isid/ms/2003/01
January 16, 2003
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On the coverage of space by random sets

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Abstract

Let $\xi_1, \xi_2, \ldots$ be a Poisson point process of density λ on $(0,\infty)^d$, $d \geq 1$ and let $\rho, \rho_1, \rho_2, \ldots$ be i.i.d. positive random variables independent of the point process. Let $C := \cup_{i\geq 1}\{\xi_i + [0,\rho_i]^d\}$. If, for some $t > 0$, $(t,\infty)^d \subseteq C$, then we say that $(0,\infty)^d$ is eventually covered. We show that the eventual coverage of $(0,\infty)^d$ depends on the behaviour of $xP(\rho > x)$ as $x \to \infty$ as well as on whether $d = 1$ or $d \geq 2$. These results are quite dissimilar to those known for complete coverage of $\mathbb{R}^d$ by such Poisson Boolean models (Hall [3]).

In addition, we consider the region $C := \cup_{\{i\geq 1:\, X_i=1\}}[i, i+\rho_i]$, where $X_1, X_2, \ldots$ is a $\{0,1\}$ valued Markov chain and $\rho, \rho_1, \rho_2, \ldots$ are i.i.d. positive integer valued random variables independent of the Markov chain. We study the eventual coverage properties of this random set C.

1 Introduction

In this paper we address two issues. One of these arises from genome analysis, while the other complements the results on complete coverage in stochastic geometry.

In genomics, contig analysis is the method employed in sequencing or identifying the nucleotides of a DNA sequence. This method involves cloning to obtain many identical copies of the sequence. Each such copy is then fragmented (by bio-chemical means) into many *contigs* or random segments, with each contig being of random length and starting from some random point of the sequence. After sequencing each of the random segments obtained from all the

*Research supported in part by the DST GRANT MS/152/01.

Keywords : Complete coverage, Renewal theorem, Markov chain, Poisson process, Boolean model

Classification: 60k35

clones, they are then 'stitched' together by stitching all pairs of segments such that the end portion of one of the pair has a significant match with the beginning portion of the other. For more details see Ewens and Grant [1]. The question of interest here is what should be the random mechanism to guarantee that the sequence is significantly covered by the contigs. Mathematically, we think of the sequence to be of infinite length and with a fixed starting point. We have a segment S_i of random length ρ_i starting from the ith point of the sequence (S_i may be empty). The question now is that given a random mechanism of choosing the points i such that S_i is non-empty, what should ρ_i be such that these S_i's together cover the sequence significantly.

More formally, let $X_1, X_2, \ldots$ be a $\{0, 1\}$ valued time homogeneous Markov chain and $\rho_1, \rho_2, \ldots$ be an i.i.d. positive integer valued sequence of random variables, independent of the Markov chain. Let

$$S_i := \begin{cases} [i, i+\rho_i] & \text{if } X_i = 1 \\ \emptyset & \text{if } X_i = 0, \end{cases}$$

and $C := \cup_{i=1}^{\infty} S_i$ be the sequence obtained by stitching the segments. Of course, C may not completely cover $\mathbb{N}$, however it may be the case that barring an initial piece, C covers the rest of $\mathbb{N}$.

Definition 1.1. *$\mathbb{N}$ is said to be eventually covered by C if there exists $t \geq 1$ such that $[t, \infty) \subseteq C$.*

Here if $X_i = 1$ then we obtain the contig $[i, i+\rho_i]$ and two overlapping contigs are assumed to be obtained from two cloned copies of the sequence. The Markovian structure generally assumed for a DNA sequence is incorporated by considering $X_1, X_2, \ldots$ to be a Markov chain.

To formulate our first result, let the transition probability matrix of the Markov chain be given by $\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$, where $p_{ij} := P(X_{n+1} = j | X_n = i)$.

Theorem 1.1. *Assume that $0 < p_{00}, p_{10} < 1$.*

(a) If $l = \liminf_{j\to\infty} jP(\rho_1 > j) > 1$, then $P\{C$ eventually covers $\mathbb{N}\} = 1$ whenever $\frac{p_{01}}{p_{10}+p_{01}} > \frac{1}{l}$.

(b) If $L = \limsup_{j\to\infty} jP(\rho_1 > j) < \infty$, then $P\{C$ eventually covers $\mathbb{N}\} = 0$ whenever $\frac{p_{01}}{p_{10}+p_{01}} < 1/L$.

In stochastic geometry a question of interest is the complete coverage of a given region by a collection of random shapes, where the random shapes are placed according to a well-behaved point process. The most common model used is the Poisson Boolean model (Ξ, λ, ρ), which consists of a Poisson point process $\xi_1, \xi_2, \ldots$ of density λ on $\mathbb{R}^d$, $d \geq 1$ and i.i.d. random variables $\rho, \rho_1, \rho_2, \ldots$ independent of the point process. Let $B(\mathbf{0}, \rho)$ denote the closed d-dimensional ball of radius ρ centred at the origin $\mathbf{0}$. One then studies the (random) covered region $\cup_{i=1}^{\infty}(\xi_i + B(\mathbf{0}, \rho_i))$ of $\mathbb{R}^d$.

We provide a brief summary of the mathematical literature. It is known that

Proposition 1.1. (Hall [3], Theorem 3.1) *For the Poisson Boolean model (Ξ, λ, ρ) on $\mathbb{R}^d$, we have $\mathbb{R}^d = \cup_{i=1}^{\infty}(\xi_i + B((\mathbf{0}, \rho_i))$ almost surely if and only if $E\rho^d = \infty$.*

When the driving process of the Boolean model is not Poisson, but an arbitrary ergodic process, $\mathbb{R}^d = \cup_{i=1}^{\infty}(\xi_i + B((\mathbf{0}, \rho_i))$ almost surely if $E\rho^d = \infty$ (Meester and Roy [6], Proposition 7.3). Also, regarding percolation properties of the Poisson Boolean model, Tanemura [8] has shown that the critical parameters of percolation are the same for both the whole space as well as the orthant.

