Decomposition of generalized polynomial symmetric matrices

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Abstract

Characterization of nonnegative matrices A which satisfy the equation

$$(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i},$$

where $p < m_1 < m_2 < \cdots < m_t$ are positive integers and $\alpha_i > 0$ for all i is given as a direct sum of matrices of special types. Applications to stochastic and {0,1} matrices are derived, thus generalizing a number of earlier results.

Keywords: Nonnegative matrices; Group inverse; Drazin pseudoinverse; Stochastic matrices; {0,1} matrices

1. Introduction

A matrix A is nonnegative, denoted $A \geqslant 0$, if each entry of the matrix is nonnegative. A nonnegative matrix is stochastic if each row sum of the matrix is one. The transpose of the matrix A will be denoted by A^{T}. A matrix A is doubly stochastic if both A and A^{T} are stochastic.

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Sinkhorn [10] characterized stochastic matrices A which satisfy the condition $A^{\mathrm{T}} = A^p$, where $p > 1$ is a positive integer. Such matrices were called power symmetric in [10]. In [1] a square matrix A was defined to be generalized power symmetric if $(A^p)^{\mathrm{T}} = A^m$, where $p < m$ are positive integers. The main result in [1] obtained a characterization of generalized power symmetric stochastic matrices, thereby generalizing earlier results.

Call a square matrix A generalized polynomial symmetric if

$$(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}, \tag{1.1}$$

where $p < m_1 < m_2 < \cdots < m_t$ are positive integers and $\alpha_i > 0$ for all i. In the present paper we first obtain a characterization of nonnegative generalized polynomial symmetric matrices of index one. The decomposition obtained in the general case is then specialized to matrices of index one which are $\{0,1\}$ matrices or are stochastic. The case of matrices of higher index is settled using the core-nilpotent decomposition. The techniques in the present paper rely heavily on the machinery of generalized inverses. Our methods considerably simplify the proofs given earlier in [1]. In particular, we take advantage of the properties of λ-monotone matrices. Monotonicity of generalized inverses has been studied by many authors (see for example [4,5,9,11,12]). We remark that since $(A^n)_{ij}$ represents the probability of an event to change from the state i to the state j in n units of time, for a stochastic matrix A, the condition (1.1) can be given a probabilistic interpretation.

Furthermore, if A is a $\{0,1\}$ matrix that represents the adjacency matrix of a graph, then (1.1) gives the family of graphs where the number of paths from $j \to i$ of length p is equal to the total number of paths from $i \to j$ of length $m_1, m_2, \ldots, m_k$ (cf. Example 4.1).

The paper is organized as follows. In Section 2 we first introduce some definitions and prove certain preliminary results. We then obtain our main result of the paper that gives a decomposition of nonnegative matrices of index one satisfying Eq. (1.1) (Theorem 3.1 in Section 3). The special cases of $\{0,1\}$ and stochastic matrices are carried out in Section 4. Finally, we give a decomposition of nonnegative matrices of arbitrary index satisfying Eq. (1.1).

2. Definitions, notation, and preliminary results

If A is an $m \times n$ matrix, then an $n \times m$ matrix G is called a generalized inverse of A if $AGA = A$. If A is a square matrix, then G is called the group inverse of A if $AGA = A, GAG = G$ and $AG = GA$. We refer to [2] for the background concerning generalized inverses. It is well known that A admits group inverse if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^2)$, in which case the group inverse, denoted by $A^{\#}$, is unique.

If A is an $n \times n$ matrix, then the index of A, denoted by index A, is the least positive integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$. Thus A has group inverse if and only if index $A = 1$.

If A is an $n \times n$ matrix of index k, then an $n \times n$ matrix G is called the Drazin inverse of A if it satisfies $GAG = G$, $AG = GA$ and $A^{k+1}G = A^k$. It is well known that the Drazin inverse exists and is unique. We denote the Drazin inverse of A by $A^{(\mathrm{d})}$. Note that if index $A = 1$, then $A^{(\mathrm{d})} = A^{\#}$.

The reader is referred to [2] for additional definitions and results on generalized inverses.

Lemma 2.1. *If $A \geqslant 0$ and $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$, where $p < m_1 < m_2 < \cdots < m_t$ are positive integers and $\alpha_i > 0$ for all i, then* index $A \leqslant p$.

Proof. $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i} \Rightarrow (A^p)^{\mathrm{T}} = A^{p+1}(\mathbf{X})$ where $\mathbf{X} = \sum_{i=1}^{t} \alpha_i A^{m_i-(p+1)}$ and because $p < m_1$, the powers of A in X are all nonnegative.

This implies $\operatorname{rank}((A^p)^{\mathrm{T}}) = \operatorname{rank}(A^p) \leqslant \operatorname{rank}(A^{p+1})$.

But we always have $\operatorname{rank}(A^p) \geqslant \operatorname{rank}(A^{p+1}) \Rightarrow \operatorname{rank}(A^p) = \operatorname{rank}(A^{p+1}) \Rightarrow$ index $A \leqslant p$. □

The lemma which follows is the key lemma.

Lemma 2.2. *If $A \geqslant 0$ and $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$, where $p < m_1 < m_2 < \cdots < m_t$ are positive integers and $\alpha_i > 0$ for all i then $A^{(\mathrm{d})} \geqslant 0$.*

Proof

$$(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i} \quad \text{implies that } A^p = \sum_{i=1}^{t} \alpha_i (A^{m_i})^{\mathrm{T}}$$

and so,

$$A^p = \sum_{i=1}^{t} \alpha_i (A^p)^{\mathrm{T}} (A^{m_i-p})^{\mathrm{T}}$$

yielding

$$A^p = \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i A^{m_i} \right) (A^{m_j-p})^{\mathrm{T}}.$$

Next, we multiply the above by $A^{(n-1)p}$ and we write $m_i + (n-1)p = m_i' n + n_i$ where $0 \leqslant n_i \leqslant n-1$. Since $m_i + (n-1)p > np$, we have that $m_i' > p$. Thus, we get the equation

$$A^{np} = \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i A^{m_i+(n-1)p} \right) (A^{m_j-p})^{\mathrm{T}}.$$

Now, suppose index $A = n$. Let $B = A^n$, then index $B = 1$ and $B^{\#}$ exists.

