On Paley-Wiener and Hardy theorems for NA groups

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Abstract Let N be a H-type group and let $S = NA$ be an one dimensional solvable extension of N. For the Helgason Fourier transform on S we prove the following analogue of Hardy's theorem. Let $\hat{f}(\lambda, Y, Z)$ stand for the Helgason Fourier transform of f and let h_α denote the heat kernel associated to the Laplace-Beltrami operator.

Suppose a function f on S satisfies the conditions $|f(x)| \leq c\, h_\alpha(x)$ and

$$\int_N |\hat{f}(\lambda, Y, Z)|^2 (1 + |Z|^2)^\gamma \, dY \, dZ \leq c\, e^{-2\beta\lambda^2}$$

for all $x \in S$, $\lambda \in \mathbb{R}$ where $\gamma > \frac{k-1}{2}$, k being the dimension of the centre of N. Then $f = 0$ or $f = ch_\alpha$ depending on whether $\alpha < \beta$ or $\alpha = \beta$.

We also establish a stronger version of Hardy's theorem and a Paley-Wiener theorem. These are generalisations of the corresponding results for rank one symmetric spaces of noncompact type.

1 Introduction and the main results

Let G be a connected, noncompact semi-simple Lie group with finite center. Let $G = NAK$ be an Iwasawa decomposition and G/K the associated symmetric space which is assumed to be of rank one. Let M be the centraliser of A in K and consider an orthonormal basis

$$\{Y_{\delta,j} : 1 \leq j \leq d_\delta,\ \delta \in \hat{K}_M\} \tag{1.1}$$

of $L^2(K/M)$ consisting of K-finite functions of type δ on K/M . For a function f on G/K let $\tilde{f}(\lambda, b)$, $\lambda \in \mathbb{R}$, $b \in K/M$ be the Helgason Fourier transform of f. Let Δ_0 be the Laplace-Beltrami operator on G/K with the associated heat kernel

$h_t^0(x)$ and let $Q_\delta(\lambda)$ be the Kostant polynomials associated to δ. In [20] we have established the following analogue of Hardy's theorem.

Theorem 1.1. *Let f be a function on G/K which satisfies the estimate $|f(x)| \leq c\, h_\alpha(x)$ for all $x \in G/K$. Further assume that for every $\delta \in \hat{K}_M$ and $1 \leq j \leq d_\delta$ the functions*

$$F^0_{\delta,j}(\lambda) = Q_\delta(\lambda)^{-1} \int_{K/M} \tilde{f}(\lambda, b) Y_{\delta,j}(b) db \tag{1.2}$$

satisfy the estimates $|F^0_{\delta,j}(\lambda)| \leq c_{\delta,j}\, e^{-\beta\lambda^2}$ for all $\lambda \in \mathbb{R}$. Then either $f = 0$ or $f = ch_\alpha$ depending on whether $\alpha < \beta$ or $\alpha = \beta$.

We refer to [20] for the analogous result for the Euclidean Fourier transform and for the original version of Hardy's theorem proved in [11] by Hardy. Our aim in this paper is to prove an analogue of Theorem 1.1 for NA groups.

Note that $G/K = NA$ can be considered as the one dimensional extension of N by A. The group N, called the Iwasawa N group, is an example of a H -type group. In recent times, such H-type groups and their one dimensional extensions have received considerable attention. See the works of Damek-Ricci [8] and Cowling et al [5]. Given such a group N, the one dimensional extension NA gives an example of a nonsymmetric harmonic manifold. This class, called harmonic NA groups, contains all the noncompact rank one symmetric spaces. Harmonic analysis on such groups has been studied by Damek-Ricci [8], Anker et al [1] and others. Helgason Fourier transform on NA groups has been studied by Astengo et al [2] and they have proved inversion and Plancherel theorems. In this paper we prove analogues of Hardy and Paley-Wiener theorems.

In the context of nonsymmetric NA groups, there is no analogue of K which acts transitively on the spheres in NA. Therefore, we do not have analogues of the spherical harmonics $Y_{\delta,j}$. To remedy this shortcoming, consider the case $G = SU(n+1, 1)$, $n \geq 1$ which gives rise to the complex hyperbolic space. Here $N = H^n$, the Heisenberg group and the symmetric space can be naturally identified with the unit ball $\mathcal{B}_{n+1}$ in $\mathbb{C}^{n+1}$. Using the Cayley transform C which takes $\mathcal{B}_{n+1}$ onto the Siegel's upper half space $\mathcal{D}_{n+1}$ we can also identify NA with $\mathcal{D}_{n+1}$. We refer to Section 3 for all this and more.

