Perron eigenvector of the Tsetlin matrix

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Abstract

We consider the move-to-position k linear search scheme where the sequence of record requests is a Markov chain. Formulas are derived for the stationary distribution of the permutation chain for $k = 1, 2, n-1$ and n, where n is the number of records. Certain identities for the Perron complement are established in the process.

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1. Introduction

Consider a collection of n books (or records) arranged in a sequence. With the ith book a weight w_i is associated which indicates the probability of requesting the ith book at any given time. We assume that each $w_i > 0$ and that $\sum_{i=1}^{n} w_i = 1$. Let k be a fixed integer, $1 \leqslant k \leqslant n$. In the *move-to-position k* scheme, at each unit of time the ith book is removed with probability w_i and is put back in the kth position. This gives a Markov chain on S_n, the set of permutations of $\{1, 2, \ldots, n\}$. If $k = 1$, the scheme is known as the *move-to-front* scheme or the *Tsetlin library* and has been extensively studied in the literature, see [1,3,6–9].

A more general and perhaps a more realistic model than the one described above assumes that the sequence of requests follow a Markovian model. Thus let P be an

$n \times n$ stochastic matrix. If the ith book is requested at a given unit of time, then we assume that at the subsequent unit of time the jth book will be requested with probability p_{ij}. The transition matrix of the associated Markov chain on S_n under the move-to-position k scheme will be called the kth *Tsetlin matrix* of P and we denote it by $\mathscr{T}_k(P)$. We denote $\mathscr{T}_1(P)$ by $\mathscr{T}(P)$ and refer to it as the Tsetlin matrix of P.

Let A be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$. Recall that by the Perron–Frobenius theorem, $\rho(A)$ is an eigenvalue of A and if A is irreducible, then it admits positive right and left eigenvectors for $\rho(A)$ which are unique up to a scalar multiple. We refer to these as Perron eigenvectors of A.

In this paper we consider the problem of finding an expression for a left Perron eigenvector of $\mathscr{T}_k(P)$ when P is an $n \times n$ irreducible stochastic matrix. Note that a normalized left Perron eigenvector of $\mathscr{T}_k(P)$ is just the stationary distribution of the Markov chain on S_n arising from the move-to-position k scheme. For $k = 1$ a formula for the stationary distribution has been given by Dobrow and Fill [1] using probability arguments. We give a more compact formula employing the notion of the Perron complement. Our proof technique is elementary and permits us to handle some other values of k as well. More remarks along these lines are given later.

The paper is organized as follows. In Section 2, we review some basic properties of the Perron complement. We then prove some identities for the Perron complement which are used in later sections. In Section 3, we obtain a formula for a left Perron eigenvector of $\mathscr{T}(P)$, where P is an irreducible stochastic matrix. It is shown that our formula is equivalent to the one given in [1]. We also show that the formula reduces to that given by Hendricks [3] when the requests are independent (this is the case when P is of rank 1). Finally, in Section 4, we consider the move-to-position k scheme for some special values of k.

As usual, let S_n denote the set of permutations of $1, \ldots, n$. Let k be fixed, $1 \leqslant k \leqslant n$. For $i = 1, \ldots, n$, let $\phi^i \in S_n$ be the permutation represented by the cycle $(k\,k-1\,k-2 \cdots i)$ if $i \leqslant k$ and by the cycle $(k\,k+1 \cdots i)$ if $i > k$. Let P be an $n \times n$ matrix. The kth Tsetlin matrix $\mathscr{T}_k(P)$ associated with P is an $n! \times n!$ matrix defined as follows. The rows and the columns of $\mathscr{T}_k(P)$ are indexed by S_n. For $\tau, \sigma \in S_n$, if $\sigma = \phi^i \circ \tau$ for some $i \in \{1, \ldots, n\}$, then the (τ, σ)-entry of $\mathscr{T}_k(P)$ is defined to be $p_{\tau(k)\tau(i)}$; otherwise it is defined to be 0. We set $\mathscr{T}(P) = \mathscr{T}_1(P)$ and call it the Tsetlin matrix of P.

