# Ergodic amenable actions of algebraic groups

C. R. E. Raja

**Abstract**

We prove that every ergodic amenable action of an algebraic group over a local field of characteristic zero is induced from an ergodic action of an amenable subgroup.

**2000 Mathematics Subject classifications**: 22D40 22F10 37A15.

It was shown by Zimmer that an ergodic amenable action of a connected locally compact group is induced by an ergodic action of an amenable subgroup, here we prove the following analogue for algebraic groups over any local field of characteristic zero. Let $\mathbb{K}$ be denote a local field of characteristic zero.

**Theorem 1** *Let $G$ be the group of $\mathbb{K}$-points of a Zariski-connected algebraic group defined over $\mathbb{K}$. Let $S$ be an ergodic amenable $G$-space. Then the action on $S$ is induced from an ergodic action of an amenable subgroup of $G$.*

In order to prove this theorem we first prove the analogue of Moore's result (see 3.2.22 and 9.2.5 of [4]) and the rest is similar to the proof of Zimmer. Our approach is different from [2] and we use a lemma of Furstenberg. Let $V$ be a finite-dimensional vector space over $\mathbb{K}$. Let $P(V)$ be the corresponding projective space. Let $\Pi\colon V\setminus(0)\to P(V)$ be the natural quotient map. It is a well known fact that any element $g$ of $GL(V)$ gives a homeomorphism $\Pi(g)$ of $P(V)$. Let $PGL(V)$ denotes the group consisting of $\Pi(g)$ for all $g\in GL(V)$: elements of $PGL(V)$ are known as projective linear transformations. A subset $L$ of $V$ is called a *quasi-linear variety* if it is a union of finitely many subspaces of $V$. We now state the following lemma due to Furstenberg (see [1]).

**Lemma 1** *Let $(g_n)$ be a sequence in $PGL(V)$ and $\mu$ and $\lambda$ be probability measures on $P(V)$. Then there is a subsequence $(g_{k_n})$ and a transformation $\tau$ of $P(V)$ onto $\Pi(L\setminus(0))$ where $L$ is a quasi-linear variety with the following properties:*

1. *$(g_{k_n})$ converges to $\tau$ pointwise on $P(V)$ and if $(g_{k_n})$ does not converge in $PGL(V)$, then $L$ is a proper quasi-linear variety of $V$;*

2. *If $g_n(\mu) \to \lambda$ in the space of probability measures on $P(V)$, equipped with weak* topology, then $\lambda$ is supported on $\Pi(L \setminus (0))$.*

Let $X$ be any locally compact topological space and $\mathcal{P}(X)$ be the space of all regular Borel probability measures on $X$ equipped with the weak* topology which is the weakest topology on $\mathcal{P}(X)$ for which the functions $\mu \mapsto \mu(f)$ are continuous for all continuous bounded functions on $X$. Let $G$ be any locally compact group acting on $X$ by homeomorphisms of $X$. Then the action of $G$ induces an action of $G$ on $\mathcal{P}(X)$. For a probability measure $\mu$ on $X$, we define the subgroups $\mathcal{I}_G(\mu) = \{g \in G \mid g\mu = \mu\}$ and $I_G(\mu) = \{\alpha \in A \mid \alpha(x) = x \text{ for all } x \in S(\mu)\}$ where $S(\mu)$ denotes the support of $\mu$.

**Proposition 1** *Let $G$ be an algebraic subgroup of $PGL(V)$ and $\mu$ be a probability measure on $P(V)$. Then $\mathcal{I}_G(\mu)/I_G(\mu)$ is a compact group.*