For the coverage of the real line $\mathbb{R}$, Mandelbrot [5] introduced the terminology *interval processes* and Shepp [7] showed that if S is an inhomogeneous Poisson point process on $\mathbb{R} \times [0, \infty)$ with density measure $\lambda \times \mu$ where λ is the Lebesgue measure on the x-axis and μ is a given measure on the y-axis, then the union of the intervals $(x, x + y)$ for Poisson points $(x, y) \in S$ covers $\mathbb{R}$ almost surely if and only if $\int_0^1 dx \exp(\int_x^{\infty}(y - x)\mu(dy)) = \infty$. Shepp also considered random Cantor sets, which is defined as follows. Let $1 \geq t_1 \geq t_2 \geq \ldots$ be a sequence of positive numbers decreasing to 0 and let $\mathcal{P}_1, \mathcal{P}_2, \ldots$ be Poisson point processes on $\mathbb{R}$, each with density λ. The set $V := \mathbb{R} \setminus (\cup_i \cup_{x \in \mathcal{P}_i} (x, x + t_i))$ is the random Cantor set. He showed that V has Lebesgue measure 0 if and only if $\sum_i t_i = \infty$. Moreover, $P(V = \emptyset) = 0$ or 1 according as $\sum_{n=1}^{\infty} n^{-2} \exp\{\lambda(t_1 + \cdots + t_n)\}$ converges or diverges (see also Hall [3], Theorem 3.20.)

In this paper we take $\mathbb{R}_+^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1, \ldots, x_d > 0\}$ and consider the random covered region $C := \cup_{\{i : \xi_i \in \mathbb{R}_+^d\}}(\xi_i + [0, \rho_i]^d)$. Clearly C will never completely cover $\mathbb{R}_+^d$ because, for any $\epsilon > 0$ $[0, \epsilon]^d$ will not be covered by C with positive probability. However it may be the case that for some $t > 0$, the region $\{(x_1, \ldots, x_d) : x_i \geq t \text{ for } i = 1, \ldots, d\}$ is entirely covered by C. Thus in analogy with the notion of complete coverage for the space $\mathbb{R}^d$, we have the following notion of eventual coverage for the orthant $\mathbb{R}_+^d$.

Definition 1.2. *$\mathbb{R}_+^d$ is said to be eventually covered by the Poisson Boolean model (Ξ, λ, ρ) if there exists $0 < t < \infty$ such that $(t, \infty)^d \subseteq C$.*

Our results on point processes focus on eventual coverage of $\mathbb{R}_+^d$ by the points of the Poisson Boolean model (Ξ, λ, ρ) situated in $\mathbb{R}_+^d$. Here even when $E\rho = \infty$, whether eventual coverage occurs or not depends on the growth rate of the distribution function of ρ as is shown in the following results.

Theorem 1.2. *For $d = 1$,*

(a) if $0 < l := \liminf_{x\to\infty} xP(\rho > x) < \infty$ then there exists $0 < \lambda_0 \leq \frac{1}{l} < \infty$ such that

$$P_\lambda(\mathbb{R}_+ \text{ is eventually covered by } C) = \begin{cases} 0 & \text{if } \lambda < \lambda_0 \\ 1 & \text{if } \lambda > \lambda_0; \end{cases}$$

(b) if $0 < L := \limsup_{x\to\infty} xP(\rho > x) < \infty$ then there exists $0 < \frac{1}{L} \leq \lambda_1 < \infty$ such that

$$P_\lambda(\mathbb{R}_+ \text{ is eventually covered by } C) = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ 1 & \text{if } \lambda > \lambda_1; \end{cases}$$

(c) if $\lim_{x\to\infty} xP(\rho > x) = \infty$ then for all $\lambda > 0$, $\mathbb{R}_+$ is eventually covered by C almost surely (P_λ); and

(d) if $\lim_{x\to\infty} xP(\rho > x) = 0$ then for any $\lambda > 0$, $\mathbb{R}_+$ is not eventually covered by C almost surely (P_λ).

In higher dimensions the notion of criticality in λ is absent. A point $(x, y) \in \mathbb{R}^2_+$ may be covered by any Poisson point in the rectangle $[0, x] \times [0, y]$. The further the point is from the origin the larger is the probability of it being covered. Therefore one would expect (x, y) to be covered for large x and y, irrespective of λ.

Theorem 1.3. *Let $d \geq 2$. For all $\lambda > 0$ we have*
(a) $P_\lambda(\mathbb{R}^d_+$ is eventually covered by $C) = 1$ whenever $\liminf_{x\to\infty} xP(\rho > x) > 0$,

(b) $P_\lambda(\mathbb{R}^d_+$ is eventually covered by $C) = 0$ whenever $\lim_{x\to\infty} xP(\rho > x) = 0$.

Remark: (i) It is not too difficult to see that Proposition 1.1 is true when $B((\mathbf{0}, \rho_i))$ is replaced by $[0, \rho_i]^d$. Comparing the above results with this we see that while $E\rho^d = \infty$ guarantees complete coverage of $\mathbb{R}^d$ by C, it is insufficient to guarantee eventual coverage for the orthant $\mathbb{R}^d_+$. This dichotomy in the coverage property arises because for the orthant $\mathbb{R}^d_+$ we have the 'boundary' effect which is, however, absent for the whole space $\mathbb{R}^d$.

(ii) If $0 < l := \lim_{x\to\infty} xP(\rho > x) < \infty$ then $P_\lambda(\mathbb{R}_+ \text{ is eventually covered by } C) = \begin{cases} 0 & \text{if } \lambda < \frac{1}{l} \\ 1 & \text{if } \lambda > \frac{1}{l} \end{cases}$.
This is an immediate consequence of the first two parts of Theorem 1.2.

The rest of the paper is organised as follows. In the next section we prove Theorem 1.1. In Section 3, we first consider an independent discrete model and then via a straight-forward comparison to the discrete model we obtain the results for the Poisson Boolean model.

Acknowledgments: We thank Ronald Meester for asking a question which led to this study.

2 The Markov model

In this section we will prove Theorem 1.1. Let F be the distribution function of ρ and, for each $k \in \mathbb{N}$, let $A_k := \{k \notin C\}$.

Proof of Theorem 1.1 (b) For $k \geq 1$, let $P_0(A_k) = P(A_k \mid X_1 = 0)$ and $P_1(A_k) = P(A_k \mid X_1 = 1)$. We first show that

$$\sum_k P_1(A_k) = \sum_k P_0(A_k) = \infty \text{ whenever } \frac{p_{01}}{p_{10} + p_{01}} < \frac{1}{L}. \tag{1}$$

To this end, we first observe that a simple conditioning argument yields the following recurrence relations:

$$P_0(A_{k+1}) = p_{00}P_0(A_k) + p_{01}P_1(A_k) \tag{2}$$

$$P_1(A_{k+1}) = F(k-1)\left[p_{10}P_0(A_k) + p_{11}P_1(A_k)\right]. \tag{3}$$

For $k_0 \geq 1$ to be chosen later in (7), let $B(s) = \sum_{k=k_0}^{\infty} P_1(A_k)s^k$, $A(s) = \sum_{k=k_0}^{\infty} P_0(A_k)s^k$. To establish (2) we need to show that $A(1) = B(1) = \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{L}$. Multiplying (2) by s^{k+1} and summing over $k \geq k_0$, we have that

$$\sum_{k=k_0}^{\infty} s^{k+1}P_0(A_{k+1}) = p_{00}\sum_{k=k_0}^{\infty} s^{k+1}P_0(A_k) + p_{01}\sum_{k=k_0}^{\infty} s^{k+1}P_1(A_k).$$