We give some examples. If P is 2×2, then $\mathscr{T}(P) = P$. If P is 3×3, then

$$\mathscr{T}_1(P) = \mathscr{T}(P) = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{pmatrix} p_{11} & 0 & p_{12} & 0 & p_{13} & 0 \\ 0 & p_{11} & p_{12} & 0 & p_{13} & 0 \\ p_{21} & 0 & p_{22} & 0 & 0 & p_{23} \\ p_{21} & 0 & 0 & p_{22} & 0 & p_{23} \\ 0 & p_{31} & 0 & p_{32} & p_{33} & 0 \\ 0 & p_{31} & 0 & p_{32} & 0 & p_{33} \end{pmatrix} \end{array}$$

while

$$\mathscr{T}_2(P) = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{pmatrix} p_{22} & p_{23} & p_{21} & 0 & 0 & 0 \\ p_{32} & p_{33} & 0 & 0 & p_{31} & 0 \\ p_{12} & 0 & p_{11} & p_{13} & 0 & 0 \\ 0 & 0 & p_{31} & p_{33} & 0 & p_{32} \\ 0 & p_{13} & 0 & 0 & p_{11} & p_{12} \\ 0 & 0 & 0 & p_{23} & p_{21} & p_{22} \end{pmatrix} \end{array}.$$

Note that $\mathscr{T}_k(P)$ is a sparse matrix having at most n nonzero entries in each row and column. If P is stochastic, then so is $\mathscr{T}_k(P)$. Furthermore, if P is positive, then $\mathscr{T}_k(P)$ is irreducible.

2. Perron complement

Let A be an $n \times n$ matrix. If S and T are nonempty subsets of $\{1, \dots, n\}$, then $A[S \mid T]$ will denote the submatrix of A formed by the rows indexed by S and the columns indexed by T. If S and T are proper subsets of $\{1, \dots, n\}$, then $A(S \mid T)$ will denote the submatrix of A formed by deleting the rows indexed by S and the columns indexed by T. The submatrices $A(S \mid T]$ and $A[S \mid T)$ are defined similarly.

Let P be an $n \times n$ stochastic matrix and let S be a nonempty, proper subset of $\{1, \dots, n\}$. If $I - P[S \mid S]$ is nonsingular, then the Perron complement of $P[S \mid S]$ in P is defined as (see [5])

$$P^S = P(S \mid S) + P(S \mid S](I - P[S \mid S])^{-1} P[S \mid S).$$

If $S = \phi$, then we define $P^S = P$. We index the rows and the columns of P^S by the complement of S in $\{1, \dots, n\}$. If $S = \{i, j, \dots, k\}$, we often write $P^{i,j,\dots,k}$ instead of $P^{\{i,j,\dots,k\}}$.

A probabilistic interpretation of P^S is as follows. Let P be the transition matrix of a Markov chain with state space $\{1, \dots, n\}$. Let S be a proper subset of $\{1, \dots, n\}$. If we ignore the transitions that occur between states within S, then we get a "reduced" Markov chain whose transition matrix is precisely P^S.

Let P be an $n \times n$ stochastic matrix. By the Perron–Frobenius theorem there exists a row vector π such that $\pi \geqslant 0$ and $\pi P = \pi$. If π is nomalized to be a probability vector so that $\sum_{i=1}^{n} \pi_i = 1$, then we call π a normalized left Perron eigenvector of P. Recall that if P is irreducible, then it has a unique normalized left Perron eigenvector and is also known as the steady-state vector or the stationary distribution of the associated Markov chain.

The following basic result will be used, see [5] for a proof.

Lemma 1. *Let P be an $n \times n$ irreducible, stochastic matrix and let S be a proper subset of $\{1, \dots, n\}$. Then P^S is well defined and is irreducible, stochastic. Furthermore, if π is a left Perron eigenvector of P, then the subvector of π indexed by $\{1, \dots, n\} \backslash S$ is a left Perron eigenvector of P^S.*

The concept of the Perron complement is closely related to that of the Schur complement (see [5] for details). The Schur complement enjoys a well-known hereditary property, see [2]. A similar property holds for the Perron complement and is stated next. We include a proof for completeness.

Lemma 2. *Let P be an $n \times n$ irreducible, stochastic matrix and let $T \subset S$ be distinct, proper subsets of $\{1, \dots, n\}$. Then the Perron complement of $P^T[S\backslash T \mid S\backslash T]$ in P^T equals P^S.*

Proof. We assume, without loss of generality, that $T = \{1, \dots, t\}$, $S = \{1, \dots, s\}$, where $1 \leqslant t < s < n$. Let P be partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix},$$

where P_{11} is $t \times t$ and P_{22} is $(s-t) \times (s-t)$. Then

$$\begin{aligned} P^T &= \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{21} \\ P_{31} \end{bmatrix} (I - P_{11})^{-1} \begin{bmatrix} P_{12} & P_{13} \end{bmatrix} \\ &= \begin{bmatrix} P_{22} + P_{21}(I - P_{11})^{-1} P_{12} & P_{23} + P_{21}(I - P_{11})^{-1} P_{13} \\ P_{32} + P_{31}(I - P_{11})^{-1} P_{12} & P_{33} + P_{31}(I - P_{11})^{-1} P_{13} \end{bmatrix} \end{aligned} \tag{1}$$

and

$$P^S = P_{33} + \begin{bmatrix} P_{31} & P_{32} \end{bmatrix} \begin{bmatrix} I - P_{11} & -P_{12} \\ -P_{21} & I - P_{22} \end{bmatrix}^{-1} \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix}. \tag{2}$$