Therefore,

$$B^p = \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i B^{m_i'} A^{n_i} \right) (A^{m_j - p})^{\mathrm{T}}.$$

We multiply by $(B^{\#})^p$ and get

$$(B^{\#})^p B^p = \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i (B^{\#})^p B^{m_i'} A^{n_i} \right) (A^{m_j - p})^{\mathrm{T}}.$$

Furthermore, $B^{\#} B^2 = B$ and $B^{\#} B = B B^{\#}$ implies $(B^{\#})^p B^p = B^{\#} B$ and $(B^{\#})^p B^{m_i'} = (B^{\#})^p B^p B^{m_i' - p} = B^{\#} B B^{m_i' - p} = B^k$, where $k = m_i' - p$. Thus,

$$B^{\#} B = \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i B^k A^{n_i} \right) (A^{m_j - p})^{\mathrm{T}} \geqslant 0.$$

Therefore, we have that $B^{\#} B \geqslant 0$.

Hence

$$\begin{aligned} B^{\#} &= (B^{\#})^2 B \\ &= B^{\#} \left(B^{\#} B \right) \\ &= B^{\#} \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i B^k A^{n_i} \right) (A^{m_j - p})^{\mathrm{T}} \\ &= \sum_{j=1}^{t} \alpha_j \left(\sum_{i=1}^{t} \alpha_i B^{\#} B^k A^{n_i} \right) (A^{m_j - p})^{\mathrm{T}} \\ &\geqslant 0 \end{aligned}$$

since $B^{\#} B \geqslant 0$.

Thus $B^{\#} = (A^n)^{\#} \geqslant 0$. It is known that if index $A = n$, then $(A^n)^{\#} = A^{(\mathrm{d})}$. Therefore, $A^{(\mathrm{d})} \geqslant 0$. □

We remark that in addition to the hypotheses of Lemma 1 if we assume that index $A = 1$, then it follows that $A^{\#} \geqslant 0$.

3. Decomposition theorem

The following is the main result of this section which gives a decomposition of a matrix of index one satisfying (1.1).

Theorem 3.1. *Let A be a nonnegative $n \times n$ matrix with* index $A = 1$. *If*

$$(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i} \tag{3.1}$$

with $p < m_1 < \cdots < m_t$ positive integers and $\alpha_i > 0$ then there exists a permutation matrix P such that PAP^{T} is a direct sum of matrices of the following types:

(I): *βxy^{T}, x, y positive unit vectors of the same size with $y^{\mathrm{T}}x = 1$ and β is the unique positive root of $1 = \sum_{i=1}^{t} \alpha_i \beta^{m_i - p}$.*

(II): *$d \times d$ block partitioned matrices of the form*

$$\begin{bmatrix} 0 & \beta_{12}x_1x_2^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2x_3^{\mathrm{T}} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \beta_{d-1}x_{d-1}x_d^{\mathrm{T}} \\ \beta_{d1}x_dx_1^{\mathrm{T}} & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

where the x_i's are positive unit vectors, not necessarily the same size, $\beta_{ij} > 0$, and $m_i + p \equiv m_j + p \equiv 0 \bmod d \ \forall i, j \leqslant t$, such that the following hold:

Case (i): $d \mid p$. Then $\beta_{12}\beta_{23}\cdots\beta_{d1}$ is the unique root of $\sum_{i=1}^{t} \alpha_i x^{q_i - q'} = 1$, where $m_i = dq_i + r$ and $p = dq' + r'$.

Case (ii): $d \nmid p$. Let $(d, r') = \delta$. Then $\beta_{i,i+1} = \beta_{i+r',i+r'+1}$ and if κ is the product of the distinct $\beta_{i,i+1}$'s, i.e. $\kappa = \beta_{12}\beta_{23}\cdots\beta_{r',r'+1}$, then κ is the unique positive root of $1 = \sum_{i=1}^{t} \alpha_i x^{(m_i - p)/\delta}$. In particular, if $(d, r') = 1$ then $\beta_{12} = \beta_{23} = \cdots = \beta_{d1} = \beta$, say, and β is the unique positive root of $1 = \sum_{i=1}^{t} \alpha_i x^{m_i - p}$.

(III): *Zero blocks of appropriate size.*

The converse also holds.

Proof. By the given hypothesis on A we may apply Lemma 2.2, and hence, $A^{\#} \geqslant 0$. Therefore, by Corollary 4.3 of [6], we have that there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where some of the rows may be absent, C, D are nonnegative matrices of suitable sizes, diagonal blocks are square matrices, and J is a direct sum of matrices of the following types,

(I′): $\beta xy^{\mathrm{T}}, \beta > 0$, x, y are positive unit vectors of the same size and $y^{\mathrm{T}}x = 1$.

(II′):

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^{\mathrm{T}} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & & & & \beta_{d-1,d}x_{d-1}y_d^{\mathrm{T}} \\ \beta_{d1}x_dy_1^{\mathrm{T}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

with $\beta_{ij} > 0, x_i, y_i$ are positive unit vectors, x_i, y_i are of the same size, x_i, y_j, $i \neq j$ are not necessarily the same size and $y_i^{\mathrm{T}}x_i = 1$.