The restriction of the Cayley transform to the boundary S^{2n+1} of $\mathcal{B}_{n+1}$ maps it onto the boundary of $\mathcal{D}_{n+1}$ which is identified with H^n. If C_b stands for this restriction then $Y_{\delta,j} \circ C_b^{-1}$ are functions on H^n denoted by $S_{\delta,j}$. The image of the Haar measure on K under the Cayley transform is $c_n\, P_1^0(\zeta, u) d\zeta\, du$ where

$$P_a^0(\zeta, u) = a^{n+1} \left((a + \frac{1}{4}|\zeta|^2)^2 + |u|^2 \right)^{-(n+1)}. \tag{1.3}$$

Then $\{S_{\delta,j} : 1 \leq j \leq d_\delta, \delta \in \hat{K}_M\}$ forms an orthonormal basis for $L^2(H^n, c_n P_1^0(\zeta, u) du\, d\zeta)$. Let $\mathcal{P}^0_\lambda(\zeta, u)$ be the kernel defined by

$$\mathcal{P}^0_\lambda(\zeta, u) = (P_1^0(\zeta, u))^{\frac{1}{2} - \frac{i\lambda}{Q_0}} \tag{1.4}$$

where $Q_0 = n + 1$. Then it is clear the $P_1^0(\zeta, u) = \mathcal{P}_\lambda^0(\zeta, u)\mathcal{P}_{-\lambda}^0(\zeta, u)$. Let $\hat{f}(\lambda, Y, Z)$, $\lambda \in \mathbb{R}$, $(Y, Z) \in N$ be the (nonnormalised) Helgason Fourier transform of a function f on NA. Let $Q = m + k$ be the homogeneous dimension of the H-type group N. Let Δ be the Laplace-Beltrami operator on NA with the associated heat kernel $h_t(x)$. Let $B_{m,k}(\lambda)$ be the meromorphic function given by

$$B_{m,k}(\lambda) = \frac{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{m+1}{2} - i\lambda\right)}{\Gamma\left(\frac{m+k}{2} - i\lambda\right)}. \tag{1.5}$$

With these preparations we now state our first version of Hardy's theorem.

In the following theorem, $K = S(U(m) \times U(1))$ is the K part of the Iwasawa decomposition of $G = SU(m+1, 1)$. M, $\hat{K}_M$ etc. are all related to this group. $\hat{f}(\lambda, Y, Z)$, $(Y, Z) \in N$ stands for the Helgason Fourier transform of f on NA (to be defined in the next section).

Theorem 1.2. *Let f be a function on the NA group which satisfies the estimate $|f(x)| \leq c\ h_\alpha(x)$ for all $x \in NA$. Further assume that for every $\delta \in \hat{K}_M$, $1 \leq j \leq d_\delta$ and $\omega \in S^{k-1}$ the functions*

$$\begin{aligned} &F_{\delta,j}(\lambda, \omega) \\ &= B_{m,k}(\lambda)^{-1} Q_\delta(\lambda)^{-1} \int_N \hat{f}(\lambda, Y, Z) S_{\delta,j}(Y, Z\cdot\omega)\mathcal{P}_{-\lambda}^0(Y, Z\cdot\omega)dZ\, dY \end{aligned}$$

satisfy the estimate $|F_{\delta,j}(\lambda, \omega)| \leq c_{\delta,j}\ e^{-\beta\lambda^2}$ for all $\lambda \in \mathbb{R}$. Then either $f = 0$ or $f = ch_\alpha$ depending on whether $\alpha < \beta$ or $\alpha = \beta$.

We also have the following Paley-Wiener theorem for the Helgason Fourier transform. Let $e = (0, 0, 1)$ denote the identity of NA and let $d(x) = d(x, e)$ stand for the geodesic distance between x and e. If $x = (X, Z, a)$, $d(x, e)$ can be written down in terms of $|X|$, $|Z|$ and a. Let $\mathcal{S}(NA)$ denote the space of Schwartz class functions. Any $f \in \mathcal{S}(NA)$ is a C^∞ function which satisfies the conditions

$$\sup_{x\in NA} e^{\frac{Q}{2}d(x)}(1 + d(x))^n |UVf(x)| < \infty \tag{1.6}$$

for all $n \geq 0$ and left (right) invariant vector field U (resp. V). In the next theorem we consider functions which have more decay than Schwartz class functions.