**Proof** Let $L$ be the smallest quasi-linear variety such that $\Pi(L \setminus (0))$ contains the support of $\mu$. Let $H$ be the algebraic subgroup of $G$ consisting of transformations that preserve $\Pi(L \setminus (0))$, that is, $H = \{g \in G \mid g(\Pi(L \setminus (0))) = \Pi(L \setminus (0))\}$. Then it is easy to see that $\mathcal{I}_G(\mu)$ is contained in $H$. Let $(g_n)$ be a sequence in $\mathcal{I}_G(\mu)$. Then by Lemma 1 and by passing to a subsequence, if necessary, we may assume that there is transformation $\tau$ of $P(V)$ onto $\Pi(U \setminus (0))$ where $U$ is a quasi-linear variety such that $\Pi(U \setminus (0))$ contains the support of $\mu$ and $g_n \to \tau$ pointwise on $P(V)$. Since $g_n(\mu) = \mu$ for all $n \geq 1$, $\tau(\mu) = \mu$ and hence for $n \geq 1$, $g_n(\Pi(L \setminus (0))) = \tau(\Pi(L \setminus (0))) = \Pi(L \setminus (0))$. Let $L = \cup_{i=1}^{k} W_i$ where each $W_i$ is a subspace of $V$. Then $H$ has a subgroup $N = \{g \in G \mid g(\Pi(W_i \setminus (0))) = \Pi(W_i \setminus (0)) \text{ for all } 1 \leq i \leq k\}$ of finite index. Now by passing to subsequence, if necessary, we may assume there is a $h \in G$ such that $g_n h \in N$ for all $n \geq 1$. Let $\alpha$ be the restriction of $\tau h$ to $\Pi(L \setminus (0))$. Then for $1 \leq i \leq k$, $\alpha(\Pi(W_i \setminus (0))) = \Pi(W_i \setminus (0))$. Let $\alpha_i$ be the restriction of $\alpha$ to $\Pi(W_i \setminus (0))$ for $1 \leq i \leq k$. Let $\Phi\colon N \to \prod_{i=1}^{k} PGL(W_i)$ be defined by $\Phi(g) = (\Phi_i(g))_{i=1}^{k}$ where $\Phi_i(g)$ is the restriction of $g$ to $\Pi(W_i \setminus (0))$ for $1 \leq i \leq k$. Since $N$ is an algebraic group, the image $\Phi(N)$ is closed and it is isomorphic to $N/\ker\Phi$. Since $\Phi(g_n h) \to (\alpha_i)_{i=1}^{k}$ and the kernel of $\Phi$ is $I_G(\mu)$, $(g_n h)$ is relatively compact in $N/I_G(\mu)$. This proves the proposition since $\mathcal{I}_G(\mu)$ is a closed subgroup of $H$ containing $I_G(\mu)$.

By an algebraic group over $\mathbb{K}$ we mean the group of $\mathbb{K}$-points of an algebraic group defined over $\mathbb{K}$.

**Corollary 1** *Let $G$ be an algebraic group over $\mathbb{K}$ and $H$ be an algebraic subgroup of $G$. Let $\mu$ be a probability measure on $G/H$. Then $G$ acts on $G/H$ in a canonical way. Let $\mathcal{I}(\mu) = \{g \in G \mid g \text{ action on } G/H \text{ preserves } \mu\}$ and $I(\mu) = \{g \in G \mid g \text{ acts trivially on support of } \mu\}$. Then $\mathcal{I}(\mu)/I(\mu)$ is a compact group.*

**Proof** Since $H$ is an algebraic subgroup of an algebraic group $G$, there is a finite-dimensional vector space $V$ on which $G$ acts by linear transformations and there is a vector $v_0 \in V$ such that $H = \{g \in G \mid gv_0 \in (v_0)\}$ where $(v_0)$ is the one-dimensional subspace of $V$ spanned by $v_0$. This implies that $G/H$ can be viewed as a Borel subset (in fact, a locally closed subset) of $P(V)$ and $G$ acts on $P(V)$ as an algebraic group. Now the result follows from Proposition 1.

**Corollary 2** *Let $G$ be the group of $\mathbb{K}$-points of a Zariski-connected algebraic group defined over $\mathbb{K}$. Let $P$ be a minimal parabolic subgroup of $G$. Then for any algebraic subgroup $H$ of $G$ and any $\mu \in \mathcal{P}(G/P)$, $\mathcal{I}_H(\mu) = \{h \in H \mid h\mu = \mu\}$ is amenable. In particular, a subgroup of $G$ is amenable if and only if it has a fixed point in the space of probability measures on $G/P$.*

**Proof** By Corollary 1, $\mathcal{I}_H(\mu)/I_H(\mu)$ is compact where $I_H(\mu) = \{h \in H \mid hx = x$ for all $x$ in support of $\mu\}$. Hence it is enough to prove $I_H(\mu)$ is amenable. Since elements of $I_H(\mu)$ fixes the support of $\mu$ pointwise, $I_H(\mu)$ is contained in a conjugate of $P$ and hence since $P$ is amenable $I_H(\mu)$ is amenable.