Now, A satisfies (3.1) and so PAP^{T} satisfies (3.1). Also, notice that

$$(PAP^{\mathrm{T}})^f = \begin{bmatrix} J^f & J^fD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^f & CJ^fD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, (3.1) gives us that

$$\begin{bmatrix} J^p & 0 & CJ^p & 0 \\ J^pD & 0 & CJ^pD & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{i=1}^{t} \alpha_i \begin{bmatrix} J^{m_i} & J^{m_i}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^{m_i} & CJ^{m_i}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that $J^{m_i}D = 0$ and $CJ^{m_i} = 0$. But then, $J \geqslant 0$ implies $D = 0$ and $C = 0$; so in fact, we have $PAP^{\mathrm{T}} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$.

Now J satisfies (3.1) implies that each Type (I′) and Type (II′) satisfies (3.1).

Consider Type (I′): Recall that x and y are unit vectors, and so $x^{\mathrm{T}}x = y^{\mathrm{T}}y = 1$, such that $y^{\mathrm{T}}x = 1$. Using these facts we obtain the following:

From the equation $((\beta xy^{\mathrm{T}})^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i (\beta xy^{\mathrm{T}})^{m_i}$, we get

$$\begin{aligned} \beta^p yx^{\mathrm{T}} &= \sum_{i=1}^{t} \alpha_i \beta^{m_i} xy^{\mathrm{T}} \\ \Rightarrow \beta^p xy^{\mathrm{T}}yx^{\mathrm{T}} &= \sum_{i=1}^{t} \alpha_i \beta^{m_i} xy^{\mathrm{T}}xy^{\mathrm{T}} \\ \Rightarrow \beta^p xx^{\mathrm{T}} &= \sum_{i=1}^{t} \alpha_i \beta^{m_i} xy^{\mathrm{T}} \\ \Rightarrow \beta^p xx^{\mathrm{T}}x &= \sum_{i=1}^{t} \alpha_i \beta^{m_i} xy^{\mathrm{T}}x. \end{aligned}$$

The above equation implies $\beta^p x = \sum_{i=1}^{t} \alpha_i \beta^{m_i} x$, and so, $\beta^p = \sum_{i=1}^{t} \alpha_i \beta^{m_i}$, yielding $1 = \sum_{i=1}^{t} \alpha_i \beta^{m_i - p}$.

Thus, β is the unique positive solution of $1 = \sum_{i=1}^{t} \alpha_i x^{m_i - p}$.

Now consider a Type (II′) summand S of size d. We have $(S^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i^{m_i} S$ which in turn yields $(S^p)(S^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i S^{m_i+p}$. But $(S^p)(S^p)^{\mathrm{T}}$ is a block diagonal matrix, so each S^{m_i+p} must be diagonal also. But for Type (II′) summands, one gets diagonal matrices when one raises the matrix to a multiple of d. So we have $m_i + p \equiv m_j + p \equiv 0 \bmod d$ and consequently, $m_i \equiv m_j \bmod d$. Let q_i, r be such that $m_i = dq_i + r$.

Now, notice that

$$S^d = \beta_{12}\beta_{23}\cdots\beta_{d1} \begin{bmatrix} x_1 y_1^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & x_2 y_2^{\mathrm{T}} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_d y_d^{\mathrm{T}} \end{bmatrix}.$$

Let $\mu = \beta_{12}\beta_{23}\cdots\beta_{d1}$ and let

$$E = \begin{bmatrix} x_1 y_1^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & x_2 y_2^{\mathrm{T}} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_d y_d^{\mathrm{T}} \end{bmatrix}.$$

Then $S^d = \mu E$, $S^{2d} = \mu^2 E$, and so on. Also note that $ES = S$.

So,

$$\begin{aligned} (S^p)^{\mathrm{T}} &= \sum_{i=1}^{t} \alpha_i S^{m_i} = \sum_{i=1}^{t} \alpha_i S^{dq_i+r} \\ &= \sum_{i=1}^{t} \alpha_i (S^d)^{q_i} S^r = \sum_{i=1}^{t} \alpha_i \mu^{q_i} S^r \\ &= \left(\sum_{i=1}^{t} \alpha_i \mu^{q_i} \right) S^r. \end{aligned}$$

Thus we have $(S^p)^{\mathrm{T}} = \lambda S^r$ where $\lambda = \sum_{i=1}^{t} \alpha_i \mu^{q_i}$.

Then $(S^p)^{\mathrm{T}} = \lambda S^r$ implies $S^p (S^p)^{\mathrm{T}} = \lambda S^{p+r}$, and $(S^p)(S^p)^{\mathrm{T}}$ being diagonal tells us that S^{p+r} is diagonal, hence $r + p \equiv 0 \bmod d$. Now, $\lambda S^{p+r} = \lambda S^{((p+r)/d)d} = \lambda \mu^{(p+r)/d} E$. So, we have that $S^p (S^p)^{\mathrm{T}} = \lambda \mu^{(p+r)/d} E$.

Let q', r' be such that $p = dq' + r'$. We next simplify $(S^p)(S^p)^{\mathrm{T}}$.

$$\begin{aligned} (S^p)(S^p)^{\mathrm{T}} &= \left(S^{dq'+r'}\right)\left(S^{dq'+r'}\right)^{\mathrm{T}} = \left(S^{dq'} S^{r'}\right)\left(S^{dq'} S^{r'}\right)^{\mathrm{T}} \\ &= \left(\mu^{q'} E S^{r'}\right)\left(\mu^{q'} E S^{r'}\right)^{\mathrm{T}} = \left(\mu^{q'} S^{r'}\right)\left(\mu^{q'} S^{r'}\right)^{\mathrm{T}} \\ &= \left(\mu^{q'}\right)^2 \left(S^{r'}\right)\left(S^{r'}\right)^{\mathrm{T}} \end{aligned}$$