Let $X = (I - P_{11})^{-1}$ and let $Y = I - P_{22} - P_{21} X P_{12}$. By a well-known formula for the inverse of a partitioned matrix we have

$$\begin{bmatrix} I - P_{11} & -P_{12} \\ -P_{21} & I - P_{22} \end{bmatrix}^{-1} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -X P_{12} \\ -I \end{bmatrix} Y^{-1} \begin{bmatrix} -P_{21} X & -I \end{bmatrix}. \tag{3}$$

Substituting (3) into (2) we get

$$P^S = P_{33} + P_{31} X P_{13} + (P_{31} X P_{12} + P_{32}) Y^{-1} (P_{21} X P_{13} + P_{23}),$$

and from (1) we see that this expression equals the Perron complement of $P^T[S\backslash T \mid S\backslash T]$ in P^T. $\square$

We now prove certain identities involving Perron complement. These will be used in subsequent sections and may also be of independent interest.

Lemma 3. *Let P be an $n \times n$ irreducible, stochastic matrix, let $S \subset \{1, \dots, n\}$ and let i, j, k be distinct integers in $\{1, \dots, n\} \backslash S$. Then*

$$p_{ij}^{S} \frac{p_{ki}^{S\cup\{j\}}}{1 - p_{ii}^{S\cup\{j\}}} + p_{kj}^{S} = \left(1 - p_{jj}^{S}\right) \frac{p_{kj}^{S\cup\{i\}}}{1 - p_{jj}^{S\cup\{i\}}}. \tag{4}$$

Proof. By Lemma 1 P^S is irreducible, stochastic. By Lemma 2 we may replace P^S by P and hence it is sufficient to prove the following assertion:

$$p_{ij} \frac{p_{ki}^{j}}{1 - p_{ii}^{j}} + p_{kj} = (1 - p_{jj}) \frac{p_{kj}^{i}}{1 - p_{jj}^{i}}. \tag{5}$$

Thus we must prove that

$$p_{ij} p_{ki}^{j} \left(1 - p_{jj}^{i}\right) + p_{kj} \left(1 - p_{ii}^{j}\right)\left(1 - p_{jj}^{i}\right) = p_{kj}^{i} \left(1 - p_{jj}\right)\left(1 - p_{ii}^{j}\right). \tag{6}$$

Note that

$$p_{ki}^{j} = p_{ki} + \frac{p_{kj} p_{ji}}{1 - p_{jj}}, \quad p_{jj}^{i} = p_{jj} + \frac{p_{ji} p_{ij}}{1 - p_{ii}} \tag{7}$$

and

$$p_{ii}^{j} = p_{ii} + \frac{p_{ij} p_{ji}}{1 - p_{jj}}, \quad p_{kj}^{i} = p_{kj} + \frac{p_{ki} p_{ij}}{1 - p_{ii}}. \tag{8}$$

It follows, using (7) and (8), that

$$\begin{aligned} & p_{ij} \frac{p_{ki}^{j}}{1 - p_{ii}^{j}} + p_{kj} \\ &= \frac{p_{ij}(p_{ki}(1 - p_{jj}) + p_{kj} p_{ji}) + p_{kj}((1 - p_{ii})(1 - p_{jj}) - p_{ij} p_{ji})}{(1 - p_{ii})(1 - p_{jj}) - p_{ij} p_{ji}} \\ &= \frac{(1 - p_{jj})(p_{ij} p_{ki} + p_{kj} - p_{kj} p_{ii})}{(1 - p_{ii})(1 - p_{jj}) - p_{ij} p_{ji}} \\ &= (1 - p_{jj}) \frac{p_{kj}^{i}}{1 - p_{jj}^{i}}, \end{aligned}$$

and the proof is complete. □

Lemma 4. *Let w be a positive probability vector of order $1 \times n$, let e be an $n \times 1$ vector of all 1's and let $P = ew$. Then for $3 \leqslant k \leqslant n - 1$,*

$$\frac{p_{kk-1}^{12\cdots k-2}}{1 - p_{k-1k-1}^{12\cdots k-2}} = \frac{w_{k-1}}{1 - w_1 - \cdots - w_{k-1}}. \tag{9}$$