So $A(s) - s^{k_0}P_0(A_{k_0}) = p_{00}sA(s) + p_{01}sB(s)$, and consequently

$$A(s) = \frac{s^{k_0}P_0(A_{k_0}) + p_{01}sB(s)}{1 - p_{00}s}. \tag{4}$$

Now multiplying (3) by s^{k+1} and summing over $k \geq k_0$ we have that

$$\sum_{k=k_0}^{\infty} s^{k+1}P_1(A_{k+1}) = p_{10}\sum_{k=k_0}^{\infty} F(k-1)s^{k+1}P_0(A_k) + p_{11}\sum_{k=k_0}^{\infty} F(k-1)s^{k+1}P_1(A_k). \tag{5}$$

Each of the the above power series is uniformly convergent for $|s| < 1$, and hence differentiating (5) term by term with respect to s we obtain, for $|s| < 1$

$$\sum_{k=k_0}^{\infty} (k+1)s^kP_1(A_{k+1}) = p_{10}\sum_{k=k_0}^{\infty} F(k-1)(k+1)s^kP_0(A_k) + p_{11}\sum_{k=k_0}^{\infty} F(k-1)(k+1)s^kP_1(A_k). \tag{6}$$

Let $\epsilon > 0$ be given such that $C_1 = L + \epsilon > 0$, where L is as in the statement of the theorem. There exists k_0 such that

$$k_0 + (1 - C_1) > 0, \quad P_0(A_{k_0}) > 0, \quad P_1(A_{k_0}) > 0, \quad \text{and } F(k-1) \geq 1 - \frac{C_1}{k+1} \quad \text{for} \quad k \geq k_0. \tag{7}$$

Our choice of k_0 above yields,

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} (k+1) s^k P_1(A_{k+1}) \\
&\geq \quad p_{10} \sum_{k=k_0}^{\infty} (1 - \frac{C_1}{k+1})(k+1) s^k P_0(A_k) + p_{11} \sum_{k=k_0}^{\infty} (1 - \frac{C_1}{k+1})(k+1) s^k P_1(A_k) \\
&= \quad p_{10} \sum_{k=k_0}^{\infty} (k+1) s^k P_0(A_k) + p_{11} \sum_{k=k_0}^{\infty} (k+1) s^k P_1(A_k) \\
&\quad -C_1 \left[p_{10} \sum_{k=k_0}^{\infty} s^k P_0(A_k) + p_{11} \sum_{k=k_0}^{\infty} s^k P_1(A_k) \right].
\end{aligned}$$

So we have

$$B'(s) - k_0 s^{k_0 - 1} P_1(A_{k_0}) \geq p_{10} s A'(s) + p_{11} s B'(s) + (1 - C_1) \left[p_{10} A(s) + p_{11} B(s) \right]. \tag{8}$$

Using (4), we have that

$$A'(s) = \frac{(1 - p_{00}s)(k_0 s^{k_0-1} P_0(A_{k_0}) + p_{01}(sB'(s) + B(s))) + p_{00}(s^{k_0} P_0(A_{k_0}) + p_{10} s B(s))}{(1 - p_{00}s)^2} \tag{9}$$

Substituting for A' and A in (8) we have

$$\begin{aligned}
&B'(s) - k_0 s^{k_0-1} P_1(A_{k_0}) \\
&\geq \quad p_{11} s B'(s) + (1 - C_1) \left[p_{10} \frac{s^{k_0} P_0(A_{k_0}) + p_{01} s B(s)}{1 - p_{00}s} + p_{11} B(s) \right] \\
&\quad + p_{10} s \frac{(1 - p_{00}s)(k_0 s^{k_0-1} P_0(A_{k_0}) + p_{01}(sB'(s) + B(s))) + p_{00}(s^{k_0} P_0(A_{k_0}) + p_{10} s B(s))}{(1 - p_{00}s)^2}
\end{aligned}$$

Multiplying both sides by $(1 - p_{00}s)^2$ we have

$$\begin{aligned}
&B'(s)(1 - p_{00}s)^2(1 - p_{11}s) \\
&\geq \quad (1 - p_{00}s)^2(1 - C_1) \left[p_{10}(1 - p_{00}s)(s^{k_0} P_0(A_{k_0}) + p_{01} s B(s)) + p_{11} B(s) \right] \\
&\quad + p_{10} s (1 - p_{00}s)(k_0 s^{k_0-1} P_0(A_{k_0}) + p_{01}(sB'(s) + B(s))) + p_{00}(s^{k_0} P_0(A_{k_0}) + p_{10} s B(s)),
\end{aligned}$$

from which we obtain

$$B'(s) P(s) \geq Q(s) B(s) + R(s), \tag{10}$$

where

$$\begin{aligned}
P(s) &= (1 - p_{00}s)^2(1 - p_{11}s) + p_{10} s (1 - p_{00}s) p_{01} s \\
&= (1 - p_{00}s)(1 - s)(1 - s(1 - p_{01} - p_{10}))
\end{aligned}$$

$$Q(s) = (1 - p_{00}s)^2(1 - C_1)p_{11} + (1 - C_1)p_{10}p_{01}s(1 - p_{00}s) + p_{10} s p_{01}(1 - p_{00}s) + p_{10}p_{00}p_{01}s^2$$

$$R(s) = (1 - p_{00}s)^2 k_0 s^{k_0-1} P_1(A_{k_0}) + (k_0 + 1 - C_1) p_{10} s^{k_0}(1 - p_{00}s) P_0(A_{k_0}) + p_{10} s^{k_0+1} p_{00} P_0(A_{k_0})$$

From (10) we have for any $0 < t < 1$

$$B(t) \geq e^{\int_0^t \frac{Q(s)}{P(s)} ds} \int_0^t e^{\int_0^s \frac{-Q(r)}{P(r)} dr} \frac{R(s)}{P(s)} ds. \tag{11}$$

Now for $s < 1$, $\frac{Q(s)}{P(s)} = \frac{D}{1-p_{00}s} + \frac{E}{1-s} + \frac{F}{1-s(1-p_{01}-p_{10})}$, for some real numbers D, E, F. Using the fact that $p_{00} < 1$ and $-1 < 1-(p_{01}+p_{10}) < 1$ we have

$$\int_0^t \frac{Q(s)}{P(s)} ds = \ln\left\{(1-p_{00}t)^{\frac{-D}{p_{00}}}(1-t)^{-E}(1-t(1-p_{01}-p_{10}))^{\frac{-F}{1-p_{01}-p_{10}}}\right\}. \tag{12}$$

