimes blocks of widely different sizes. The necessity of constructing symmetrical incomplete block arrangements with blocks of unequal size is, therefore, evident, because these would doubtless be the most efficient arragements in such cases.

In the present paper, attention is mainly confined to symmetrical incomplete block arrangements with blocks of two different sizes. It will be seen that the most general class of designs from which all of Yates's designs including the two-dimensional factorial arrangements in two unequal groups of sets (but not the three-dimensional designs in three equal or unequal groups of sets) are derivable as particular cases are the partially balanced incomplete block designs* of the $(p-1)$th order (*i. e.*, involving p different λ's) with blocks of q different sizes (*i. e.*, also having q different k's). However, the higher the order of the partially balanced incomplete block design and the larger the number of the different k's involved, the more inefficient inevitably is the resulting arrangement and the more involved the corresponding expressions for the estimates of varietal effects, the sums of squares for varieties for testing the significance of the differences between varietal means and variances of the differences between varietal means. It has, therefore, been considered desirable in the first instance to consider only designs which are completely balanced (in the sense of having a single λ) but which involve blocks of different sizes.

In arrangements with blocks of unequal sizes, however, it is necessary that the differences between the various sizes should not be large and that all the blocks should be completely randomized in the field. Otherwise, the weights to be used to take account of the different error variances for plots from blocks of different sizes become important, necessitating the calculation of weighted estimates of varietal effects. In what follows, the analysis has throughout been performed on the assumption that the weights are all equal. This is an approximate procedure, but would do as a first approximation in the absence of any '*a priori*' knowledge of the hypothetical weights. The assumption has the merit of simplifying the whole analysis and is necessary for the formulation of a unified theory without those complexities which are best avoided at this stage of the investigation.

§1. DEFINITIONS

Let v be the number of varieties, b the number of blocks, r the number of replications of a variety, λ the number of times a pair occurs within a block, k_1, $k_2, \ldots\ldots\ldots k_p$ $(k_1<k_2 \ldots\ldots\ldots <k_p)$ the p different sizes of the blocks, $n_1, n_2, \ldots\ldots\ldots\ldots n_p$ respectively being their number. Also let $\mu_1, \mu_2, \ldots\ldots\ldots \mu_p$ be the number of varieties which occur with a given variety in blocks of the first, second, ... and pth types respectively. If N be the total number of plots, we have the relations

$$b = n_1 + n_2 + \ldots\ldots\ldots\ldots + n_p \qquad \ldots \quad (1)$$

$$N = n_1\, k_1 + n_2\, k_2 + \ldots\ldots\ldots + n_p\, k_p \qquad \ldots \quad (2)$$

$$\lambda = \frac{n_1\, k_1(k_1-1) + \ldots\ldots\ldots\ldots + n_p\, k_p(k_p-1)}{r(v-1)} \qquad \ldots \quad (3)$$

$$= \frac{r(k_1-1)}{v-1} - (k_1-k_2)\, \frac{n_2\, k_2}{v(v-1)} \ldots\ldots\ldots - (k_1-k_p)\, \frac{n_p\, k_p}{v(v-1)} \qquad \ldots \quad (3{\cdot}1)$$

so that for $p=1$, the equations (2) and (3) reduce to:

$$N = bk = rv \qquad \ldots \quad (4)$$

and

$$\lambda = \frac{b\, k(k-1)}{v(v-1)} = r(k-1)/(v-1) \qquad \ldots \quad (5)$$

which are the algebraical relations which hold between the parameters of a symmetrical incomplete block arrangement.

Also for $\lambda = 1$, the case where there is no intermixture of any of the block associates of a variety with the others, it is easily seen that

$$v = 1 + \mu_1 + \mu_2 + \ldots\ldots\ldots + \mu_p \qquad \ldots \quad (6)$$

Also, from equation (2),

$$n_1 k_1/v + n_2 k_2/v + \;+ \ldots\ldots\ldots + n_p k_p/v = r \qquad \ldots \quad (7)$$

Only such designs have been found to exist, in the sense of being conveniently analysable, for which each of $n_1 k_1/v$, $n_2 k_2/v$.........$n_p k_p/v$ is an integer. That implies the existence of designs in which each variety is replicated the same number of times within any one of the group of blocks of equal size. The equations above, however, only enumerate the algebraical as distinguished from the combinatorial possibilities. For the latter an internal symmetry in the design is necessary which is ensured only when certain other parametric relations to be discussed in the succeeding sections hold.

§2. DESIGNS WITH $\lambda = 1$

We shall first consider the simplest type of these arrangements. They form a class apart, viz., those for which each pair of varieties occurs once within a block. Among these, only those with two unequal k's have been extensively investigated here. Designs involving three or more different sizes are easily constructed by algebraical and other methods. They will be considered in detail in a subsequent paper.

Considering, then, the designs with blocks of sizes k_1 and $k_2 (k_1 < k_2)$, we have the relations

$$b = n_1 + n_2 \qquad \ldots \quad (8)$$

$$N = n_1 k_1 + n_2 k_2 = r\, v \qquad \ldots (9)$$

$$\lambda = \frac{n_1 k_1(k_1-1) + n_2 k_2(k_2-1)}{v(v-1)} = \frac{r(k_1-1)}{v-1} - (k_1 - k_2)\frac{n_2 k_2}{v(v-1)} \qquad \ldots \quad (10)$$

$n_1 k_1/v$ and $n_2 k_2/v$ being both integers.

We may then define as the first block associates of a variety those varieties which occur with the given variety in blocks of the first type, i.e. those of size k_1; and as its second block associates, those varieties which occur with the given variety in blocks of the second type, viz., of size k_2. This definition is perfectly general, so that in the case of designs with blocks of p unequal sizes, the ith block associates of a variety are the varieties which occur with the given variety in blocks of the i-th type viz., of the size $k_i (i = 1, 2, \ldots\ldots \ldots p)$.

It will be seen that when $\lambda = 1$, there is no intermixture among the first, second, ...pth block associates of a given variety, so that those varieties which are the first block associates of a variety cannot simultaneously be any other block associates. This definition will consequently have to be modified when we come to the consideration of designs in which $\lambda > 1$.

We may now state the parametric relations which must be satisfied if an omplete block design wi blocks of two different sizes k_1 and k_2 is to exist, in the sense of lending itself to the finding of general expressions for the estimation of the varietal effects and sum of squares for varieties for tests of significance.