$$= \left(\mu^{q'}\right)^2 \begin{bmatrix} (\beta_1)^2 x_1 x_1^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & (\beta_2)^2 x_2 x_2^{\mathrm{T}} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (\beta_d)^2 x_{d1} x_{d1}^{\mathrm{T}} \end{bmatrix},$$

where $\beta_1 = \beta_{12}\beta_{23}\cdots\beta_{r',r'+1}, \beta_2 = \beta_{23}\cdots\beta_{r'+1,r'+2}, \dots, \beta_d = \beta_{d1}\cdots\beta_{r'+1,r'}$ and all indices are taken modulo d from here on. So, we have that $S^p(S^p)^{\mathrm{T}} = \lambda S^{p+r}$ simplifies to the following:

$$\left(\mu^{q'}\right)^2 \begin{bmatrix} (\beta_1)^2 x_1 x_1^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & (\beta_2)^2 x_2 x_2^{\mathrm{T}} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (\beta_d)^2 x_{d1} x_{d1}^{\mathrm{T}} \end{bmatrix} = \lambda \mu^{(p+r)/d} E.$$

Also, because both sides are symmetric E is symmetric, thus $x_i = a_i y_i$ for some $a_i \in \mathbb{R}^+$. But $y_i^{\mathrm{T}} x_i = 1$ implies $y_i^{\mathrm{T}} a_i y_i = 1$, which yields that $a_i = 1$ and hence $x_i = y_i$ for all i.

If $d \mid p$ then $r' = 0$ and $d \mid m$. Thus we get $(\mu^{q'})^2 E = \lambda \mu^{q'} E$ which gives us that $\mu^{q'} = \lambda$. We know $\lambda = \sum_{i=1}^{t} \alpha_i \mu^{q_i}$, and so $\mu^{q'} = \sum_{i=1}^{t} \alpha_i \mu^{q_i}$. Because $p < m_i$ we must have that $q' \leqslant q_i$. Dividing both sides by $\mu^{q'}$ gives us $1 = \sum_{i=1}^{t} \alpha_i \mu^{q_i - q'}$. So we have that $\mu = \beta_{12}\cdots\beta_{d1}$ must be the unique positive solution to $\sum_{i=1}^{t} \alpha_i x^{q_i - q'} = 1$.

Now, suppose $d \nmid p$. We get the following system of equations by comparing corresponding entries of $(S^p)(S^p)^{\mathrm{T}} = \lambda S^{p+r}$.

$$(1): \ (\mu^{q'})^2 (\beta_{12} \dots \beta_{r',r'+1})^2 = \lambda \mu^{(p+r)/d}$$

$$(2): \ (\mu^{q'})^2 (\beta_{23} \dots \beta_{r'+1,r'+2})^2 = \lambda \mu^{(p+r)/d}$$

$$\vdots$$

$$(d): \ (\mu^{q'})^2 (\beta_{d1}\beta_{12} \dots \beta_{r'+1,r'})^2 = \lambda \mu^{(p+r)/d}.$$

We get from (1) and (2) that $(\beta_{12}\beta_{23}\cdots\beta_{r',r'+1})^2 = (\beta_{23}\cdots\beta_{r'+1,r'+2})^2$ and thus, $\beta_{12} = \beta_{r'+1,r'+2}$. From (2) and (3) we get $\beta_{23} = \beta_{r'+2,r'+3}, \dots$ Finally, from (d) and (1) we get $\beta_{d1} = \beta_{r',r'+1}$. Now, if $(d, r') = 1$ then we have $\beta_{12} = \beta_{23} = \cdots = \beta_{d1}$, say β. So we have

$$\beta \begin{bmatrix} 0 & x_1 x_2^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & 0 & x_2 x_3^{\mathrm{T}} & \ddots & 0 \\ 0 & \dots & \ddots & \ddots & x_{d-1} x_d^{\mathrm{T}} \\ x_d x_1^{\mathrm{T}} & 0 & \dots & \dots & 0 \end{bmatrix} = S,$$

which implies, using the equation $(X^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i X^{m_i}$, that $\beta^p = \sum_{i=1}^{t} \alpha_i \beta^{m_i}$. Divide both sides of the equation by β^p, and obtain $1 = \sum_{i=1}^{t} \alpha_i \beta^{m_i - p}$. Therefore, if $(d, r') = 1$ then β is the unique positive solution of $1 = \sum_{i=1}^{t} \alpha_i x^{m_i - p}$.

Now, if $(d, r') = \delta \neq 1$, then taking all indices modulo d we get $\beta_{12} = \beta_{1+r',2+r'} = \beta_{1+2r',2+2r'} = \cdots$, until β_{12} appears again and this will happen after d/δ times; $\beta_{23} = \beta_{2+r',3+r'} = \beta_{2+2r',3+2r'} = \cdots$, until β_{23} appears again, $\beta_{r',r'+1} = \beta_{r'+r',r'+1+r'} = \beta_{r'+2r',r'+1+2r'} = \cdots$, until $\beta_{r',r'+1}$ appears again. So we have δ sets of $\beta_{i,i+1}$ repeated d/δ times. Now, we solve the equation $(X^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i X^{m_i}$ using $S^d = \mu E$ along with the fact that $\mu = \kappa^{d/\delta}$ where $\kappa = \beta_{12}\beta_{23} \cdots \beta_{r',r'+1}$. Also, note that $S^{r'} = \kappa^{r'/\delta} S'$. So, on the left side we have $(S^p)^{\mathrm{T}} = \left(S^{dq'+r'}\right)^{\mathrm{T}} = \left(S^{dq'} S^{r'}\right)^{\mathrm{T}} = \left(\left(\kappa^{(d/\delta)} E\right)^{q'} \kappa^{r'/\delta} S'\right)^{\mathrm{T}} = \left(\kappa^{(d/\delta)q' + (r'/\delta)} S'\right)^{\mathrm{T}} = \kappa^{(d/\delta)q' + (r'/\delta)} \left(S'\right)^{\mathrm{T}}$. On the right side we have $\sum_{i=1}^{t} \alpha_i S^{m_i} = \sum_{i=1}^{t} \alpha_i S^{dq_1 + r} = \sum_{i=1}^{t} \alpha_i S^{dq_1} S^r = \sum_{i=1}^{t} \alpha_i \left(\kappa^{d/\delta} E\right)^{q_1} \kappa^{r/\delta} \left(S'\right)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i \kappa^{(d/\delta)q_1 + (r/\delta)} \left(S'\right)^{\mathrm{T}}$.