Using (12), we have

$$\begin{aligned} B(t) &\geq (1-p_{00}t)^{\frac{-D}{p_{00}}}(1-t)^{-E}(1-t(1-p_{01}-p_{10}))^{\frac{-F}{1-p_{01}-p_{10}}} \times \\ &\quad \times \int_0^t (1-p_{00}s)^{\frac{D}{p_{00}}}(1-s)^{E}(1-s(1-p_{01}-p_{10}))^{\frac{F}{1-p_{01}-p_{10}}} \frac{R(s)}{P(s)} ds \\ &\geq (1-p_{00}t)^{\frac{-D}{p_{00}}}(1-t)^{-E}(1-t(1-p_{01}-p_{10}))^{\frac{-F}{1-p_{01}-p_{10}}} \times \\ &\quad \times \int_0^t (1-p_{00}s)^{\frac{D}{p_{00}}-1}(1-s)^{E-1}(1-s(1-p_{01}-p_{10}))^{\frac{F}{1-p_{01}-p_{10}}-1} R(s) ds \end{aligned}$$

By our choice of k_0, each of the summands in expression for $R(s)$ is non-negative. Thus for α such that $0 < \alpha \leq s < t < 1$ we see that

$$R(s) \geq (1-p_{00})^2 k_0 \alpha^{k_0-1} P_1(A_{k_0}) + (k_0+1-C_1)p_{10}\alpha^{k_0}(1-p_{00})P_0(A_{k_0}) + p_{00}P_0(A_{k_0}). \tag{13}$$

Since $0 < p_{00} < 1$ and $-1 < 1-(p_{01}+p_{10}) < 1$, for all $0 < s < 1$, both $1-p_{00}s$ and $1-s(1-p_{01}-p_{10})$ are bounded above by 1 and below by a strictly positive constant. Hence regardless of the values of D and F, using (13) we have that for $\alpha < t < 1$

$$\begin{aligned} B(t) &\geq c_2(1-t)^{-E} \int_\alpha^t (1-s)^{E-1} R(s) ds \\ &\geq c_3(1-t)^{-E} \int_\alpha^t (1-s)^{E-1} ds \\ &= c_3(1-t)^{-E}\left[\frac{(1-\alpha)^E}{E} - \frac{(1-t)^E}{E}\right] \\ &= c_3\left[\frac{(1-t)^{-E}(1-\alpha)^E}{E} - \frac{1}{E}\right], \end{aligned}$$

for some $0 < c_3 < \infty$. Since $B(\cdot)$ is a power series, this implies $B(1) = \infty$ whenever $E > 0$. Observe that $E = \frac{Q(1)}{(1-p_{00})(p_{01}+p_{10})}$. Therefore $(1-p_{00})(p_{01}+p_{10})E = p_{00}p_{10}p_{01} + (1-p_{00})p_{10}p_{01} + (1-C_1)(p_{11}(1-p_{00})^2 + p_{10}p_{01}(1-p_{00})) = (1-p_{00})(p_{10}+(1-C_1)p_{01})$. Now,

$$\begin{aligned} E > 0 &\iff p_{10} + (1-C_1)p_{01} > 0 \\ &\iff C_1 < \frac{p_{10}+p_{01}}{p_{01}} \\ &\iff \frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{C_1} = \frac{1}{L+\epsilon}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have that $B(1) = \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{L}$ which implies, from (4), that $A(1) = \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{L}$.

The events $\{A_k : k \geq 1\}$ are not independent and hence Borel-Cantelli lemma cannot be applied, however they are delayed renewal events as will be shown in (14). Let μ and ν be two probability measures on $\{0,1\}$ with $\nu(1) = p_{01}$ and $\nu(0) = p_{00}$. Let P_μ and P_ν denote the probability distributions governing the Markov chains starting with the initial distribution μ and ν, respectively, and the transition probabilities as given earlier. Observe that

$$P_\mu(A_{i+j+k} \cap A_{i+j} \mid A_i) = P_\nu(A_k)P_\nu(A_j) \text{ for all } i, j, k \geq 1, \tag{14}$$

which establishes that $\{A_k : k \geq 1\}$ are delayed renewal events (see Feller [2], page 317). Thus, for $f_0(k) := P_\mu(A_k \cap \cap_{j=1}^{k-1} A_j^c)$, if we show that

$$P_\mu\{k \notin C \text{ for some } k \geq 1\} = \sum_{k=1}^{\infty} f_0(k) = 1 \text{ whenever } \frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{L}, \tag{15}$$

then along with (1), Theorems 2 and 1 (pages 312 and 318 respectively, Feller [2]) we have $P_\mu\{k \notin C \text{ for infinitely many } k \geq 1\} = 1$.

To prove (15) we observe that

$$P_\mu(A_k) = \mu(1)P_1(A_k) + \mu(0)P_0(A_k). \tag{16}$$

It is easy to see that for all $k \geq 1$ we have $P_\mu(A_k) = f_0(k) + \sum_{j=1}^{k-1} f_0(j)P_\nu(A_{k-j})$. Set $P_\mu(A_0) = 1$ and $f_0(0) = 0$. Then, we can write $P_\mu(A_k) = \sum_{j=0}^{k} f_0(j)P_{p_{01}}(A_{k-j})$ for all $k \geq 1$. Thus multiplying both sides by s^k and summing over k, we have,

$$G_\mu(s) - 1 = F_0(s)G_\nu(s)$$

where $G_\mu(s) = \sum_{k=0}^{\infty} P_\mu(A_k)s^k$ and $F_0(s) = \sum_{k=0}^{\infty} f_0(k)s^k$. Since $G_\nu(s) \neq 0$, we have,

$$F_0(s) = \frac{G_\mu(s) - 1}{G_\nu(s)}. \tag{17}$$

Now, taking $A_1(s) := \sum_{k=0}^{\infty} P_0(A_k)s^k$ and $B_1(s) := \sum_{k=0}^{\infty} P_1(A_k)s^k$ we have $G_\mu(s) = \mu(0)A_1(s) + \mu(1)B_1(s)$ and $G_\nu(s) = p_{00}A_1(s) + p_{01}B_1(s)$. Substituting this in (17) we have that

$$\begin{aligned} F_0(s) &= \frac{\mu(0) + ((\mu(1)B_1(s) - 1)/A_1(s))}{p_{00} + p_{01}B_1(s)/A_1(s)} \\ &\to 1 \text{ as } s \to 1, \end{aligned}$$

where the limit above holds because both the numerator and denominator in the first equality converge to 1, as may be seen easily from (4) for $k_0 = 1$ and the fact that $A(1) = \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} < \frac{1}{L}$. □

Proof of Theorem 1.1(a) The proof of part (a) is similar. Using the hypothesis, and an analogous calculation as in the previous part, it may be seen that $B(1) < \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} > \frac{1}{l}$. Also, by (4), $A(1) < \infty$ and here the Borel-Cantelli lemma will imply the result. $\square$

3 The Poisson Boolean model

In this section we prove Theorem 1.2 and Theorem 1.3. We first prove it for a discrete model on $\mathbb{N}^d$ and then for the point process via a straight forward comparison with the discrete model.