Let μ_1 and μ_2 be the number of the first and second block associates of a variety. Let us now consider two varieties which are first block associates. We may denote by a_{111} the number of varieties common to their first block associates, the first suffix denoting the type of association between the two varieties and the next two, standing for the type associates of the two varieties; by a_{112} the number of varieties common to the first block associates of the first and the second block associates of the second; by a_{121} the number of varieties common to the second block associates of the first and the first block associates of the second; and by a_{122} the number of varieties common to their second block associates. Similarly, considering two varieties which are second block associates, we define a_{211}, a_{212}, a_{221}, a_{222}.

From symmetrical considerations, we have $a_{112}=a_{121}$ and $a_{212}=a_{221}$.

Also it follows that

$$v=1+\mu_1+\mu_2 \qquad \dots \quad (11)$$

Further, the sufficient conditions for such a design to exist, in the sense of being analysable, are the following (not all independent):—

$$a_{111}+a_{112}=\mu_1-1 \qquad \dots \quad (12)$$

$$a_{212}+a_{222}=\mu_2-1 \qquad \dots \quad (13)$$

$$a_{211}+a_{212}=\mu_1 \qquad \dots \quad (14)$$

$$a_{112}+a_{122}=\mu_2 \qquad \dots \quad (15)$$

$$\mu_1\, a_{111}+\mu_2\, a_{211}=\mu_1(\mu_1-1) \qquad \dots \quad (16)$$

$$\mu_1\, a_{122}+\mu_2\, a_{222}=\mu_2(\mu_2-1) \qquad \dots \quad (17)$$

$$\mu_1\, a_{112}+\mu_2\, a_{212}=\mu_1\ \mu_2 \qquad \dots \quad (18)$$

Symmetrical incomplete block designs with blocks of two different sizes for all values of λ are most easily obtained by cutting out one or more varieties from all the blocks of a known Yates's symmetrical incomplete block design. Mr. R. C. Bose suggested to me that an alternative method of obtaining these designs is by utilizing the properties of finite hyper-dimensional projective and affine geometries allied to the Galois Fields. The useful infinite series of designs given in a later section (§7) have all been worked out by this method.

Coming now to the case $\lambda=1$, we may obtain designs with two unequal block sizes by cutting out a single variety from all the blocks. It is easily seen that in this case

$$k_1=k_2-1 \qquad \dots \quad (19)$$

$$n_1=r \qquad \dots \quad (20)$$

$$n_1\, k_1/v=\lambda=1 \qquad \dots \quad (21)$$

$$\mu_1=\lambda(k_1-1)=k_1-1 \qquad \dots \quad (22)$$

These, however, constitute the simplest type of such designs. For want of space we shall not consider them or any other particular case here and will proceed to obtain the expressions for the general case which considerably simplify for the above class of designs.

(2·2) *General Case*: In the general case, let r, v, k_1, k_2, b, n_1, n_2, μ_1, μ_2, a_{111}, a_{112}, a_{121}, a_{122}, a_{211}, a_{212}, a_{221} and a_{222} have the usual meaning. Also, let m, b_1, b_2, ... b_b $(\sum_1^b b_j = 0)$, v_1, v_2, ... v_v $(\sum_1^v v_j = 0)$ be the constants for the mean, block and varietal effects respectively. For brevity we shall denote the sum of the constants for varietal effects of varieties which are the first block associates of variety i by $\Sigma v_{(\mu_1,\, i)}$ and for varieties which are the second block associates of variety i by $\Sigma v_{(\mu_2,\, i)}$. Similarly, $\Sigma_{\mu_1} \Sigma v_{(\mu_1,\, i)}$ stands for the sum of constants for the varietal effects of the first block associates of the first block associates of variety i, the other expressions, viz., $\Sigma_{\mu_2} \Sigma v_{(\mu_1,\, i)}$, $\Sigma_{\mu_1} \Sigma v_{(\mu_2,\, i)}$ and $\Sigma_{\mu_2} \Sigma v_{(\mu_2,\, i)}$ having a similar meaning. Also, V_i denotes the total yield for the i-th variety and B_i the total yield for the i-th block.

The sum of squares due to varieties

$$= \frac{1}{k_1 k_2} (v_1 Q_1 + v_2 Q_2 + \dots\dots\dots + v_i Q_i + \dots\dots\dots + v_v Q_v) \qquad \dots \quad (23)$$

where $Q_i = k_1 k_2 V_i - k_2$ (sum of the totals for Blocks of the first type, *i. e.*, with k_1 plots, containing variety i) $-k_1$ (sum of the totals of Blocks of the second type, *i. e.*, with k_2 plots, containing variety i)

$$= \left(k_1 k_2 r - \frac{k_1 k_2 \mu_1}{v} - \frac{k_1 k_2 \mu_2}{v}\right) v_i - k_2 \Sigma v_{(\mu_1,\, i)} - k_1 \Sigma v_{(\mu_2,\, i)}$$

$$= (k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1) v_i + (k_1 - k_2) \Sigma v_{(\mu_1,\, i)} \qquad \dots \quad (24)$$

$$= (k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_2) v_i - (k_1 - k_2) \Sigma v_{(\mu_2,\, i)} \qquad \dots \quad (25)$$

since

$$v_i + \Sigma v_{(\mu_1,\, i)} + \Sigma v_{(\mu_2,\, i)} = 0 \qquad \dots \quad (26)$$

Denote by $\Sigma Q_{(\mu_1,\, i)}$, the sum of the Q's, μ_1 in number, corresponding to the first block associates of variety i; and by $\Sigma Q_{(\mu_2,\, i)}$, the sum of the Q's, μ_2 in number, corresponding to the second block associates of the variety i.