And therefore, we have from the above equation $\kappa^{(d/\delta)q' + (r'/\delta)} \left(S'\right)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i \kappa^{(d/\delta)q_1 + (r/\delta)} \left(S'\right)^{\mathrm{T}}$, $\kappa^{(d/\delta)q' + (r'/\delta)} = \sum_{i=1}^{t} \alpha_i \kappa^{(d/\delta)q_1 + (r/\delta)}$. Then

$$1 = \sum_{i=1}^{t} \alpha_i \kappa^{(d/\delta)(q_i - q') + ((r - r')/\delta)} = \sum_{i=1}^{t} \alpha_i \kappa^{(m_i - p)/\delta}.$$

The converse is clear. This completes the proof. □

4. {0,1} matrices and stochastic matrices

As a first application of Theorem 3.1, we give the following characterization of {0,1} matrices of index one satisfying Eq. (1.1).

Theorem 4.1. *If A is a $\{0,1\}$ matrix with index $A = 1$ and $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$, with $p < m_1 < m_2 < \cdots < m_t$ and $\alpha_i > 0$ then there exists a permutation matrix P such that PAP^{T} is a direct sum of matrices of the following types*:

(I′): $J_{m \times m}$, *the $m \times m$ matrix, all of whose entries are* 1, *where m is the root of* $1 = \sum_{i=1}^{t} \alpha_i x^{m_i - p}$

(II′):

$$\begin{bmatrix} 0 & J_{l_1 \times l_2} & 0 & \cdots & 0 \\ \vdots & \ddots & J_{l_2 \times l_3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & J_{l_{d-1} \times l_d} \\ J_{l_d \times l_1} & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

where $J_{l_i \times l_j}$ is the $l_i \times l_j$ matrix, all of whose entries are 1.

Furthermore, if $d \mid p$ *then* $l_1 l_2 \cdots l_d$ *is the solution of* $1 = \sum_{i=1}^{t} \alpha_i x^{q_i - q'}$ *where* $m_i = dq_i + r$, $p = dq'$. *If* $d \nmid p$ *then we write* $p = dq' + r'$ *and* $(d, r') = \delta$, *then* $l_{i,i+1} = l_{i+r',i+r'+1}$ *and* $\kappa = \left(\sqrt{l_1}\sqrt{l_2}\right)\left(\sqrt{l_2}\sqrt{l_3}\right)\cdots\left(\sqrt{l_{r'}}\sqrt{l_{r'+1}}\right)$ *is the unique positive root of* $1 = \sum_{i=1}^{t} \alpha_i x^{(m_i - p)/\delta}$.

In particular, if $(d, r') = 1$, *then the only possible blocks in the above representation are* $J_{l_1 \times l_2}$ *and* $J_{l_2 \times l_1}$. *Moreover,* $\sqrt{l_1}\sqrt{l_2}$ *is the unique positive root of* $1 = \sum_{i=1}^{t} \alpha_i x^{m_i - p}$.

(III)*: Zero blocks of appropriate size.*

Proof. We have by Theorem 3.1 that there is a permutation matrix P such that PAP^{T} is a direct sum of Types (I), (II), (III).

Consider first the Type (I) $\beta x y^{\mathrm{T}}, \beta > 0, x, y$ positive unit vectors of the same length and $y^{\mathrm{T}} x = 1$. Let $x = \begin{bmatrix} x_1 & \cdots & x_l \end{bmatrix}^{\mathrm{T}}$ and $y = \begin{bmatrix} y_1 & \cdots & y_l \end{bmatrix}^{\mathrm{T}}$. Then

$$\beta x y^{\mathrm{T}} = \beta \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_l \\ \vdots & & \vdots \\ x_l y_1 & \cdots & x_l y_l \end{bmatrix}.$$

Because $\beta x y^{\mathrm{T}} > 0$ and A is a {0,1} matrix, $\beta x_1 y_1 = 1 = \beta x_1 y_2 = \cdots = \beta x_1 y_l$. This gives $y_1 = y_2 = \cdots = y_l$.

Furthermore, $\beta x_1 y_1 = 1 = \beta x_2 y_1 = \cdots = \beta x_l y_l$, and so $x_1 = x_2 = \cdots = x_l$.

Now x is a unit vector, and so $x = \begin{bmatrix} a & \cdots & a \end{bmatrix}^{\mathrm{T}}$, with $la^2 = 1$ and similarly $y = x = \begin{bmatrix} a & \cdots & a \end{bmatrix}^{\mathrm{T}}$ an $l \times 1$ vector. So $\beta x y^{\mathrm{T}} = 1$ gives $\beta = l$. From Theorem 3.1, β is the unique root of $1 = \sum_{i=1}^{t} \alpha_i \beta^{m_i - p}$. Then the Type (I) summands are those $\beta x y^{\mathrm{T}}$ for which β satisfies $1 = \sum_{i=1}^{t} \alpha_i \beta^{m_i - p}$ and x is the $\beta \times 1$ vector $\begin{bmatrix} \frac{1}{\sqrt{\beta}} & \cdots & \frac{1}{\sqrt{\beta}} \end{bmatrix}^{\mathrm{T}}$. This implies then that $\beta x y^{\mathrm{T}} = J_{m \times m}$.