3.1 Independent discrete model

In this subsection we take $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$ to be an i.i.d. collection of $\{0,1\}$ valued random variables with $p = P(X_{\mathbf{i}} = 1)$ and $\{\rho_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$ to be another i.i.d. collection of positive integer valued random variables with distribution function F and independent of $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$. Let $C := \cup_{\mathbf{i}\in\mathbb{N}^d |\, X_{\mathbf{i}}=1}(\mathbf{i} + [0, \rho_{\mathbf{i}}]^d)$. We first consider eventual coverage of $\mathbb{N}^d$ by $X_{\mathbf{i}}$. Although, for $d = 1$, Theorem 1.1 holds for this set-up, we present an alternate simpler proof as it indicates the method needed for $d \geq 2$.

Proposition 3.1. *(a) If $l = \liminf_{j\to\infty} jP(\rho > j) > 1$, then for all $p > 1/l$,*

$$P_p\{C \textit{ eventually covers } \mathbb{N}\} = 1.$$

(b) If $L = \limsup_{j\to\infty} jP(\rho > j) < \infty$, then for all $p < 1/L$,

$$P_p\{C \textit{ eventually covers } \mathbb{N}\} = 0.$$

Proof : Define $A_i := \{i \notin C\}$ for $i \geq 1$ and let $G := 1 - F$.

(a) Fix $p > 1/l$. Since $l > 1/p$, we can choose $\delta > 0$ so that $l > (1+\delta)/p$. Now, we can choose j_0 so large that $jG(j) > (1+\delta)/p$ for all $j \geq j_0$. Thus, for $j \geq j_0$, we have $pG(j) > (1+\delta)/j$. Therefore, we have, for all $j \geq j_0$, $P_p(A_j) = (1-p)\prod_{k=1}^{j-1}(1-pG(k)) \leq (1-p)\prod_{k=1}^{j_0}(1-pG(k))\prod_{k=j_0+1}^{j-1}(1-(1+\delta)/k) = a_j$ (say). Observe that $a_{j+1}/a_j = 1-(1+\delta)/j$, hence by Gauss' test (see Knopp [4], page 288) we have $\sum_{j=1}^{\infty} P_p(A_j) < \infty$. An application of the Borel-Cantelli lemma proves part (a).

(b) The proof is similar. One calculates $P_p(A_j)$ as above. An application of Gauss' test shows that $\sum_i P_p(A_i) = \infty$ for $p < \frac{1}{L} = \frac{1}{\limsup_{j\to\infty} jG(j)}$. The A_k's are not independent and hence

Borel-Cantelli lemma cannot be applied. However using conditional independence one can show that

$$P_p(A_k \cap A_i) = P_p(A_{k-i})P_p(A_i)$$

and therefore, A_i's satisfy the definition of a renewal event in Feller [2], page 308. So from Theorem 2, on page 312 in Feller [2] if $\sum_{i=1}^{\infty} P(A_i) = \infty$ then A_i occurs for infinitely many $i's$ with probability one. $\square$

For higher dimensions, we have

Proposition 3.2. *Let $d \geq 2$ and $0 < p < 1$.*

(a) if $\lim_{j\to\infty} jP(\rho > j) = 0$ then $P_p(C$ eventually covers $\mathbb{N}^d) = 0$

(b) if $\liminf_{j\to\infty} jP(\rho > j) > 0$ then $P_p(C$ eventually covers $\mathbb{N}^d) = 1$

Proof : Recall that $G(j) = P(\rho > j)$

(a) Let $d = 2$ and fix $0 < p < 1$. For $i, j \in \mathbb{N}$ let $A(i,j) := \{(i,j) \notin C\}$. Now observe that for each fixed j,

$$P(A(k,j) \cap A(i,j)) = P(A(k-i,j))P(A(i,j)),$$

i.e., for each fixed j the event $A(i,j)$ is a renewal event. Thus, if, for every $j \geq 1$, $\sum_{i=1}^{\infty} P(A(i,j)) = \infty$ then, on every line $\{y = j\}$, $j \geq 1$, we have infinitely many i's for which (i,j) is uncovered with probability one and hence would imply part (a).

We proceed to show that $\sum_{i=1}^{\infty} P(A(i,j)) = \infty$. Let $i \geq j+1$. To calculate $P_p(A(i,j))$ we divide the rectangle into a square of length j and a rectangle as in Figure 1. For any point (k,l), $1 \leq k \leq i-j$ and $1 \leq l \leq j$, in the shaded region of Figure 1, we ensure that either $X_{(k,l)} = 0$ or $\rho_{(k,l)} \leq k+j-1$. The remaining square region in Figure 1 is decomposed into j sub squares of length t, $1 \leq t \leq j-1$ and we ensure that for each point (k,l) on the section of the boundary of the sub square t given by the dotted lines either $X_{(k,l)} = 0$ or $\rho_{(k,l)} \leq t$. So,

$$\begin{aligned} P_p(A(i,j)) &= (1-p)\prod_{t=1}^{j-1}(1-p+pF(t-1))^{2t+1}\prod_{k=1}^{i-j}(1-p+pF(k+j-1))^j \\ &= (1-p)\prod_{t=1}^{j-1}(1-pG(t))^{2t+1}\prod_{k=j+1}^{i}(1-pG(k))^j. \qquad (18) \end{aligned}$$

Now choose $\epsilon > 0$ such that $pj\epsilon < 1$. By assumption there exists N such that, for all $i \geq N$,