Then it immediately follows that $\Sigma Q_{(\mu_1,\, i)}$ is equal to :

$$(k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1) \Sigma v_{(\mu_1,\, i)} + (k_1 - k_2) \left[\mu_1 v_i + a_{111} \Sigma v_{(\mu_1,\, i)} + a_{211} \Sigma v_{(\mu_2,\, i)}\right], \quad (27)$$

(utilizing the parametric relations (12) to (18) above)

$$= (k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1) \Sigma v_{(\mu_1,\, i)} + (k_1 - k_2) \left[a_{212} v_i + (a_{111} - a_{211}) \Sigma v_{(\mu_1,\, i)}\right]; \quad (28)$$

whence, substituting for $\Sigma Q_{(\mu_1, i)}$ from equation (28) in equation (24), we obtain

$$v_i = \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)\ (a_{111} - a_{211})\right]\ Q_i - (k_1 - k_2)\ \Sigma Q_{(\mu_1, i)}}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right]\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)\ (a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212}} \qquad (29)$$

Similarly, working in terms of the second block associates, *i.e.*, using now equation (25), we obtain after simplification

$$v_i = \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_2 - (k_1 - k_2)\ (a_{222} - a_{122})\right] Q_i + (k_1 - k_2)\ \Sigma Q_{(\mu_2, i)}}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_2\right]\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_2 - (k_1 - k_2)\ (a_{222} - a_{122})\right] - (k_1 - k_2)^2 a_{112}}$$

$$(30)$$

Now, Sum of Squares due to Blocks

$$= \frac{1}{k_1}\left(B^2_1 + B^2_2 + \ldots\ldots\ldots + B^2_{n_1}\right) + \frac{1}{k_2}\left(B^2_{n_1+1} + \ldots\ldots + B^2_{n_1+n_2}\right) - G^2/N \quad \ldots \quad (31)$$

where $B_1, B_2, \ldots\ldots\ldots B_{n_1}$ denote total yields for blocks of the first type and $B_{n_1+1}, \ldots\ldots B_{n_1+n_2}$ those for blocks of the second type.

Also Sum of Squares due to Varieties

$$= \frac{1}{k_1 k_2} \sum_{i=1}^{v} v_i Q_i$$

$$= \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)\ (a_{111} - a_{211})\right] \sum\limits_{i=1}^{v} Q_i^2 - (k_1 - k_2) \sum\limits_{i=1}^{v} Q_i \{\Sigma Q_{(\mu_1, i)}\}}{k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right]\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)\ (a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212}}$$

$$\ldots \quad (32)$$

using equation (29). Another expression for the same sum of squares is obtained by using equation (30).

For analysis of covariance, Sum of Products due to Blocks

$$= \frac{1}{k_1}\left(B_1 B'_1 + B_2 B'_2 + \ldots\ldots\ldots + B_{n_1} B'_{n_1}\right) + \frac{1}{k_2}\left(B_{n_1+1} B'_{n_1+1} + \ldots\ldots\ldots + B_{n_1+n_2} B'_{n_1+n_2}\right) - G\,G'/N \quad \ldots \quad (33)$$

where dashes indicate the corresponding expressions for the other set of data.

Also Sum of Products due to Varieties

$$= \frac{1}{k_1 k_2} \sum_{i=1}^{v} v_i Q'_i$$

$$= \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211})\right] \sum_{i=1}^{v} Q_i Q'_i - (k_1 - k_2) \sum_{i=1}^{v} Q'_i \{\Sigma Q_{(\mu_1,\ i)}\}}{k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{11} - a_{211})\right] - (k_1 - k_2) a_{112}} \qquad \ldots \quad (34)$$

using equation (29). Another expression for the same sum of products is easily obtained by using equation (30).

Similarly, using the other expression

$$\frac{1}{k_1 k_2} \sum_{i=1}^{v} v'_i Q_i$$

we obtain two other similar expressions for the same sum of products.

Again, the variance of the difference between two varieties which are first block associates, using the equation (29),

$$= \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (a_{111} - a_{211} + 1)(k_1 - k_2)\right]}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{112}} \qquad (35)$$

Also, the variance of the difference between two second block associates

$$= \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211})\right]}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{112}} \qquad \ldots \quad (36)$$

Another similar expression may, as before, be obtained by using equation (30).

Now the total number of first block associates is $v\mu_1/2$ and that of second block associates $v\,\mu_2/2$, so that the mean variance of all comparisons is given by

$$V_m = \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211}) + \frac{\mu_1 (k_1 - k_2)}{v - 1}\right]}{\left(k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right) \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{112}} \qquad \ldots \quad (37)$$

Thus the efficiency factor in this case is

$$I = \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{112}}{k_1 k_2 r \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 + (k_1 - k_2)(a_{111} - a_{211}) + \frac{\mu_1 (k_1 - k_2)}{v - 1}\right]} \qquad \ldots \quad (38)$$

Another similar expression for the same can be easily worked out.

§3. DESIGNS WITH $\lambda \geqslant 2$.

Attention may now be directed to the other, and wider, class of symmetrical incomplete block arrangements with blocks of two unequal sizes for which $\lambda > 1$. It will be seen that these form a class apart in that in all these cases there will be intermixture among the first and second block associates of a variety. It is, therefore, necessary to modify our previous definition of the two block associates to meet this contingency.

It will be convenient to classify these arrangements under the two main heads, (i) those for which $n_1\, k_1/v = 1$, and (ii) those for which $n_1\, k_1/v > 1$, for any value of λ. In case (i), when (a) $\lambda = 2$, provided intermixture among the ordinary first and second block associates of a variety, as defined in §2, is not absent, or (b) when $\lambda > 2$, so that each of the ordinary first block associates occurs only once among blocks of the first type, but is repeated $(\lambda - 1)$ times in blocks of the second type, (besides the remaining varieties, which are each repeated λ times in blocks of the second type) we define the ordinary first block associates of a variety as its *intrinsic* first block associates, and the remaining varieties its *intrinsic* second block associates. Considering now case (ii), where a wide diversity in the possible types of designs prevails, it is easily seen that for (a) $\lambda = 2$, the designs existent are of exactly the same type as in (i), case (a) above, provided there is no absence of intermixture among the ordinary first and second block associates of a variety. As for (b) $\lambda > 2$, either the ordinary first block associates of a variety or its second block associates are constituted by a certain number of varieties, each occurring u times. These associates are repeated $(\lambda - u)$ times among blocks of the second or first type, and the remaining varieties occur λ times in blocks of that type; in this case the definition of the *intrinsic* first and second block associates is clear. For further elucidating our definition, two illustrative examples follow.

(3·1) Illustrative examples: The annexed table shows a design for which $v = 5$, $b = 10$, $k_1 = 2$, $k_2 = 3$, $n_1 = 5$, $n_2 = 5$, $r = 5$, $\mu_1 = 2$, $\mu_2 = 2$, $\lambda = 2$.