Proof. Suppose P is partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{10}$$

where P_{11} is $r \times r$, $1 \leqslant r < n$. We first derive an expression for the Perron complement $P^{12\cdots r}$. To this end, let $e_{(r)}$, $e_{(n-r)}$ be column vectors of all 1's of order r, $n-r$, respectively, and let

$$w_{(r)} = [w_1, \ldots, w_r] \quad \text{and} \quad w_{(n-r)} = [w_{r+1}, \ldots, w_n].$$

Then (10) can be expressed as

$$P = \begin{bmatrix} e_{(r)} w_{(r)} & e_{(r)} w_{(n-r)} \\ e_{(n-r)} w_{(r)} & e_{(n-r)} w_{(n-r)} \end{bmatrix}. \tag{11}$$

It follows from (11) that

$$P^{12\cdots r} = e_{(n-r)} w_{(n-r)} + e_{(n-r)} w_{(r)} \left(I - e_{(r)} w_{(r)}\right)^{-1} e_{(r)} w_{(n-r)}. \tag{12}$$

It is easily verified that

$$\left(I - e_{(r)} w_{(r)}\right)^{-1} = I + \frac{e_{(r)} w_{(r)}}{1 - w_{(r)} e_{(r)}}. \tag{13}$$

Substituting (13) into (12) and simplifying, we get

$$P^{12\cdots r} = \frac{e_{(n-r)} w_{(n-r)}}{1 - w_{(r)} e_{(r)}}. \tag{14}$$

It follows from (14) that

$$p_{kk-1}^{12\cdots k-2} = p_{k-1k-1}^{12\cdots k-2} = \frac{w_{k-1}}{1 - w_1 - \cdots - w_{k-2}}. \tag{15}$$

Clearly (9) follows from (15) and the proof is complete. □

In what follows, we will denote the (i, j)-entry of the matrix A by $A(i, j)$ as well.

Lemma 5. *Let P be an $n \times n$ irreducible, stochastic matrix and let $2 \leqslant v \leqslant n-1$. Set $S = \{1, 2, \ldots, v-1\}$. Then*

$$\sum_{i=1}^{v} p_{v+1,i} (I - P[S \cup \{v\} \mid S \cup \{v\}])^{-1}(i, v) = \frac{p_{v+1,v}^{S}}{1 - p_{vv}^{S}}. \tag{16}$$

Proof. Let $T = I - P[S \mid S]$. Then

$$p_{v+1,v}^{S} = p_{v+1,v} + [p_{v+1,1}, \ldots, p_{v+1,v-1}] T^{-1} \begin{bmatrix} p_{1v} \\ \vdots \\ p_{v-1,v} \end{bmatrix}$$

$$= -\frac{1}{|T|}\begin{vmatrix} & T & & -p_{1v} \\ & & & \vdots \\ & & & -p_{v-1,v} \\ -p_{v+1,1} & \cdots & -p_{v+1,v-1} & -p_{v+1,v} \end{vmatrix}. \tag{17}$$

Also

$$1 - p^S_{vv} = 1 - p_{vv} - [p_{v1}, \dots, p_{v,v-1}]T^{-1}\begin{bmatrix} p_{1v} \\ \vdots \\ p_{v-1,v} \end{bmatrix}$$

$$= \frac{1}{|T|}\begin{vmatrix} & T & & -p_{1v} \\ & & & \vdots \\ & & & -p_{v-1,v} \\ -p_{v1} & \cdots & -p_{v,v-1} & 1 - p_{vv} \end{vmatrix}$$

$$= \frac{|I - P[S \cup \{v\} \mid S \cup \{v\}]|}{|T|}. \tag{18}$$

From (17) and (18) we have

$$\frac{p^S_{v+1,v}}{1 - p^S_{vv}} = -\frac{\begin{vmatrix} & T & & -p_{1v} \\ & & & \vdots \\ & & & -p_{v-1,v} \\ -p_{v+1,1} & \cdots & -p_{v+1,v-1} & -p_{v+1,v} \end{vmatrix}}{|I - P[S \cup \{v\} \mid S \cup \{v\}]|}. \tag{19}$$

Expand the determinant in the numerator in (19) in terms of its last row. Then it can be seen that the right-hand side of (19) equals the left-hand side of (16) and the result is proved. □

3. Move-to-front scheme

We introduce some notation. For $2 \leqslant r \leqslant n$, let S^r_n denote the set of r-permutations of $1, 2, \dots, n$. Thus $|S^r_n| = n!/(n-r)!$ and $S^n_n = S_n$. Let P be an $n \times n$ irreducible, stochastic matrix and let π be the normalized left Perron eigenvector of P. For $\tau = \tau(1) \cdots \tau(r) \in S^r_n$, define

$$f(\tau) = \pi_{\tau(r)} \frac{p_{\tau(2)\tau(1)}}{1 - p_{\tau(1)\tau(1)}} \prod_{k=3}^{r} \frac{p^{\tau(1)\tau(2)\cdots\tau(k-2)}_{\tau(k)\tau(k-1)}}{1 - p^{\tau(1)\tau(2)\cdots\tau(k-2)}_{\tau(k-1)\tau(k-1)}}. \tag{20}$$

It may be remarked that since P is irreducible, any Perron complement in P is irreducible and does not have a diagonal entry equal to 1. Thus $f(\tau)$ is well defined.