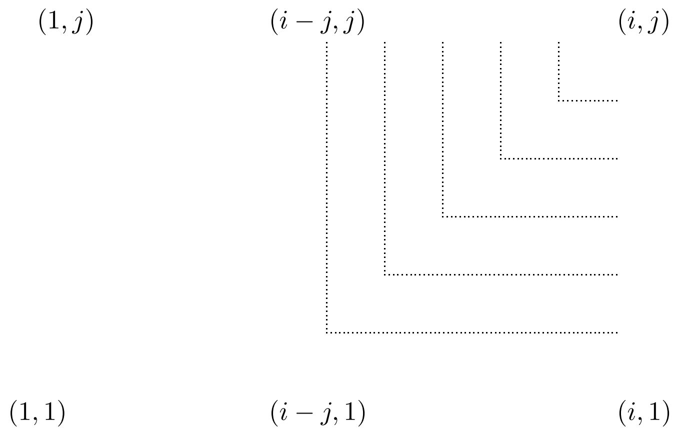

Figure 1: Division of the rectangle formed by $[1, i] \times [1, j]$

$iG(i) < \epsilon$. Taking $c_j := \prod_{t=1}^{j-1} (1 - pG(t))^{2t+1}$, we have from (18) we have that

$$\begin{aligned} \sum_{i=N}^{\infty} P_p(A(i,j)) &= (1-p)c_j \sum_{i=N}^{\infty} \prod_{k=1}^{i-j} (1 - pG(k+j))^j \\ &= (1-p)c_j \sum_{i=N}^{\infty} e_i \quad \text{(say)}. \end{aligned} \tag{19}$$

For $m \geq N$ we have

$$\begin{aligned} \frac{e_{m+1}}{e_m} &= (1 - pG(m+1))^j \\ &\geq \left(1 - \frac{p\epsilon}{m+1}\right)^j \\ &= 1 - j\frac{p\epsilon}{m+1} + \sum_{k=2}^{j} (-p)^k \binom{j}{k} \frac{\epsilon^k}{(m+1)^k} \\ &= 1 - \frac{pj\epsilon}{m+1} + \frac{g(m,p,j,\epsilon)}{(m+1)^2}, \end{aligned} \tag{20}$$

for some function $g(m,p,j,\epsilon)$ bounded in m. Thus by Gauss' test, as $pj\epsilon < 1$ we have $\sum_{i=N}^{\infty} e_i = \infty$ and hence, $\sum_{i=1}^{\infty} P_p(A(i,j)) = \infty$. This completes the proof of part (a) for $d = 2$.

We turn our attention to $d = 3$. Fix $l_2 \geq l_3 \in \mathbb{N}$. For $i \geq l_2, l_3$, consider the event $A(i, l_2, l_3) = \{(i, l_2, l_3) \notin C\}$. We decompose the cube $[1, i] \times [1, l_2] \times [1, l_3]$ into rectangles $[1, i] \times [1, l_2] \times \{m\}$, $1 \leq m \leq l_3$. We divide each such rectangle as in Figure 2. Now in the shaded region we need to ensure that at each point $X_{\mathbf{i}} = 0$ or $\rho_{\mathbf{i}} \leq l_3 - m - 1$ and for the rest we proceed exactly as in the $d = 2$ case. We have

$$\begin{aligned} &P_p(A(i, l_2, l_3)) \\ &= \prod_{m=1}^{l_3} [1 - pG(l_3 - m)]^{(l_3 - m + 1)^2} \prod_{k=1}^{i-l_2} [1 - pG(k + l_2)]^{l_2} \prod_{t=l_3-m-1}^{l_2} [1 - pG(t)]^{2t+1} . \end{aligned}$$

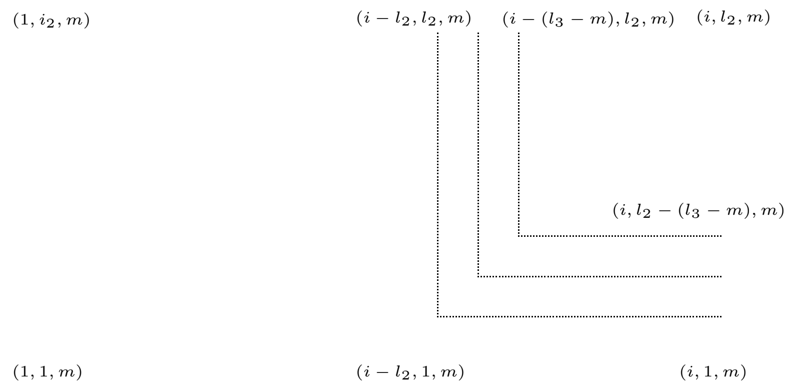

Figure 2: Division of the rectangle formed by $[1, i] \times [1, l_2] \times m$

Note that there is only one product involving i, while the other two products form a constant say d_{l_2,l_3}. Therefore, as in (19),

$$\sum_{i=N}^{\infty} P_p(A(i, l_2, l_3)) = (1-p)d_{l_2,l_3} \sum_{i=N}^{\infty} \prod_{k=1}^{i-l_2} [1 - pG(k + l_2)]^{l_2} .$$

Proceeding as in (20) $\sum_{i=1}^{\infty} P_p(A(i, l_2, l_3)) = \infty$.

By the renewal argument stated at the beginning, $A(i, l_2, l_3)$ occurs for infinitely many i. Hence, along every line $\{(i, l_2, l_3), i \in \mathbb{N}\}$, $l_2 \geq l_3$ there are infinitely many vertices on it which are not in C. The same argument holds for $l_2 \leq l_3$. As in $d = 2$, this shows that C does not eventually cover $\mathbb{N}^3$.

For $d \geq 4$, the argument follows along similar lines.

(b) We prove this part of the proposition for $d = 2$, the proof for $d \geq 3$ is similar and we omit it. Fix $\eta > 0$ such that $\eta < \liminf_{j\to\infty} jG(j)$. Let N_1 be such that for all $i \geq N_1$ we have $iG(i) > \eta$. Also, we fix $0 < p < 1$ and choose a such that $0 < \exp(-p\eta) < a < 1$. Let N_2 be such that for all $j \geq N_2$ we have $(1 - p\eta j^{-1})^j < a$. For $N := \max\{N_1, N_2\}$, let $i, j \in \mathbb{N}$ be such that $j \geq N$ and $i > j$. Define $A(i, j) := \{(i, j) \notin C\}$. As in (18) we have

$$\begin{aligned} P_p(A(i,j)) &= (1-p) \prod_{k=1}^{i-j} ((1-p) + pF(j+k-1))^j \prod_{t=1}^{j-1} ((1-p) + pF(t-1))^{2t+1} \\ &= (1-p) \prod_{k=1}^{i-j} (1 - pG(j+k))^j \prod_{t=1}^{j-1} (1 - pG(t))^{2t+1} . \end{aligned} \quad (21)$$

Taking $c_j := \prod_{t=1}^{j-1}(1-pG(t))^{2t+1}$, we have from (21) and by our choice of j,

$$\sum_{i=N}^{\infty} P_p(A(i,j)) = (1-p)c_j \sum_{i=N}^{\infty}\prod_{k=1}^{i-j}(1-pG(k+j))^j \tag{22}$$

$$= (1-p)c_j \sum_{i=N}^{\infty} b_i \text{ (say)}. \tag{23}$$

Now $m \geq N$

$$\begin{aligned}
\frac{b_{m+1}}{b_m} &= (1-pG(m+1))^j \\
&\leq \left(1-p\frac{\eta}{m+1}\right)^j \\
&= 1-pj\frac{\eta}{m+1}+\sum_{k=2}^{j}(-p)^k\binom{j}{k}\frac{\eta^k}{(m+1)^k} \\
&= 1-\frac{pj\eta}{m+1}+\frac{h(m,p,j,\eta)}{(m+1)^2}
\end{aligned} \tag{24}$$

for some function $h(m,p,j,\eta)$ bounded in m.