TABLE 1. SYMMETRICAL UNEQUAL BLOCK DESIGN

Block No.	Treatments: ($k_1 = 2$)		Block No.	Treatments: ($k_2 = 3$)		
1	1	2	6	1	2	5
2	1	3	7	1	3	4
3	2	4	8	1	4	5
4	3	5	9	2	3	4
5	4	5	10	2	3	5

Here varieties 2 and 3 are the intrinsic first block associates of variety 1, and varieties 4 and 5 its intrinsic second block associates. Similarly varieties 1, 4; 1, 5; 2, 5; 3, 4 are respectively the intrinsic first block associates of varieties 2, 3, 4 and 5, whilst varieties 3, 5; 2, 4; 1, 3; 1, 2 are respectively their intrinsic second block associates.

Table 2 shows another design for which $v=14$, $b=15$, $k_1=6$, $k_2=7$, $n_1=7$, $n_2=8$, $r=7$, $\mu_1=1$, $\mu_2=2$, $\lambda=3$.

TABLE 2. SYMMETRICAL UNEQUAL BLOCK DESIGN

Block No.	Treatments: ($k_1=6$)					
1	1	2	3	4	5	6
2	1	2	7	8	9	10
3	3	4	7	8	11	12
4	1	2	11	12	13	14
5	3	4	9	10	13	14
6	5	6	7	8	13	14
7	5	6	9	10	11	12

Block No.	Treatments: ($k_2=7$)						
8	1	3	5	8	9	12	14
9	1	4	6	7	10	12	14
10	2	3	6	7	11	11	14
11	1	3	5	7	10	11	13
12	1	4	6	8	9	11	13
13	2	3	6	8	9	12	13
14	2	4	5	7	10	12	13
15	2	4	5	8	9	11	14

In this case variety 2 is the *intrinsic* first block associate of variety 1, and varieties 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14, its *intrinsic* second block associates. Similarly, varieties 1, 4, 3 and 6 are respectively the *intrinsic* first block associates of varieties 2, 3, 4, and 5, whilst the remaining varieties in each case are respectively their *intrinsic* second block associates; and so on for the other varieties.

Then the sufficient condition for the existence of such a design, in the sense of its being amenable to generalization in regard to expressions for the estimates of the varietal effects, etc., is that, μ_1 and μ_2 being respectively the number of the *intrinsic* first and second block associates of any variety, and a_{111}, a_{112} $(=a_{121})$, a_{122}, a_{211}, a_{212} $(=a_{221})$, a_{222}, having precisely the same meaning as in §2, but now referring to *intrinsic* block associates, the parametric relations (12) to (18) should hold. The equation (11) is, of course, true. All these conditions are satisfied in the case of both the above designs. For the first design (vide Table 1) $a_{111}=0$, a_{112} $(=a_{121})=1$, $a_{122}=1$, $a_{211}=1$, a_{212} $(=a_{221})=1$, $a_{222}=0$.

For the second design (vide Table 2) $a_{111}=0$, a_{112} $(=a_{121})=0$, $a_{122}=12$, $a_{211}=0$, a_{212} $(=a_{221})=1$, $a_{222}=10$.

§ 4. ANALYSIS OF THE GENERAL SYMMETRICAL UNEQUAL BLOCK ARRANGEMENT

We may now proceed to perform the analysis in the general case where r, v, k_1, k_2, b, n_1, n_2, μ_1, μ_2, λ, a_{111}, a_{112} $(=a_{121})$, a_{122}, a_{211}, a_{212} $(=a_{221})$, a_{222} have the usual meaning, relevant parameters referring now to *intrinsic* block associates. Similarly, $\Sigma v_{(\mu_1,\, i)}$, $\underset{\mu_1}{\Sigma} \Sigma v_{(\mu_1,\, i)}$, etc., have the usual meaning explained in §2·2, reference, of course, being now to *intrinsic* block associates.

As already pointed out above, designs for which $n_1 k_1/v>1$ and $\lambda>2$ (vide §3, case ii (*b*)) are of an extensively diversified character. These comprise various classes of designs which it is not possible here for want of space to consider in their entirety. We would, therefore, content ourselves with dealing with the general class of designs for which either (a) λ $(=w)=2$, $n_1 k_1/v$ being any integer whatever (excepting for cases in which intermixture among the two ordinary block associates of a variety is absent); or (*b*) $\lambda=w$ (>2) and $n_1 k_1/v=1$. This is the class of designs which is of frequent occurrence and includes as a particular case that for which $\lambda=1$.

In either case the sum of squares for varieties

$$= \frac{1}{k_1 k_2}(v_1 Q_1 + v_2 Q_2 + \ldots + v_i Q_i + \ldots\ldots + v_v Q_v) \qquad \ldots \quad (39)$$

where $Q_i = k_1 k_2 V_i - k_2$ (Sum of Totals for Blocks of the first type containing variety i) $-k_1$ (Sum of the Totals for Blocks of the second type containing variety i)

$= \left(k_1 k_2 r - \frac{k_1 k_2 b}{v}\right) v_i - k_2$ (Sum of the constants for varietal effects corresponding to the ordinary first block associates of variety i) $- k_1$ (Sum of the constants for varietal effects corresponding to the ordinary second block associates of variety i)

$$= (k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w) v_i + (k_1 - k_2)\ \Sigma v_{(\mu_1, i)} \qquad \ldots \quad (40\cdot1)$$

$$= \{k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_2 + (w-1)k_1\}\ v_i - (k_1 - k_2)\ \Sigma v_{(\mu_2, i)} \qquad \ldots \quad (40\cdot2)$$

since,

$$v_i + \Sigma v_{(\mu_1, i)} + \Sigma v_{(\mu_2, i)} = 0 \qquad \ldots \quad (41)$$

Firstly, then, utilizing equation (41), and denoting as before by $\Sigma Q_{(\mu_1, i)}$ and $\Sigma Q_{(\mu_2, i)}$ the sums of the Q's, μ_1 and μ_2 in number respectively, corresponding to the *intrinsic* first and second block associates of variety i, we obtain

$$\Sigma\, Q_{(\mu_1, i)} = (k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w)\ \Sigma v_{(\mu_1, i)} + (k_1 - k_2) \sum_{\mu_1} \Sigma v_{(\mu_1, i)}$$

$$= [k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)\ (a_{111} - a_{211})]\ \Sigma v_{(\mu_1, i)}.$$

$$+ (k_1 - k_2)\ a_{212}\ v_i \qquad \ldots \quad (42)$$

whence, after substitution in equation (40·1) and some simplification, it immediately follows that $v_i = U/V$ ... (43)

where,

$$U = [k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)\ (a_{111} - a_{211})]\ Q_i - (k_1 - k_2)\ \Sigma Q_{(\mu_1, i)} \qquad \ldots \quad (43\cdot1)$$

$$V = [k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w]\ [k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)\ (a_{111} - a_{211})]$$

$$- (k_1 - k_2)^2\, a_{212} \qquad \ldots \quad (43\cdot2)$$

Now, Sum of Squares due to Blocks

$$= \frac{1}{k_1}(B^2_1 + B^2_2 + \ldots\ldots\ldots + B^2_{n_1}) + \frac{1}{k_2}(B^2_{n_1} + \ldots\ldots\ldots\ B^2_{n_1+n_2}) - \frac{G^2}{N} \qquad \ldots \quad (44)$$