Let f^r be the row vector of order $n!/(n-r)!$ indexed by S_n^r with the element corresponding to $\tau \in S_n^r$ given by $f(\tau)$. With this notation we have the following:

Lemma 6. *For $2 \leqslant r \leqslant n$, f^r is a probability vector.*

Proof. We prove the result by induction on r. First, let $r=2$. If ij is a 2-permutations of $1,\ldots,n$, then

$$f(ij) = \pi_j \frac{p_{ji}}{1-p_{ii}}. \tag{21}$$

Since

$$\sum_{j=1}^{n} \pi_j p_{ji} = \pi_i,$$

then

$$\sum_{j \neq i} \pi_j p_{ji} = (1-p_{ii})\pi_i. \tag{22}$$

It follows from (21) and (22) that

$$\sum_{i=1}^{n} \sum_{j \neq i} f(ij) = \sum_{i=1}^{n} \pi_i = 1,$$

and the result is proved for $r=2$.

Assume the result to be true for $r-1$ and proceed by induction. Let $\tau \in S_n^{r-1}$ and let $T = \{\tau(1),\ldots,\tau(r-1)\}$. Then

$$\sum_{i \notin T} f(\tau(1)\cdots\tau(r-1)i) = \frac{p_{\tau(2)\tau(1)}}{1-p_{\tau(1)\tau(1)}} \prod_{k=3}^{r} \frac{p_{\tau(k)\tau(k-1)}^{\tau(1)\tau(2)\cdots\tau(k-2)}}{1-p_{\tau(k-1)\tau(k-1)}^{\tau(1)\tau(2)\cdots\tau(k-2)}} \times \sum_{i \notin T} \frac{p_{i\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)}}{1-p_{\tau(r)\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)}} \pi_i. \tag{23}$$

By Lemma 1,

$$\sum_{i \notin T} p_{i\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)} \pi_i + p_{\tau(r)\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)} \pi_{\tau(r)} = \pi_{\tau(r)},$$

and hence

$$\sum_{i \notin T} \frac{p_{i\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)}}{1-p_{\tau(r)\tau(r)}^{\tau(1)\tau(2)\cdots\tau(r-1)}} \pi_i = \pi_{\tau(r)}. \tag{24}$$

Substituting (24) into (23) we see that

$$\sum_{i \notin T} f(\tau(1)\cdots\tau(r)i) = f(\tau). \tag{25}$$

It follows from (25) and the induction hypothesis that

$$\sum_{\tau \in S_n^r} f(\tau) = \sum_{\tau \in S_n^{r-1}} f(\tau) = 1,$$

and the result is proved. □

In the next result we give an explicit formula for the normalized left Perron eigenvector of $\mathscr{T}(P)$ in terms of the corresponding vector of P.

Theorem 7. *Let P be an $n \times n$ irreducible, stochastic matrix, $n \geqslant 3$, with the normalized left Perron eigenvector π. For $\tau \in S_n$, let*

$$f(\tau) = \pi_{\tau(n-1)} \frac{p_{\tau(2)\tau(1)}}{1 - p_{\tau(1)\tau(1)}} \prod_{j=3}^{n-1} \frac{p_{\tau(j)\tau(j-1)}^{\tau(1)\tau(2)\cdots\tau(j-2)}}{1 - p_{\tau(j-1)\tau(j-1)}^{\tau(1)\tau(2)\cdots\tau(j-2)}}. \tag{26}$$

Then the row vector $f = (f(\tau))_{\tau \in S_n}$ is the normalized left Perron eigenvector of the Tsetlin matrix $\mathscr{T}(P)$.

Proof. We remark that we always set the product over an empty set (for example, when $n = 3$ in (26)) to be 1. If $\tau \in S_n$, then we associate with it the unique $\hat{\tau} \in S_n^{n-1}$ given by $\hat{\tau} = \tau(1) \cdots \tau(n-1)$. Then $f(\tau)$ equals $f(\hat{\tau})$ defined in (20). It follows from Lemma 4 that f is a probability vector.

We now must show that $f\mathscr{T}(P) = f$. Let id $\in S_n$ be the identity permutation. We will show that the inner product of f with the column of $\mathscr{T}(P)$ indexed by id equals $f(\mathrm{id})$. The case of any other column is similar.