Consider now Type (II) summands. Each $\beta_{i,i+1} x_i x_{i+1}^{\mathrm{T}}$ must have all entries equal to 1 as above. For convenience, call $\beta = \beta_{i,i+1}$, $x = x_i$, $y = x_{i+1}$.

So again x, y are unit vectors with all terms equal, therefore x is the $f \times 1$ vector with all entries equal to $1/\sqrt{f}$ and y is the $l \times 1$ vector with all entries equal to $1/\sqrt{l}$. Then $\beta(1/\sqrt{f})(1/\sqrt{l}) = 1 \Rightarrow \beta = \sqrt{f}\sqrt{l}$.

So in general if x_i is an $l_i \times 1$ vector we have that $\beta_{i,i+1} = \sqrt{l_i}\sqrt{l_{i+1}}$, and x_i is the $l_i \times 1$ vector with all entries equal to $1/\sqrt{l_i}$. But then, $\beta_{i,i+1} x_i x_{i+1}^{\mathrm{T}} = J_{l_i \times l_{i+1}}$.

If $d \mid p$ then $\beta_{12} \cdots \beta_{d1} = \sqrt{l_1}\sqrt{l_2} \cdots \sqrt{l_d}\sqrt{l_1} = l_1 \times \cdots \times l_d$ must be the positive solution of the equation $1 = \sum_{i=1}^{t} \alpha_i x^{q_i - q'}$.

If $d \nmid p$ and $(d, r') = 1$ then $\beta_{12} = \beta_{23} = \cdots = \beta_{dl} = \sqrt{l_1}\sqrt{l_2}$. Then the only possible blocks are $J_{l_1 \times l_2}$ and $J_{l_2 \times l_1}$.

If $(d, r') = \delta \neq 1$ then we have δ sets of $\beta_{i,i+1}$ repeated d/δ times. Therefore, $\beta_{12} = \sqrt{l_1}\sqrt{l_2} = \beta_{1+r',2+r'} = \cdots$ as in Theorem 3.1. $\beta_{23} = \sqrt{l_2}\sqrt{l_3} = \beta_{2+r',3+r'} = \cdots$ as in Theorem 3.1.

And, as in Theorem 3.1 if we let $\kappa = \beta_{12}\beta_{23}\cdots\beta_{r',r'+1}$ then κ is the unique positive root of $1 = \sum_{i=1}^{t} \alpha_i x^{(d/\delta)(q_i - q') + ((r-r')/\delta)}$.

This completes the proof. □

Clearly, the matrices satisfying these criteria are indeed very specialized and we make no claim that this can be applied to all cases, and so we present one set of data for which the criteria do apply and we now illustrate Theorem 4.1 with an example.

Example 4.1. Consider the flight map shown in Fig. 1. The matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

would be the adjacency matrix of the directed graph representing these flights with the rows representing Seattle, Columbus, Chicago, Los Angeles, Washington DC and Dallas respectively. This is a {0,1} matrix that satisfies the equation $18A^{\mathrm{T}} = A^2 + A^5$. Notice that A is a Type (II″) matrix. We can see that $d = 3$, hence $d \nmid p$ and $(d, r') = 1$, and $\beta = \sqrt{4} = 2$. So by Theorem 4.1, we get the equation $1 = \frac{1}{18}(2) + \frac{1}{18}(2)^4$. Notice that as expected, we have that m_1 and m_2 are congruent modulo d. So we consider the equation $18A^{\mathrm{T}} = A^2 + A^5$. We have for example the number of direct flights from Seattle to Chicago multiplied by 18 is the same as the total number of flights from Chicago to Seattle with 1 stop plus the number of flights from Chicago to Seattle with four stops. We also remark that because $d = 3$, it is possible to leave any city and come back to it with a total of three flights, making a total of two stops.

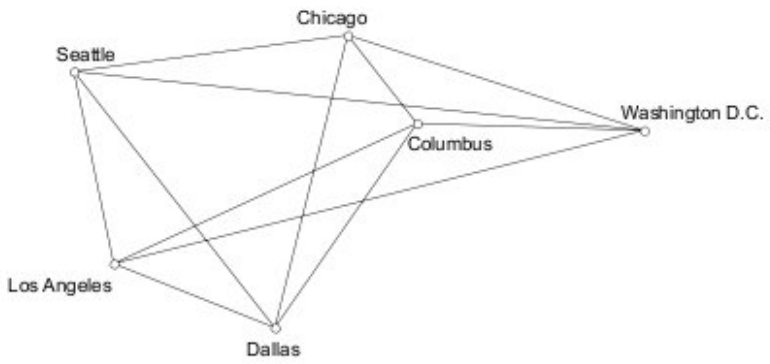

Fig. 1.

Another application of Theorem 3.1 gives us the following characterization of stochastic matrices of index one.

Theorem 4.2. *If $A \geqslant 0$ is a stochastic matrix with* index $A = 1$ *such that* $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$ *with* $p < m_1 < m_2 < \cdots < m_t$ *positive integers and* $\alpha_i > 0$, *then there exists a permutation matrix* P *such that* PAP^{T} *is a direct sum of matrices of the following types*:

(I^{rr}): $(1/l)J_{l\times l}$ *and* $1 = \sum_{i=1}^{t} \alpha_i$

(II^{rn}): $\hat{J}_{n_1,\ldots,n_d}$ *where*

$$\hat{J}_{n_1,\ldots,n_d} = \begin{bmatrix} 0 & \frac{1}{n_2}J_{n_1\times n_2} & 0 & \cdots & 0 \\ \vdots & \ddots & \frac{1}{n_3}J_{n_2\times n_3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \frac{1}{n_d}J_{n_{d-1}\times n_d} \\ \frac{1}{n_2}J_{n_d\times n_1} & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

where $J_{n_i\times n_j}$ *is the* $n_i \times n_j$ *matrix, all of whose entries are* 1.