Thus by Gauss' test, if $pj\eta > 1$ then $\sum_{i=N}^{\infty} b_i < \infty$ and hence, $\sum_{i=1}^{\infty} P_p(A(i,j)) < \infty$.

Now, for a given p, let $j' := \sup\{j : pj\eta < 1\}$ and $j_0 := \max\{j'+1, N\}$. We next show that the region $Q_{j_0} := \{(i_1,i_2) \in \mathbb{N}^2 : i_1, i_2 \geq j_0\}$ has at most finitely many points that are not covered by C almost surely; there by proving that C eventually covers $\mathbb{N}^2$. For this we apply Borel-Cantelli lemma after showing that $\sum_{(i_1,i_2)\in Q_{j_0}} P_p(A(i_1,i_2)) < \infty$.

Towards this end we have

$$\begin{aligned}
&\sum_{i_1,i_2\geq j_0} P_p(A(i_1,i_2)) \\
&= \sum_{k=1}^{\infty}\left(2\sum_{m=1}^{k-1}P_p(A(j_0+k,j_0+m))+P_p(A(j_0+k,j_0+k))\right) \\
&= 2\sum_{k=1}^{\infty}\sum_{m=1}^{k-1}(1-p)\prod_{i=1}^{k-m}(1-pG(j_0+m+i))^{j_0+m}\prod_{t=1}^{j_0+m-1}(1-pG(t))^{2t+1} \\
&\qquad +\sum_{k=1}^{\infty}\prod_{t=1}^{j_0+k-1}(1-pG(t))^{2t+1} \\
&= 2(1-p)\sum_{m=1}^{\infty}\left(\prod_{t=1}^{j_0+m-1}(1-pG(t))^{2t+1}\sum_{k=m+1}^{\infty}\prod_{i=1}^{k-m}(1-pG(j_0+m+i))^{j_0+m}\right) \\
&\qquad +\sum_{k=1}^{\infty}\prod_{t=1}^{j_0+k-1}(1-pG(t))^{2t+1}.
\end{aligned}$$

Observe that

$$
\begin{aligned}
\sigma_m &:= \sum_{k=m+1}^{\infty} \prod_{i=1}^{k-m} (1 - pG(j_0 + m + i))^{j_0+m} \\
&= \sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + i))^{j_0+m} \qquad (25)\\
&\leq \sum_{s=1}^{\infty} \prod_{i=1}^{s} \left(1 - \frac{p\eta}{j_0 + m + s}\right)^{j_0+m},
\end{aligned}
$$

hence as in (24) and the subsequent application of Gauss' test, we have that, for every $m \geq 1$, $\sigma_m < \infty$.

Now let $\gamma_m := \prod_{t=1}^{j_0+m-1}(1 - pG(t))^{2t+1}\sigma_m$. Note that an application of the ratio test yields $\sum_{m=1}^{\infty} \gamma_m < \infty$; indeed from (25),

$$
\begin{aligned}
\frac{\gamma_{m+1}}{\gamma_m} &= (1 - pG(j_0 + m))^{2j_0+2m+1} \frac{\sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + 1 + i))^{j_0+m+1}}{\sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + i))^{j_0+m}} \\
&= \frac{(1 - pG(j_0 + m))^{2j_0+2m+1}}{(1 - pG(j_0 + m + 1))^{j_0+m}} \frac{\sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + 1 + i))^{j_0+m+1}}{1 + \sum_{s=2}^{\infty} \prod_{i=2}^{s} (1 - pG(j_0 + m + i))^{j_0+m}} \\
&\leq (1 - pG(j_0 + m))^{j_0+m+1} \frac{\sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + 1 + i))^{j_0+m+1}}{1 + \sum_{s=1}^{\infty} \prod_{i=1}^{s} (1 - pG(j_0 + m + 1 + i))^{j_0+m}}.
\end{aligned}
$$

Since $\sigma_m < \infty$ for all $m \geq 1$, in the fraction on the right side of the above inequality both the numerator and the denominator are finite. Moreover, each term in the sum of the numerator is less than the corresponding term in the sum of the denominator; yielding that the fraction is at most 1. Hence, for $0 < a < 1$ as chosen earlier

$$
\begin{aligned}
\frac{\gamma_{m+1}}{\gamma_m} &\leq (1 - pG(j_0 + m))^{j_0+m+1} \\
&\leq \left(1 - \frac{p\eta}{j_0 + m}\right)^{j_0+m+1} \\
&\leq a.
\end{aligned}
$$

This shows that $\sum_{m=1}^{\infty} \gamma_m < \infty$ and completes the proof of part (a). □

3.2 Proofs of Theorems 1.2 and 1.3

Consider (Ξ, λ, ρ) the Poisson Boolean model on $\mathbb{R}_+^d$ as in Section 1. Let C be the covered region as defined in Section 1.

Fix $\mathbf{i} := (i_1, \ldots, i_d)$ and consider $\xi_{\mathbf{i}} := \Xi \cap [i_1 - 1, i_1) \times \cdots \times [i_d - 1, i_d)$. Let $\xi_{\mathbf{i}1}, \ldots, \xi_{\mathbf{i}N_{\mathbf{i}}}$ be an enumeration of all the points of $\xi_{\mathbf{i}}$. Note that $N_{\mathbf{i}}$ is a Poisson random variable with mean λ

and that this enumeration is possible only if N_i is positive. Let $\rho_{\mathbf{i}1}, \ldots, \rho_{\mathbf{i}N_{\mathbf{i}}}$ be the associated random variables with these points. Define

$$\rho_{\text{red}}(\mathbf{i}) := 2 + \lfloor \max\{\rho_{\mathbf{i}1}, \ldots, \rho_{\mathbf{i}N_i}\} \rfloor \text{ and } \rho_{\text{green}}(\mathbf{i}) = \max\{0, \lfloor \max\{\rho_{\mathbf{i}1}, \ldots, \rho_{\mathbf{i}N_{\mathbf{i}}}\} \rfloor - 2\}.$$

Now we consider two discrete models – (a) the *red* model, where for $\mathbf{i} \in \mathbb{N}^d$, we call $\mathbf{i}$ *red* if $\xi_{\mathbf{i}} \neq \emptyset$ and place a cube of length $\rho_{\text{red}}(\mathbf{i})$ at $\mathbf{i}$; and (b) the *green* model where we call $\mathbf{i}$ *green* if in addition to $\xi_{\mathbf{i}} \neq \emptyset$ we have $\rho_{\text{green}}(\mathbf{i}) \geq 1$ and place a cube of length $\rho_{\text{green}}(\mathbf{i})$ at $\mathbf{i}$.