The column of $\mathscr{T}(P)$ indexed by id has p_{11} in the row corresponding to id, p_{21} in the rows corresponding to the permutations

$$213 \cdots n, 2314 \cdots n, \ldots, 23 \cdots n1$$

and 0's elsewhere. Thus we must show that

$$p_{11} f(12 \cdots n) + p_{21} \sum_{i=3}^{n+1} f(23 \cdots i-1 1 i \cdots n) = f(12 \cdots n). \tag{27}$$

We set up the following auxiliary assertion:

$$\sum_{i=s}^{n} f(23 \cdots i 1 i+1 \cdots n) = \pi_{n-1} \frac{p_{32}}{1 - p_{22}} \prod_{i=4}^{s} \frac{p_{i i-1}^{23\cdots i-2}}{1 - p_{i-1 i-1}^{23\cdots i-2}} \times \prod_{i=s}^{n-2} \frac{p_{i+1 i}^{12\cdots i-1}}{1 - p_{ii}^{12\cdots i-1}}, \quad 3 \leqslant s \leqslant n-1. \tag{28}$$

We will prove (28) by backward induction on s. For $s = n-1$, we have, using the definition of f,

$$
\begin{aligned}
&f(23\cdots n-1\,1n)+f(23\cdots n\,1)\\
&=\frac{p_{32}}{1-p_{22}}\left(\prod_{i=4}^{n-1}\frac{p_{i\,i-1}^{23\cdots i-2}}{1-p_{i-1\,i-1}^{23\cdots i-2}}\right)\\
&\quad\times\left(\frac{p_{1\,n-1}^{23\cdots n-2}}{1-p_{n-1\,n-1}^{23\cdots n-2}}\pi_1+\frac{p_{n\,n-1}^{23\cdots n-2}}{1-p_{n-1\,n-1}^{23\cdots n-2}}\pi_n\right).
\end{aligned}
\tag{29}
$$

By Lemma 1,

$$
p_{1\,n-1}^{23\cdots n-2}\pi_1+p_{n\,n-1}^{23\cdots n-2}\pi_n+p_{n-1\,n-1}^{23\cdots n-2}\pi_{n-1}=\pi_{n-1},
$$

and hence

$$
p_{1\,n-1}^{23\cdots n-2}\pi_1+p_{n\,n-1}^{23\cdots n-2}\pi_n=\left(1-p_{n-1\,n-1}^{23\cdots n-2}\right)\pi_{n-1}.
\tag{30}
$$

Substituting (30) into (29) we see that (28) is proved for $s=n-1$.

Now assume that (28) holds for some s, $3<s\leqslant n-1$ and we will prove it for $s-1$. Using the induction hypothesis,

$$
\begin{aligned}
&\sum_{i=s-1}^{n}f(23\cdots i\,1\,i+1\cdots n)\\
&=f(23\cdots s-1\,1s\cdots n)+\sum_{i=s}^{n}f(23\cdots i\,1\,i+1\cdots n)\\
&=\pi_{n-1}\frac{p_{32}}{1-p_{22}}\prod_{i=4}^{s-1}\frac{p_{i\,i-1}^{23\cdots i-2}}{1-p_{i-1\,i-1}^{23\cdots i-2}}\prod_{i=s}^{n-2}\frac{p_{i+1\,i}^{12\cdots i-1}}{1-p_{ii}^{12\cdots i-1}}\\
&\quad\times\frac{1}{1-p_{s-1\,s-1}^{23\cdots s-1}}\left(p_{1\,s-1}^{23\cdots s-2}\frac{p_{s1}^{23\cdots s-1}}{1-p_{11}^{23\cdots s-1}}+p_{s\,s-1}^{23\cdots s-2}\right).
\end{aligned}
\tag{31}
$$

By Lemma 3,

$$
p_{1\,s-1}^{23\cdots s-2}\frac{p_{s1}^{23\cdots s-1}}{1-p_{11}^{23\cdots s-1}}+p_{s\,s-1}^{23\cdots s-2}=\left(1-p_{s-1\,s-1}^{23\cdots s-2}\right)\frac{p_{s\,s-1}^{12\cdots s-1}}{1-p_{s-1\,s-1}^{12\cdots s-1}}.
\tag{32}
$$

Substituting (32) into (31) we see that (28) is proved for $s-1$. This completes the proof of (28) for $3\leqslant s\leqslant n-1$. Setting $s=3$, Eq. (28) reduces to

$$
\sum_{i=3}^{n}f(23\cdots i\,1\,i+1\cdots n)=\pi_{n-1}\frac{p_{32}}{1-p_{22}}\prod_{i=3}^{n-2}\frac{p_{i+1\,i}^{12\cdots i-1}}{1-p_{ii}^{12\cdots i-1}}.
\tag{33}
$$