If $d \mid p$ *(equivalently,* $d \mid m_i$*), then* $\sum_{i=1}^{t} \alpha_i = 1$. *If* $d \nmid p$, *write* $m_i = dq_i + r$, $p = dq' + r'$, *and* $(d, r') = \delta$. *Then we have* $l_{i,i+1} = l_{i+r',i+r'+1}$. *Moreover,* $\kappa = \left(\sqrt{l_1}\sqrt{l_2}\right)\left(\sqrt{l_2}\sqrt{l_3}\right)\cdots\left(\sqrt{l_{r'}}\sqrt{l_{r'+1}}\right)$ *is the unique positive root of* $1 = \sum_{i=1}^{t} \alpha_i x^{(m_i-p)/\delta}$. *In particular, if* $(d, r') = 1$, *then each block is* $(1/n_1)J_{n_1\times n_1}$ *and* n_1 *is the unique positive root of* $1 = \sum_{i=1}^{t} \alpha_i x^{m_i-p}$.

Proof. By Theorem 3.1, we get that there is a permutation matrix P such that PAP^{T} is a direct sum of matrices of Types (I), (II), (III). But because A is stochastic we have no Type (III) zero blocks.

So, consider Type (I) blocks βxy^{T}, $\beta > 0$, x, y positive unit vectors, $y^{\mathrm{T}}x = 1$. Now, because βxy^{T} is stochastic, we have $\beta xy^{\mathrm{T}}e = e$, where e is the vector all of whose entries are one. This yields $\beta y^{\mathrm{T}}xy^{\mathrm{T}}e = y^{\mathrm{T}}e$, and so, $\beta y^{\mathrm{T}}e = y^{\mathrm{T}}e$. Thus $\beta = 1$, because $y \neq 0$.

Now, xy^{T} satisfies $((xy^{\mathrm{T}})^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i (xy^{\mathrm{T}})^{m_i}$. But xy^{T} being idempotent, $yx^{\mathrm{T}} = (\sum_{i=1}^{t} \alpha_i)xy^{\mathrm{T}}$. Thus, $yx^{\mathrm{T}}e = (\sum_{i=1}^{t} \alpha_i)xy^{\mathrm{T}}e$. This implies that $yx^{\mathrm{T}}e = (\sum_{i=1}^{t} \alpha_i)e$, which in turn yields the equation $x^{\mathrm{T}}yx^{\mathrm{T}}e = (\sum_{i=1}^{t} \alpha_i)x^{\mathrm{T}}e$, implying $x^{\mathrm{T}}e = (\sum_{i=1}^{t} \alpha_i)x^{\mathrm{T}}e$. Finally, we obtain $\sum_{i=1}^{t} \alpha_i = 1$, because $x \neq 0$.

So $yx^{\mathrm{T}} = xy^{\mathrm{T}}$ is doubly stochastic. It is well known that a positive rank one doubly stochastic matrix is of the form $(1/l)J_{l\times l}$.

Now, consider Type (II) summands. Each block $\beta_{i,i+1}x_i x_{i+1}^{\mathrm{T}}$ of Type (II) must be stochastic. Consider $\beta_{12}x_1x_2^{\mathrm{T}}$. Then if $x_1 = \begin{bmatrix} x_{11} & \cdots & x_{1n} \end{bmatrix}^{\mathrm{T}}$ and $x_2 = \begin{bmatrix} x_{21} & \cdots & x_{2m} \end{bmatrix}^{\mathrm{T}}$ we have

$$1=\beta(x_{11}x_{21}+x_{11}x_{22}+\cdots+x_{11}x_{2n})=\beta x_{11}(x_{21}+\cdots+x_{2n})$$
$$\vdots$$
$$1=\beta(x_{1n}x_{21}+x_{1n}x_{22}+\cdots+x_{1n}x_{2n}=\beta x_{1n}(x_{21}+\cdots+x_{2n})$$

So we have $x_{11}=x_{12}=\cdots=x_{1n}$. Thus $x_1=\begin{bmatrix}a_1 & \cdots & a_1\end{bmatrix}^{\mathrm{T}}$, an $n_1\times 1$ vector. Then considering $\beta_{23}x_2x_3^{\mathrm{T}}$ we get the similar form for x_2, i.e. $x_2=\begin{bmatrix}a_2 & \cdots & a_2\end{bmatrix}^{\mathrm{T}}$, an $n_2\times 1$ vector. Repeating this gives us that each x_i is, say, $\begin{bmatrix}a_i & \cdots & a_i\end{bmatrix}^{\mathrm{T}}$ an $n_i\times 1$ vector. But also, each x_i is a unit vector. Therefore, $x_i=(1/\sqrt{n_i})\begin{bmatrix}1 & \cdots & 1\end{bmatrix}^{\mathrm{T}}$ an $n_i\times 1$ vector.

Thus $\beta_{i,i+1}x_ix_{i+1}^{\mathrm{T}}$ being stochastic tells us that $\beta_{i,i+1}(1/\sqrt{n_i})(1/\sqrt{n_{i+1}})J_{n_i\times n_{i+1}}$, is stochastic. But the row sum of $J_{n_i\times n_{i+1}}$ is n_{i+1}. So $\beta_{i,i+1}(1/\sqrt{n_i})(1/\sqrt{n_{i+1}})$ $n_{i+1}=1$, yielding $\beta_{i,i+1}=(\sqrt{n_i}\sqrt{n_{i+1}})/n_{i+1}$. Therefore, $\beta_{i,i+1}x_ix_{i+1}^{\mathrm{T}}=(1/n_{i+1})$ $J_{n_i\times n_{i+1}}$.