And Sum of Squares due to Varieties

$$= \frac{1}{k_1\ k_2} \sum_{i=1}^{v} v_i\, Q_i = \frac{U'}{V'}, \qquad \ldots \quad (45)$$

where

$$U' = [k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)\ (a_{111} - a_{211})] \sum_{i=1}^{v} Q_i^2 - (k_1 - k_2) \sum_{i=1}^{v} Q_i \{\Sigma Q(\mu_1\ i)\}$$

$$\ldots \quad (45\cdot1)$$

$$V' = k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212} \quad \ldots \quad (45{\cdot}2)$$

For analysis of covariance, Sum of Products due to Blocks

$$= \frac{1}{k_1}(B_1 B'_1 + \ldots \ldots\ldots B_{n_1} B'_{n_1}) + \frac{1}{k_2}(B_{n_1+1} B'_{n_1+1} + \ldots\ldots\ldots + B_{n_1+n_2} B'_{n_1+n_2}) - \frac{G\,G'}{N}. \quad \ldots \quad (46)$$

as in (33) above, and the Sum of Products due to Varieties

$$= \frac{1}{k_1 k_2} \sum_{i=1}^{v} v_i\, Q'_i = \frac{U''}{V''}, \quad \ldots \quad (47)$$

where

$$U'' = \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] \sum_{i=1}^{v} Q_i\, Q'_i - (k_1 - k_2) \sum_{i=1}^{v} Q'_i \{\Sigma\, Q(\mu_1, i) \quad \ldots \quad (47{\cdot}1)$$

$$V'' = k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212} \quad \ldots \quad (47{\cdot}2)$$

Again, the variance of the difference between two varieties which are first block associates

$$= \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211} + 1)\right]}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212}} \quad (48)$$

and the variance of the difference between two second block associates

$$= \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right]}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212}} \quad (49)$$

Now $v\mu_1/2$ and $v\mu_2/2$ being respectively the total number of *intrinsic* first and second block associates, we have, for the mean variance of all comparisons,

$$V_m = \frac{2\sigma^2 k_1 k_2 \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211}) + \frac{\mu_1 (k_1 - k_2)}{v - 1}\right]}{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_2} \quad \ldots \quad (50)$$

Hence the relative efficiency is given by

$$I = \frac{\left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w\right] \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211})\right] - (k_1 - k_2)^2 a_{212}.}{k_1 k_2 r \left[k_1 k_2 r - \frac{k_1 k_2 b}{v} + k_1 w + (k_1 - k_2)(a_{111} - a_{211}) + \frac{\mu_1 (k_1 - k_2)}{v - 1}\right]} \quad \ldots \quad (51)$$

Another set of expressions corresponding to (43), (45), (47), (48), (49), (50) and (51) can also be worked out, but is omitted here for want of space.

§5. AN INTERESTING RESULT

The present writer's investigations led him on to an interesting result on which a few observations may not be out of place here. The procedure of cutting out a variety from a symmetrical incomplete block arrangement gives two groups of different block sizes, each of which in itself constitutes in general a partially balanced incomplete block design, and in some special cases, a symmetrical incomplete block design, which is only a particular case of partially balanced incomplete block arrangements. In the latter case, the unequal block design has identical *intrinsic* first and second block associates; and if the hypothetical weights for the different error variances for plots from different block sizes are taken as equal in accordance with the assumption made throughout this paper, the differences between any two varietal means will have the same error variance. Thus for $s=2$, the infinite series of designs given in § 7.33 breaks into the two symmetrical incomplete block arrangements:

(*i*) $v=7,\ b=7,\ k=3,\ r=3,\ \lambda=1$ and (*ii*) $v=7,\ b=7,\ k=4,\ r=4,\ \lambda=2$. For values of $s>2$, this series yields a design in which each of the two sets of block sizes individually constitutes a partially balanced incomplete block design. As an interesting way of constructing some special cases of partially balanced incomplete block designs and symmetrical incomplete block designs this method may be noted.

§6. CONSTRUCTION OF DESIGNS BY VARIETY-CUTTING

We finally come to the consideration of methods of construction of symmetrical incomplete block arrangements with blocks of unequal sizes. No attempt is made to exhaustively enumerate all possible methods of obtaining the various combinatorial solutions and in this section only a few of those will be given which are directly derivable from a known incomplete randomized block arrangement by cutting out a single variety from all the blocks. In the next section a few infinite series of designs obtained by utilizing the properties of finite hyperdimensional projective and affine geometries constructed from the Galois fields will be given. The algebraical, configurational and other methods of constructing these designs will be considered in a subsequent communication. The following are some of the designs obtained by the process of variety-cutting :-

(6·1) $v=6,\ b=7,\ k_1=2,\ k_2=3,\ n_1=3,\ n_2=4,\ r=3,\ a_{111}=0,\ a_{112}=a_{121}=0,$
$a_{122}=4,\ a_{211}=0,\ a_{212}=a_{221}=1,\ a_{222}=2,\ \mu_1=1,\ \mu_2=4,\ \lambda=1.$

(6·2) $v=12,\ b=13,\ k_1=3,\ k_2=4,\ n_1=4,\ n_2=9,\ r=4,\ a_{111}=1,\ a_{112}=a_{121}=0,$
$a_{122}=9,\ a_{211}=0\ \ a_{212}=a_{221}=2,\ a_{222}=6,\ \mu_1=2,\ \mu_2=9.\ \ \lambda=1.$

(6·3) $v=20,\ b=21,\ k_1=4,\ k_2=5,\ n_1=5,\ n_2=16,\ r=5,\ a_{111}=2,\ a_{112}=a_{121}=0,$
$a_{122}=16,\ a_{211}=0,\ a_{212}=a_{221}=3,\ a_{222}=12,\ \mu_1=3,\ \mu_2=16.\ \ \lambda=1.$