Hence

$$f(213\cdots n)+\sum_{i=3}^{n} f(23\cdots i1i+1\cdots n)$$
$$=\pi_{n-1}\frac{1}{1-p_{22}}\prod_{i=3}^{n-2}\frac{p_{i+1i}^{12\cdots i-1}}{1-p_{ii}^{12\cdots i-1}}\left(p_{12}\frac{p_{31}^{2}}{1-p_{11}^{2}}+p_{32}\right). \tag{34}$$

By Lemma 3,

$$p_{12}\frac{p_{31}^{2}}{1-p_{11}^{2}}+p_{32}=(1-p_{22})\frac{p_{32}^{1}}{1-p_{22}^{1}}. \tag{35}$$

Substituting (35) into (34) and rearranging the terms we see that (27) is proved. That completes the proof. □

Dobrow and Fill [1, Theorem 4.1] have obtained an expression for a normalized left Perron eigenvector of $\mathscr{T}(P)$ using probabilistic arguments. Their formula for $f(\tau)$ is given by

$$\pi_{\tau(n)}\prod_{v=1}^{n-1}\left[\sum_{i=1}^{v} p_{\tau(v+1)\tau(i)}\Big(I-P[\{\tau(1),\ldots,\tau(v)\}\,\big|\{\tau(1),\ldots,\tau(v)\}]\Big)^{-1}(\tau(i),\tau(v))\right]. \tag{36}$$

We indicate that the formulas in (26) and (36) are equivalent. Using Lemma 5 we may rewrite (36) as

$$f(\tau)=\pi_{\tau(n)}\frac{p_{\tau(2)\tau(1)}}{1-p_{\tau(1)\tau(1)}}\prod_{j=3}^{n}\frac{p_{\tau(j)\tau(j-1)}^{\tau(1)\tau(2)\cdots\tau(j-2)}}{1-p_{\tau(j-1)\tau(j-1)}^{\tau(1)\tau(2)\cdots\tau(j-2)}}. \tag{37}$$

We claim that

$$\pi_{\tau(n)}\frac{p_{\tau(n)\tau(n-1)}^{\tau(1)\tau(2)\cdots\tau(n-2)}}{1-p_{\tau(n-1)\tau(n-1)}^{\tau(1)\tau(2)\cdots\tau(n-2)}}=\pi_{\tau(n-1)}. \tag{38}$$

Observe that $P^{\tau(1)\tau(2)\cdots\tau(n-2)}$ is a 2×2 stochastic matrix and by Lemma 1 it has $[\pi_{\tau(n-1)},\pi_{\tau(n)}]$ as a left Perron eigenvector. Therefore,

$$\pi_{\tau(n-1)}p_{\tau(n-1)\tau(n-1)}^{\tau(1)\tau(2)\cdots\tau(n-2)}+\pi_{\tau(n)}p_{\tau(n)\tau(n-1)}^{\tau(1)\tau(2)\cdots\tau(n-2)}=\pi_{\tau(n-1)},$$

which yields (38). The equivalence of (26) and (36) now follows from (37) and (38).

The proof in [1] is based on nontrivial results such as the strong Markov property and the bounded convergence theorem while our proof is elementary. Furthermore, our proof can be easily adapted to give a left Perron eigenvector of $\mathscr{T}(P)$ when P is just assumed to be irreducible, nonnegative but not necessarily stochastic. The corresponding expression is just an obvious modification of (26). The formula in

(26) is more compact than the one in (36). Finally, our proof technique can handle some other related schemes as discussed in Section 4.

We would like to mention at least in passing that $\mathscr{T}(P)$ has a remarkable spectral structure. For example, the eigenvalues of $\mathscr{T}(P)$ are precisely the eigenvalues of principal submatrices of P, with certain multiplicities (see [6] for details). A description of the principal idempotents of $\mathscr{T}(P)$ is given in [1].

When the requests for the books do not depend upon the previous requests we get a situation where the stochastic matrix has identical rows. In this event Theorem 7 reduces to a well-known result of Hendricks [3] as we show next.

Theorem 8. *Let w be a positive probability vector of order $1 \times n$, let e be an $n \times 1$ vector of all ones and let $P = ew$. For $\tau \in S_n$, let*

$$g(\tau) = w_{\tau(n-1)} \frac{w_{\tau(1)}}{1 - w_{\tau(1)}} \prod_{k=3}^{n-1} \frac{w_{\tau(k-1)}}{1 - w_{\tau(1)} - \cdots - w_{\tau(k-1)}}.$$

Then the vector $(g(\tau))_{\tau \in S_n}$ is the normalized left Perron eigenvector of $\mathscr{T}(P)$.