Now if $d\mid p$ then

$$\beta_{12}\beta_{23}\cdots\beta_{d1}=\frac{\sqrt{n_1},\sqrt{n_2}}{n_2}\times\frac{\sqrt{n_2}\sqrt{n_3}}{n_3}\times\cdots\times\frac{\sqrt{n_d}\sqrt{n_1}}{n_1}=1$$

must be a solution of the equation $1=\sum_{i=1}^{t}\alpha_i x^{q_i-q'}$. This implies $1=\sum_{i=1}^{t}\alpha_i$.

If $d\nmid p$ and $(d,r')=1$ then $\beta_{12}=\beta_{23}=\cdots=\beta_{d1}$. This gives

$$\frac{\sqrt{n_1}\sqrt{n_2}}{n_2}=\frac{\sqrt{n_2}\sqrt{n_3}}{n_3}=\cdots=\frac{\sqrt{n_d}\sqrt{n_1}}{n_1}$$

and so, $n_1=n_2=\cdots=n_d$.

If $(d,r')=\delta\neq 1$ then we have $\beta_{i,i+1}=\beta_{i+r',i+r'+1}$ and $\kappa=\beta_{12}\beta_{23}\cdots\beta_{r',r'+1}$ is the unique positive root of $1=\sum_{i=1}^{t}\alpha_i x^{(d/\delta)(q_i-q')+((r-r')/\delta)}$.

That completes the proof. □

We offer some remarks regarding the stochastic case.

1. It is known (see e.g. Theorem 4 in [7], and Corollary 4 in [8]) that if a row stochastic matrix A satisfies $A^{(\mathrm{d})}\geqslant 0$ (as in our case $A^{\#}\geqslant 0$) then $A^{(\mathrm{d})}=A^u$ for some positive integer u.
2. In [1], Theorem 1, stochastic matrices with index one satisfying $(A^p)^{\mathrm{T}}=A^m$ are characterized. A small verification reveals that the general result given in Theorem 4.2 includes the main result obtained in [1].

5. Decomposition of matrices of higher index

We finally consider the case of matrices of arbitrary index. If A is an $n\times n$ matrix, then recall that A can be expressed as $A=C_A+N_A$ where C_A, the core part of A

is of index one, N_A is nilpotent and $C_A N_A = N_A C_A = 0$. This is referred to as the core-nilpotent decomposition of A, see [3].

Lemma 5.1. *If* index $A = n$, $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$, $\alpha_i > 0$, $p < m_1 < m_2 < \cdots < m_t$ *integers and* $A = C_A + N_A$ *is the core-nilpotent decomposition, then* $C_A \geqslant 0$.

Proof. We have index $A \leqslant p \Rightarrow A^P = C_A^p$ and since index $C_A = 1$, we can write $C_A = C_A^p X$ for some matrix X.

Then

$$\begin{aligned} A^p (A^p)^{\#} C_A &= A^p (A^p)^{\#} C_A^p X = A^p (A^p)^{\#} A^p X \\ &= A^p X = C_A^p X = C_A. \end{aligned}$$

Therefore

$$\begin{aligned} A^p (A^p)^{\#} A &= A^p (A^p)^{\#} (C_A + N_A) = C_A + A^p (A^p)^{\#} N_A \\ &= C_A + (A^p)^{\#} C_A^p N_A = C_A. \end{aligned}$$

Now since $(A^p)^{\#} \geqslant 0$, we have $C_A \geqslant 0$. □

Theorem 5.2. *Let A be a $d \times d$ nonnegative matrix. Then*

$$(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i},$$

where $\alpha_i > 0$ and $p < m_1 < m_2 < \cdots < m_t$ are positive integers if and only if there exists a permutation matrix P such that PAP^{T} is a direct sum of matrices $C_{ii} + N_{ii}$, $1 \leqslant i \leqslant k$, where

(1): *Each C_{ii} is nonnegative of index* 1, *N_{ii} are nilpotent matrices of index $\leqslant p$, and $C_{ii} N_{ii} = N_{ii} C_{ii} = 0$.*

(2): *Each C_{ii} is a direct sum of matrices of Type* (I), (II), *and* (III) *described in Theorem* 3.1.

Proof. Because $(A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i A^{m_i}$, we have

$$((C_A + N_A)^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i (C_A + N_A)^{m_i} \quad \text{and hence}$$

$$(C_A^p)^{\mathrm{T}} = \sum_{i=1}^{t} \alpha_i C_A^{m_i}.$$

Now we have index $C_A = 1$ and $C_A \geqslant 0$. These observations and (1.1) imply that $(C_A)^{\#} \geqslant 0$. So by Theorem 3.1, the exists a permutation matrix P such that PC_AP^{T} is a direct sum of matrices of Types (I), (II), (III) as described in the statement

of that theorem. So, let $PAP^{\mathrm{T}} = [A_{ij}]$, $PC_AP^{\mathrm{T}} = [(C_A)_{ij}]$, $PN_AP^{\mathrm{T}} = [(N_A)_{ij}]$ be compatible partitions. Also $(C_A)_{ij} = 0$ if $i \neq j$. Since $A_{ij} = (C_A)_{ij} + (N_A)_{ij}$ we get $(N_A)_{ij} \geqslant 0$ for $i \neq j$. But $C_AN_A = 0$ implies that $(C_A)_{ii}(N_A)_{ij} = 0$, $i \neq j$. Now by the fact that, for some positive integer s, $(C_A)_{ii}^s$ has positive diagonal entries, $(C_A)_{ii}(N_A)_{ij} = 0$, and $(N_A)_{ij} \geqslant 0$ for $i \neq j$, we obtain $(N_A)_{ij} = 0$. So $A_{ij} = 0$, $i \neq j$. By setting $C_{ii} = (C_A)_{ii}$, $N_{ii} = (N_A)_{ii}$ for all i, the proof is completed. □

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