(6·4) $v=30,\ b=31,\ k_1=5,\ k_2=6,\ n_1=6,\ n_2=25,\ r=6,\ a_{111}=3,\ a_{112}=a_{121}=0,$
$a_{122}=25,\ a_{211}=0,\ a_{212}=a_{221}=4,\ a_{222}=20,\ \mu_1=4,\ \mu_2=25,\ \lambda=1.$

(6·5) $v=8,\ b=12,\ k_1=2,\ k_2=3,\ n_1=4,\ n_2=8,\ r=4,\ a_{111}=0,\ a_{112}=a_{121}=0,\ a_{122}=6,$
$a_{211}=0,\ a_{212}=a_{221}=1,\ a_{222}=4,\ \mu_1=1,\ \mu_2=6,\ \lambda=1.$

(6·6) $v=15,\ b=20,\ k_1=3,\ k_2=4,\ n_1=5,\ n_2=15,\ r=5,\ a_{111}=1,\ a_{112}=a_{121}=0,$
$a_{122}=12,\ a_{211}=0,\ a_{212}=a_{221}=2,\ a_{222}=9,\ \mu_1=2,\ \mu_2=12,\ \lambda=1.$

(6·7) $v=63,\ b=72,\ k_1=7,\ k_2=8,\ n_1=9,\ n_2=63,\ r=9,\ a_{111}=5,\ a_{112}=a_{121}=0,$
$a_{122}=56,\ a_{211}=0,\ a_{212}=a_{221}=6,\ a_{222}=49,\ \mu_1=6,\ \mu_2=56,\ \lambda=1.$

(6·8) $v=15,\ b=16,\ k_1=5,\ k_2=6,\ n_1=6,\ n_2=10,\ r=6,\ a_{111}=4,\ a_{112}=a_{121}=3,$
$a_{122}=3,\ a_{211}=4,\ a_{212}=a_{221}=4,\ a_{222}=1,\ \mu_1=8,\ \mu_2=6,\ \lambda=2.$

(6·9) $v=9$, $b=15$, $k_1=3$, $k_2=4$, $n_1=6$, $n_2=9$, $r=6$, $a_{111}=1$, $a_{112}=a_{121}=2$, $a_{122}=2$, $a_{211}=2$, $a_{212}=a_{221}=2$, $a_{222}=1$, $\mu_1=4$, $\mu_2=4$, $\lambda=2$.

§7. Construction of Designs by The Method of Finite Geometries

It is not possible in the limited space at our disposal to do more than discuss very briefly the definition, incidence relations and other properties of the finite hyper-dimensional projective and affine geometries PG(m, s) and EG(m, s) respectively constructed from the Galois fields. It is, however, hoped that what is given here will suffice to elucidate the method of obtaining the various infinite series of symmetrical unequal block designs given in this section.

(7·1) *Definitions*: The following are the five postulates by which a finite projective geometry may be defined:—

1. The set has a finite number of points, and contains one or more sub-sets, each of which has at least three points on it.

2. There exists at least one line passing through each of two distinct points, and there is one and only one such line.

3. If A, B, C are points not all incident with the same line, and if a line u passes through a point D of the line AB and a point E of the line BC, but does not contain A or B or C, then the line u passes through a point F of the line CA.

4. Not all the points in a given m-space are contained in a sub-space of it.

5. There does not exist an $(m+1)$-space in the set of points defining a given m-space.

The geometry defined in this manner is spoken of as a geometry of m-dimensional space, a point being a 0-space and a line a 1-space.

The Galois fields always enable us to construct such a geometry, satisfying postulates 1 – 5. The geometries thus obtained are finite Desarguesian geometries, but there exist others, Desarguesian and non-Desarguesian, which are not derivable from the Galois fields. These latter, however, will not be considered here.

(7·2) *Finite geometries constructed from the Galois Fields*: We represent the points of the m-dimensional geometry uniquely by the set of homogenous co-ordinates $(u_0, u_1, \ldots\ldots\ldots\ldots u_m)$ where $u_0, u_1, \ldots\ldots\ldots u_m$ are elements of GF(s), at least one of which is different from zero, and s is a prime number, or a power of a prime. It is understood that the set $(u_0, u_1, u_2 \ldots\ldots\ldots u_m)$ denotes the same point as $(uu_0, uu_1, \ldots\ldots\ldots uu_m)$, where u is any of the $(s-1)$ non-zero elements of the field. Now the number of ordered sets of elements $(u_0, u_1 \ldots\ldots\ldots u_m)$ is $s^{m+1}-1$, of which each of $(s-1)$ sets of elements represents the same point. Thus the totality of points in the given m-space

$$=\frac{s^{m+1}-1}{s-1}=s^m+s^{m-1}+\ldots\ldots\ldots+s^3+s^2+s+1.$$

All the five posulates 1 – 5 are then easily seen to hold for the finite Desarguesian geometries thus constructed, which are denoted symbolically by PG(m, s). When, however, only the finite elements of the geometries are considered, the affine Desarguesian geometries denoted by EG(m, s) are obtained.

Also the number of l-dimensional spaces PG(l, s) in the given PG(m, s) $(l<m)$

$$=\frac{(s^{m+1}-1)(s^m-1)(s^{m-1}-1)\ldots\ldots\ldots\ldots(s^{m-l+1}-1)}{(s^{l+1}-1)(s^l-1)(s^{l-1}-1)\ldots\ldots\ldots\ldots(s-1)} \quad \ldots \quad (52)$$

Again, a k-space and l-space in the given m-space $(k, l, < m)$ intersect in a $(k+l-m)$-space. Also, by the principle of duality, a 0-space (or point) has the same properties as an $(m-1)$-space; a 1-space (or line) has the same properties as an $(m-2)$-space. In general, an l-space possesses the same properties as an $(m-l-1)$-space. When m is odd, the $\left(\frac{m-1}{2}\right)$-space is self-dual.

It follows that from every theorem of PG(m, s) can be derived another new theorem of PG(m, s) by substituting $(m-1)$-space for 0-space, $(m-2)$-space for 1-space and, in general $(m-l-1)$-space for l-space, for all values of $l < m$. By utilizing this principle of duality extensively in conjunction with the formula (52) above, all the various incidence relations subsisting in the given m-space are easily derived.

Omitting the infinite entities in the given PG(m, s), all the incidence relations existing in EG(m, s) are easily deducible.

(7·3) *Designs*: We give below the infinite series of designs derived from PG(m, s) and EG(m, s) discussed above.