Proof. The result follows from Theorem 7, Lemma 4 and the observation that w is the normalized left Perron eigenvector of P. $\square$

4. Move-to-position k scheme

Besides the move-to-front scheme, a host of other schemes have been considered in the literature. Of particular interest to us will be the move-to-position k scheme, described in Section 1, which is a generalization of the move-to-front scheme.

In the move-to-position k scheme, by inspecting the current order of the books we can infer that the book chosen by the previous user must have been the book in position k. In addition, when $k = 1$, we can also conclude that the penultimate distinct book chosen must have been the book in position 2 and so on. This feature of the move-to-front scheme has been crucially used in devising the probabilistic proofs given in [1]. For move-to-position k scheme with $k > 1$ it is no longer possible to specify the distinct book chosen before the book in position k and therefore the arguments of [1] are not suited to handle move-to-position k scheme for $k > 1$.

In the next result, we give a formula for the normalized left Perron eigenvector of $\mathscr{T}_2(P)$, where P is an irreducible, stochastic matrix.

Theorem 9. *Let P be an $n \times n$ irreducible, stochastic matrix, $n \geqslant 3$, with the normalized left Perron eigenvector π. For $\tau \in S_n$, let*

$$f(\tau) = \pi_{\tau(n-1)} \prod_{k=3}^{n-1} \frac{p_{\tau(k)\tau(k-1)}^{\tau(1)\tau(2)\cdots\tau(k-2)}}{1 - p_{\tau(k-1)\tau(k-1)}^{\tau(1)\tau(2)\cdots\tau(k-2)}}. \tag{39}$$

Then the row vector $f = \left(\frac{1}{n-1} f(\tau)\right)_{\tau \in S_n}$ is the normalized left Perron eigenvector of the Tsetlin matrix $\mathscr{T}_2(P)$.

The proof of Theorem 9 is similar to that of Theorem 7 and is omitted. The fact that $\left(\frac{1}{n-1} f(\tau)\right)$ is a probability vector can be proved as in Lemma 6.

As an example, when $n = 3$ and $k = 2$, we conclude by Theorem 9 that the vector

$$\frac{1}{2}\left[\pi_2, \pi_3, \pi_1, \pi_3, \pi_1, \pi_2\right]$$

is the normalized left Perron eigenvector of $\mathscr{T}_2(P)$, where π is the normalized left Perron eigenvector of P. This fact can be verified directly using the expression for $\mathscr{T}_2(P)$ given in Section 1.

We remark that for $k = n$ and $k = n - 1$, a formula similar to that in Theorems 7 and 9 holds. We state it next. Again the proof is omitted since it is similar to that of Theorem 7.

Theorem 10. *Let P be an $n \times n$ irreducible, stochastic matrix, $n \geqslant 3$, with the normalized left Perron eigenvector π. For $\tau \in S_n$, let*

$$f(\tau) = \pi_{\tau(2)} \frac{p_{\tau(n-1)\tau(n)}}{1 - p_{\tau(n)\tau(n)}} \prod_{k=3}^{n-1} \frac{p_{\tau(k-1)\tau(k)}^{\tau(k+1)\tau(k+2)\cdots\tau(n)}}{1 - p_{\tau(k)\tau(k)}^{\tau(k+1)\tau(k+2)\cdots\tau(n)}}. \tag{40}$$

Then the row vector $f = (f(\tau))_{\tau \in S_n}$ is the normalized left Perron eigenvector of the Tsetlin matrix $\mathscr{T}_n(P)$.

Theorem 11. *Let P be an $n \times n$ irreducible, stochastic matrix, $n \geqslant 3$, with the normalized left Perron eigenvector π. For $\tau \in S_n$, let*

$$f(\tau) = \pi_{\tau(2)} \prod_{k=3}^{n-1} \frac{p_{\tau(k-1)\tau(k)}^{\tau(k+1)\tau(k+2)\cdots\tau(n)}}{1 - p_{\tau(k)\tau(k)}^{\tau(k+1)\tau(k+2)\cdots\tau(n)}}. \tag{41}$$

Then the row vector $f = \left(\frac{1}{n-1} f(\tau)\right)_{\tau \in S_n}$ is the normalized left Perron eigenvector of the Tsetlin matrix $\mathscr{T}_{n-1}(P)$.

The problem of finding an expression for a left Perron eigenvector of $\mathscr{T}_k(P)$ for all k, $3 \leqslant k \leqslant n - 2$, is posed as an open problem. When the requests are independent (that is, when P is of rank 1) an expression for a left Perron eigenvector of $\mathscr{T}_k(P)$ has been given by Hendricks [4].

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