**Proof** (**Proof of Theorem 1**) Consider the cocycle $\alpha$ defined by $\alpha(s, g) = g$ for all $s \in S$ and $g \in G$. Let $N$ be the solvable radical of $G$. Then $H = G/N$ is semisimple and let $P$ be the minimal parabolic subgroup of $H$. By amenability there is a $\alpha$-invariant function $f: S \to \mathcal{P}(H/P)$. Since the action of $G$ on $\mathcal{P}(H/P)$ is smooth (see Corollary 3.2.17 of [4]) and by Corollary 2, stabilizers are amenable. By cocycle reduction Lemma 5.2.11 of [4], $\alpha$ is equivalent to a cocycle taking values in an amenable subgroup, say, $M$ of $G$. We now claim that the action of $G$ on $S$ is induced from action of $M$. By 4.2.18 of [4], there is an $\alpha$-invariant function $\phi: S \to G/M$. This implies by Theorem 2.5 or Corollary 2.6 of [3] that the action on $S$ is induced from an ergodic amenable (because $M$ is amenable) action of $M$.

It may be recalled that any local field $\mathbb{K}$ of characteristic zero is either $\mathbb{R}$ or $\mathbb{C}$ or finite extension of a $p$-adic field. Zimmer's result Theorem 5.7 of [3] covers the case of real algebraic groups and all connected groups (see 9.2.5 of [4]). We denote by $\mathbb{Q}_p$ the $p$-adic field. Let $\tilde{G}$ be an algebraic group defined over $\mathbb{Q}_p$ and $G = \tilde{G}(\mathbb{Q}_p)$, the group of $\mathbb{Q}_p$-points of $\tilde{G}$. Then it may be easily seen that $G$ is a totally disconnected group, hence these groups are not covered by Zimmer's result. Now we will present a few explicit examples of these groups.

1. The elementary examples are the additive group $\mathbb{Q}_p$ and the multiplicative group $\mathbb{Q}_p \setminus (0)$ known as the one-dimensional split torus.

2. $GL_n(\mathbb{Q}_p)$ be the group of all invertible $n \times n$ matrices with entries in $\mathbb{Q}_p$, in general, all invertible elements in a finite-dimensional algebra over $\mathbb{Q}_p$.

3. $SL_n(\mathbb{Q}_p) = \{A \in GL_n(\mathbb{Q}_p) \mid \det(A) = 1\}$, the special linear group.

4. For any $m \in \mathbb{N}$ and any $2m \times 2m$ skew-symmetric matrix $E$, that is $E^t = -E$, define the symplectic group $SP_{2m}(E) = \{A \in GL_{2m}(\mathbb{Q}_p) \mid A^t E A = E\}$.

5. The group of upper triangular matrices, $UT_n(\mathbb{Q}_p) = \{(g_{ij}) \in GL_n(\mathbb{Q}_p) \mid g_{ij} = 0 \quad \text{if} \quad j < i\}$ and the group of all unipotent martices, $U_n(\mathbb{Q}_p) = \{(g_{ij}) \in UT_n(\mathbb{Q}_p) \mid g_{ii} = 1\}$.

These groups and their direct and semi-direct products (in some cases) and many other classical algebraic groups are covered by our result but not by Zimmer's result. Though the groups in examples 1 and 5 are amenable but their direct and semidirect products with $GL_n$ and $SL_n$ or some suitable subgroups of $GL_n$ and $SL_n$ are not amenable, for example, the general affine group $\mathbb{Q}_p^n \times_s GL_n$ is not amenable.

We will end the article with few consequences of the main result. In [3], it is shown that for any lattice $\Gamma$ in $SL(2,\mathbb{C})$, the $\Gamma$-space $SL(2,\mathbb{C})/N$ (where $N$ is the group of all upper triangular matrices in $SL(2,\mathbb{C})$) is ergodic amenable but it is not induced from an amenable subgroup. In fact, Theorem 1 combined with arguments preceding Theorem 5.11 of [3], we conclude that any ergodic amenable $\Gamma$-space has a factor of the form $SL(2,\mathbb{C})/R$ for some amenable subgroup $R$ of $SL(2,\mathbb{C})$. Also, results regarding minimal ergodic amenable actions of closed subgroups of $GL_n(\mathbb{K})$ may also be obtained as in Theorem 5.9 and Corollary 5.12 of [3].

**Acknowledgement** I wish to thank the referee for useful remarks.

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C. Robinson Edward Raja,
Indian Statistical Institute,
Statistics and Mathematics Unit,
8th Mile Mysore Road,