(7·31) *Designs derivable from* EG$(2, s)$: Taking lines to represent blocks, points to represent varieties and cutting out all points on any line, we obtain the infinite series of designs given by $v=s^2-s$, $b=s^2+s-1$, $k_1=s-1$, $k_2=s$, $n_1=s^2$, $n_2=s-1$, $r=s+1$. $a_{111}=s(s-3)$, $a_{112}=a_{121}=s-1$, $a_{122}=0$, $a_{211}=s(s-2)$, $a_{212}=a_{221}=0$. $a_{222}=s-2$, $\mu_1=s(s-2)$, $\mu_2=s-1$, $\lambda=1$.

The design obtained by setting $s=2$ is trivial.

(*i*) For $s=3$, : $v=6$, $b=11$, $k_1=2$, $k_2=3$, $n_1=9$, $n_2=2$, $r=4$, $a_{111}=0$, $a_{112}=a_{121}=2$, $a_{122}=0$, $a_{211}=3$, $a_{212}=a_{221}=0$, $a_{222}=1$, $\mu_1=3$, $\mu_2=2$, $\lambda=1$.

(*ii*) For $s=4$: $v=12$, $b=19$, $k_1=3$, $k_2=4$, $n_1=16$, $n_2=3$, $r=5$, $a_{111}=4$, $a_{112}=a_{121}=3$, $a_{122}=0$, $a_{211}=8$, $a_{212}=a_{221}=0$, $a_{222}=2$, $\mu_1=8$, $\mu_2=3$, $\lambda=1$.

(*iii*) For $s=5$: $v=20$, $b=29$, $k_1=4$, $k_2=5$, $n_1=25$, $n_2=4$, $r=6$, $a_{111}=10$, $a_{112}=a_{121}=4$, $a_{122}=0$, $a_{211}=15$, $a_{212}=a_{221}=0$, $a_{222}=3$, $\mu_1=15$, $\mu_2=4$, $\lambda=1$.

In the same way we could consider the cases $s=7,8,9$.

(7·32) *Designs derivable from* PG$(3, s)$: (i) Taking planes as blocks and points as varieties, the cutting of a single line yields the finite series of designs given by $v=s^3+s^2$, $b=s^3+s^2+s+1$, $k_1=s^2$, $k_2=s^2+s$, $n_1=s+1$, $n_2=s^3+s^2$, $r=s^2+s+1$, $a_{111}=s^2-2$, $a_{112}=a_{121}=0$, $a_{122}=s^2$, $a_{211}=0$, $a_{212}=a_{221}=s^2-1$, $a_{222}=s^2(s-1)$, $\mu_1=s^2-1$, $\mu_2=s^2$, $\lambda=s+1$.

(*ii*) Taking lines to represent blocks and points to represent varieties, we obtain, by cutting out a single point, the infinite series of designs given by $v=s^3+s^2+s$, $b=(s^2+1)$, (s^2+s+1), $k_1=s$, $k_2=s+1$, $n_1=s^2+s+1$, $n=s^2(s^2+s+1)$, $r=s^2+s+1$, $a_{111}=s-2$, $a_{112}=a_{121}=0$, $a_{122}=s^3+s^2$, $a_{211}=0$, $a_{212}=a_{221}=s-1$, $a_{222}=(s^2+s-1)s$, $\mu_1=s-1$, $\mu_2=s^3+s^2$, $\lambda=1$.

(*iii*) Taking planes to represent blocks and points to represent varieties, we have, after cutting out a single point standing for a variety, the infinitie series of designs given by $v=s^3+s^2+s$, $b=s^3+s^2+s+1$, $k_1=s^2+s$, $k_2=s^2+s+1$, $n_1=s^2+s+1$, $n_2=s^3$, $r=s^2+s+1$, $a_{111}=s-2$, $a_{112}=a_{121}=0$, $a_{122}=s^3+s^2$, $a_{211}=0$, $a_{212}=a_{221}=s-1$, $a_{222}=s(s^2+s-1)$, $\mu_1=s-1$, $\mu_2=s^3+s^2$, $\lambda=s+1$.

(iv) Taking lines as blocks and points as before to represent varieties, we obtain, after cutting out all points on a single line, the infinite series of designs given by $v=s^3+s^2$, $b=(s^2+1)\ (s^2+s+1)-1$, $k_1=s$, $k_2=s+1$, $n_1=s(s+1)^2$, $n_2=s^4$, $r=s^2+s+1$, $a_{111}=s^2-2$, $a_{112}=a_{121}=0$, $a_{122}=s^2$, $a_{211}=0$, $a_{212}=a_{221}=s^2-1$, $a_{222}=s^2(s-1)$, $\mu_1=s^2-1$, $\mu_2=s^2$, $\lambda=1$.

In each of the above casses (*i*), (*ii*), (*iii*), (*iv*) we get special designs by putting $s=2$ or 3. Designs with higher values of s are of more academic interest having as they do large sizes of blocks and very high replications.

(7·33) *Designs derivable from* EG(3, *s*): (*i*) Taking planes as blocks, points as varieties and cutting out a single point, we obtain the infinite series of designs $v=s^3-1$, $b=s^3+s^2+s$, $k_1=s^2-1$, $k_2=s^2$, $n_1=s^2+s+1$, $n_2=s^3-1$, $r=s^2+s+1$, $a_{111}=s-3$, $a_{112}=a_{121}=0$, $a_{122}=s(s^2-1)$, $a_{211}=0$, $a_{212}=a_{221}=s-2$, $a_{222}=(s-1)\ (s^2+s-1)$, $\mu_1=s-2$, $\mu_2=s(s^2-1)$, $\lambda=s+1$.

(*ii*) Taking planes as before to represent blocks, points to represent varieties and cutting out a single plane, we obtain the infinite series of designs given by $v=s^3-s^2$, $b=s^3+s^2+s-1$, $k_1=s^2-s$, $k_2=s^2$, $n_1=s^3+s^2$, $n_2=s-1$, $r=s^2+s+1$, $a_{111}=s^2(s-3)$, $a_{112}=a_{121}=s^2-1$, $a_{122}=0$, $a_{211}=s^2(s-2)$, $a_{212}=a_{221}=0$, $a_{222}=s^2-2$, $\mu_1=s^2(s-2)$, $\mu_2=s^2-1$, $\lambda=s+1$.