For the red model consider the region

$$C_{\text{red}} := \bigcup_{\{\mathbf{i}:\mathbf{i} \text{ is red}\}} [i_1, i_1 + \rho_{\text{red}}(\mathbf{i})] \times \cdots \times [i_d, i_d + \rho_{\text{red}}(\mathbf{i})].$$

Similarly define the region C_{green} for the green model.

We observe that

$$\text{eventual coverage of the Poisson model ensures the same for the red model;} \tag{26}$$

$$\text{eventual coverage of the green model ensures the same for the Poisson model.} \tag{27}$$

Moreover, the red model is equivalent in law to a discrete model on $\mathbb{N}^d$ where for a vertex $\mathbf{i}$, $P(X_{\mathbf{i}} = 1) = 1 - \exp(-\lambda)$, independent of other vertices; and the length of the cube, ρ_{red}, associated with such a vertex is independent of the length associated with other such vertices and has the distribution given by

$$\begin{aligned} & P(\rho_{\text{red}} \leq m) \\ &= P\{\max\{\rho_1, \ldots, \rho_N\} < m - 1 \mid N \geq 1\} \\ &= \sum_{j=1}^{\infty} \frac{\exp(-\lambda)\lambda^j}{(1 - \exp(-\lambda))j!} P(\rho < m - 1)^j \\ &= \exp(-\lambda) \frac{\exp(\lambda P(\rho < m - 1)) - 1}{1 - \exp(-\lambda)} \\ &= \frac{\exp(-\lambda P(\rho \geq m - 1)) - \exp(-\lambda)}{1 - \exp(-\lambda)}, \end{aligned} \tag{28}$$

where N is a Poisson random variable with mean λ. Similarly, the green model is equivalent in law to a discrete model on $\mathbb{N}^d$ where for a vertex $\mathbf{i}$, $P(X_{\mathbf{i}} = 1) = 1 - \exp(-\lambda P(\rho \geq 3))$, independent of other vertices; and the length of the cube ρ_{green} associated with such a vertex is independent of the length associated with other such vertices and has the distribution given by

$$
\begin{aligned}
&P(\rho_{\text{green}} \le m) \\
&\quad = \quad P\{\max\{\rho_1, \ldots, \rho_N\} < m+3 \mid N \ge 1 \text{ and } \max\{\rho_1, \ldots, \rho_N\} \ge 3\} \\
&\quad = \quad \sum_{j=1}^{\infty} \frac{\exp(-\lambda)\lambda^j}{(1-\exp(-\lambda P(\rho \ge 3)))j!}[P(\rho < m+3)^j - P(\rho < 3)^j] \\
&\quad = \quad \frac{\exp(-\lambda P(\rho \ge m+3)) - \exp(-\lambda P(\rho \ge 3))}{1-\exp(-\lambda P(\rho \ge 3))}. \qquad (29)
\end{aligned}
$$

From (28) and (29) the proofs of the Theorem 1.2 and Theorem 1.3 easily follow. We illustrate below.

Proof of Theorems 1.2 and 1.3 Using the inequality $x - x^2/2 \le 1 - e^{-x} \le x$, we have from (28),

$$
\begin{aligned}
\frac{m}{1-e^{-\lambda}}\left[\lambda P(\rho \ge m-1) - \frac{(\lambda P(\rho \ge m-1))^2}{2}\right] \quad &\le \quad mP(\rho_{\text{red}} \ge m) \\
&\le \quad \frac{m}{1-e^{-\lambda}}\lambda P(\rho \ge m-1). \qquad (30)
\end{aligned}
$$

Now given an $\epsilon > 0$ let m_0 be such that for all $m \ge m_0$ we have $\lambda P(\rho \ge m-1)/2 < \epsilon$, then for all $m \ge m_0$,

$$
\frac{m}{1-e^{-\lambda}}\lambda P(\rho \ge m-1)(1-\epsilon) \le mP(\rho_{\text{red}} \ge m) \le \frac{m}{1-e^{-\lambda}}\lambda P(\rho \ge m-1). \qquad (31)
$$

Since this is true for all $\epsilon > 0$, if $0 < l := \liminf_{x} xP(\rho \ge x) \le \limsup_{x\to\infty} xP(\rho \ge x) =: L$, then we have

$$
\liminf_{m\to\infty} mP(\rho_{\text{red}} \ge m) = \frac{\lambda l}{1-e^{-\lambda}} \quad \text{and} \quad \limsup_{m\to\infty} mP(\rho_{\text{red}} \ge m) = \frac{\lambda L}{1-e^{-\lambda}}. \qquad (32)
$$

A similar calculation yields

$$
\liminf_{m\to\infty} mP(\rho_{\text{green}} \ge m) = \frac{\lambda l}{1-e^{-\lambda P(\rho \ge 3)}} \quad \text{and} \quad \limsup_{m\to\infty} mP(\rho_{\text{green}} \ge m) = \frac{\lambda L}{1-e^{-\lambda P(\rho \ge 3)}}. \qquad (33)
$$

Having established (33), from Propositions 3.1(a) we have that for all λ such that

$$
\liminf_{m\to\infty} mP(\rho_{\text{green}} \ge m)P(\text{a site is green}) = \lambda l > 1
$$

there is eventual coverage in the green model. Thus from (26) we have that for sufficiently large λ the Poisson model eventually covers $\mathbb{R}_+$ with probability 1. Similarly from (32), Propositions 3.1(b) and (27) we have that for λ such that $\lambda L < 1$, with probability 1 the Poisson model never eventually covers $\mathbb{R}_+$. This proves Theorem 1.2.

Using (32), observe that if $\lim_{x\to\infty} xP(\rho > x) = 0$ then $\lim_{m\to\infty} mP(\rho_{\text{red}} \geq m) = 0$; and thus from Proposition 3.2 (a), together with (27), we have that Theorem 1.3(b) holds.

To prove Theorem 1.3(a), observe that, Proposition 3.2(b) together with (26) now yields that $\mathbb{R}_+^d$ is eventually covered by the Poisson model with probability 1. This completes the proof of Theorem 1.3. $\square$

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