(*iii*) Taking lines for blocks, varieties for points and cutting out a single point, we obtain the infinite series $v=s^3-1$, $b=s^2(s^2+s+1)$, $k_1=s-1$, $k_2=s$, $n_1=s^2+s+1$, $n_2=(s^2-1).\ (s^2+s+1)$, $r=s^2+s+1$, $a_{111}=s-3$, $a_{112}=a_{121}=0$, $a_{122}=s(s^2-1,)$ $a_{211}=0$, $a_{212}=a_{221}=s-2$, $a_{222}=(s-1)\ (s^2+s-1)$, $\mu_1=s-2$, $\mu_2=s(s^2-1)$, $\lambda=1$.

(*iv*) Taking lines for blocks, points for varieties and cutting out all points on a single line, we obtain in the series $v=s^3-s$, $b=s^2(s^2+s+1)-1$, $k_1=s-1$, $k_2=s$, $n_1=(s^2+s)s$, $n_2=s^4-1$, $r=s^2+s+1$, $a_{111}=s^2-2s-1$, $a_{112}=a_{121}=0$, $a_{122}=(s^2+1)(s-1)$, $a_{211}=0$, $a_{212}=a_{221}=s(s-2)$, $a_{222}=(s-1)\ (s^2-s+2)$, $\mu_1=s(s-2)$, $\mu_2=(s^2+1)\ (s-1)$, $\lambda=1$.

(*v*) Take lines as blocks, points as varieties and cut out all points on a single plane. This yields the series $v=s^3-s^2$, $b=s(s^3+s^2-1)$, $k_1=s-1$, $k_2=s$, $n_1=s^4$, $n_2=s(s^2-1)$, $r=s^2+s+1$, $a_{111}=s(s-3)$, $a_{112}=a_{121}=s^2-1$, $a_{122}=0$, $a_{211}=s^2(s-2)$, $a_{212}=a_{221}=0$, $a_{222}=s^2-2$, $\mu_1=s^2(s-2)$, $\mu_2=s^2-1$, $\lambda=1$.

In each of the above cases (*i*), (*ii*), (*iii*), (*iv*) and (*v*), we get designs of practical interest by putting $s=2$ and 3 (some of the designs for $s=2$ being trivial).

(7·34) *Designs derivable from* PG(4, *s*): Take hyperplanes to represent blocks, points to represent varieties and cut out all points on a single plane. This gives the infinite series $v=s^4+s^3$, $b=s^4+s^3+s^2+s+1$, $k_1=s^3$, $k_2=s^3+s^2$, $n_1=s+1$, $n_2=s^4+s^3+s^2$, $r=s^3+s^2+s+1$, $a_{111}=s^2-2$, $a_{112}=a_{121}=0$, $a_{122}=s^4$, $a_{211}=0$, $a_{212}=a_{221}=s^3-1$, $a_{222}=s^3(s-1)$, $\mu_1=s^3-1$, $\mu_2=s^4$, $\lambda=s^2+s+1$.

Giving to s the value 2, we obtain

(i) $v=24$, $b=31$, $k_1=8$, $k_2=12$, $n_1=3$, $n_2=28$, $r=15$, $a_{111}=0$, $a_{112}=a_{121}=0$, $a_{122}=16$, $a_{211}=0$, $a_{212}=a_{221}=7$, $a_{222}=8$, $\mu_1=7$, $\mu_2=16$, $\lambda=7$.

(7.35) Various other designs, purely of academic interest on account of involving large sizes for blocks and very high replications, are deducible in a similar manner from four and higher dimensional projective and affine Desarguesian geometries. These, therefore, will not be considered here.

§8. CONCLUSION

Investigations of unequal block designs with blocks of three or more different sizes have also been undertaken, and the author has already obtained a number of such arrangements satisfying the requisite parametric conditions. In this paper only the more useful of the symmetrical unequal block arrangements, viz., those involving two different *k's*, have been considered and the author hopes to discuss the others at length in subsequent papers.

SUMMARY

In this paper the problem of the construction and analysis of symmetrical incomplete block arrangements with blocks of two unequal sizes has been investigated in detail. All such designs have been classified under the two main heads, (i) those for which $\lambda = 1$ and (ii) those for which $\lambda \geqslant 2$. The parametric relations sufficient for the existence of both these types of designs have been stated. General expressions for varietal effects, sum of squares and products for blocks and varieties (for analysis of variance and covariance), mean variance and relative efficiency have been worked out for designs (i); and in case of (ii), for designs for which either (a) $\lambda = 2$, n_1k_1/v being any integer, intermixture among the first and second block associates of a variety not being absent, or (b) $\lambda > 2$, $n_1k_1/v = 1$. Methods of construction of these designs either by variety-cutting or by utilizing the properties of finite hyper-dimensional projective and affine geometries associated to the Galois Fields have also been given and a number of infinite series of these designs have been enumerated.

REFERENCES

1. BOSE, R. C. AND NAIR, K. R.: Partially Balanced Incomplete Block Designs. *Sankhyā: The Indian Journal of Statistics*, Vol. 4, Part 3. 1939, pp. 337-372
2. CARMICHÆL, ROBERT D.: *Introduction to the Theory of Groups of Finite Order.* Boston, U. S. A. and London: Ginn & Co., 1937.
3. FISHER, R. A. AND YATES, F.: *Statistical Tables for Biological, Agricultural and Medical Research.* London and Edinburgh: Oliver Boyd, 1938.
4. WILKS, S. S.: Analysis of Variance and Covariance In Non-orthogonal Data. *Metron*, Vol. XIII, No. 2, 1938, pp. 141-154.
5. WILKS, S. S.: The Analysis of Variance For Two Or More Variables. *Report of the Third Annual Research in Economics*, Colorado Springs, U. S. A., 1937.
6. YATES, F.: The Principles Of Orthogonality And Confounding In Replicated Experiments. *J. Agri. Sci.*, Vol. XXIII, Part 1, pp. 108-145.
7. YATES, F.: Incomplete Latin Squares. *J. Agri. Sci.*, Vol. XXVI, Part II, pp. 301-315.
8. YATES, F.: Incomplete Randomized Blocks. *Annals of Eugenics*, Vol. VII, Part II, pp. 121-140.
9. YATES, F.: A New Method Of Arranging Variety Trials Involving A Large Number Of Varieties. *J. Agri. Sci.*, Vol. XXVI, Part II, pp. 301-315.
10. YATES, F.: The Analysis Of Multiple Classifications With Unequal Number In The Different Classes. *J. Amer. Stat. Ass.*, Vol. XXIX, 1934, pp. 51-66.