Theorem 1.3. *Let $f \in \mathcal{S}(NA)$ be very rapidly decreasing in the sense that*

$$|f(x)| \leq c_n\ e^{-nd(x)}, x \in NA$$

for all $n \geq 0$. Then f is compactly supported in $d(x) \leq B$ if and only if for every $\delta \in \hat{K}_M$, $1 \leq j \leq d_\delta$ and $\omega \in S^{k-1}$ the functions $F_{\delta,j}(\lambda, \omega)$ defined in the previous theorem extend to entire functions of $\lambda \in \mathbb{C}$ satisfying the estimates

$$|F_{\delta,j}(\lambda, \omega)| \leq c_{\delta j}(1 + |\lambda|)^{-n} e^{B|Im\lambda|}$$

for all $\lambda \in \mathbb{C}$ and $n \geq 0$.

Returning to Hardy's theorem for rank one symmetric spaces, consider the following version which is weaker than Theorem 1.1.

Theorem 1.4. *Let a function f on G/K satisfy the condition $|f(x)| \leq ch_\alpha^0(x)$. Further assume that*

$$\int_{K/M} |\tilde{f}(\lambda, b)|^2 db \leq c(1 + |\lambda|)^n e^{-2\beta\lambda^2}$$

for some $n \geq 0$. Then either $f = 0$ or $f = ch_\alpha^0$ depending on whether $\alpha < \beta$ or $\alpha = \beta$.

The case $n = 0$ of this theorem is due to Narayanan-Ray [17]. We indicate a proof of the case $n > 0$ in Section 3 which requires the stronger version, viz. Theorem 1.1. In this paper we prove the following result which is the analogue of Theorem 1.4 for NA groups.

Theorem 1.5. *Let f be a function on the NA group which satisfies the estimate $|f(x)| \leq ch_\alpha(x)$. Further assume that*

$$\int_N |\hat{f}(\lambda, Y, Z)|^2 (1 + |Z|^2)^\gamma dY\, dZ \leq c\, e^{-2\beta\lambda^2}$$

for all $\lambda \in \mathbb{R}$ and some $\gamma > \frac{k-1}{2}$. Then the conclusions of Theorem 1.2 hold.

This theorem should be compared with the following result of Astengo et al [3]. They have shown that if $|f(x)| \leq ch_\alpha(x)$ and if

$$\int_N |\hat{f}(\lambda, Y, Z)|^2 dY\, dZ \leq c\, e^{-2\beta\lambda^2}$$

with $\alpha < \beta$ then $f = 0$. The equality case was left open and the above theorem treats that case. When $f = h_\alpha$ we have

$$\hat{f}(\lambda, Y, Z) = \mathcal{P}_\lambda(e, (Y, Z))e^{-\alpha\lambda^2} \tag{1.7}$$

where $\mathcal{P}_\lambda$ is a power of the Poisson kernel on NA and so it can be shown that the hypothesis on $\hat{f}$ in Theorem 1.5 is satisfied if $\gamma < \frac{m+k}{2}$. Thus the hypothesis in our theorem is only but natural. In order to prove Theorem 1.5, we need to use the stronger result, viz Theorem 1.1 whereas the case $n = 0$ of Theorem 1.4 can be proved without appealing to Theorem 1.1 as was done in Narayanan-Ray [17].

This paper is organised as follows. In Section 2 we recall the definition of the Helgason Fourier transform on NA groups. In Section 3 we consider Hardy's theorem for the complex hyperbolic space and restate Theorem 1.1 in the non-compact picture. We also give a proof of Theorem 1.4. In Section 4 we introduce the partial Radon transform and show how it can be used to reduce matters from general NA groups to the case of complex hyperbolic space. Finally, in Section 5 we prove all our main results.

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2 Fourier transform on NA groups

The aim of this section is to recall the definition of the Helgason Fourier transform on NA groups. We begin with the definition of H-type groups and their solvable extensions. Using the Poisson kernel, we define the Helgason Fourier transform as in Astengo et al [2] and collect some important properties of the same.

Let $\mathfrak{n}$ be a two step nilpotent Lie algebra equipped with an inner product $\langle , \rangle$. Let $\mathbf{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement so that $\mathfrak{n} = \mathfrak{v} \oplus \mathbf{z}$. Following Kaplan [15] we say that $\mathfrak{n}$ is a H-type algebra if for every $Z \in \mathbf{z}$ the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_Z X, Y \rangle = \langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v} \tag{2.1}$$

satisfies the condition $J_Z^2 = -|Z|^2 I$, I being the identity on $\mathfrak{v}$. A connected and simply connencted Lie group N is called a H -type group if its Lie algebra is a H-type algebra. Since $\mathfrak{n}$ is nilpotent, the exponential map is surjective and hence we can parametrise elements of $N = \exp \mathfrak{n}$ by (X, Z) where $X \in \mathfrak{v}$, $Z \in \mathbf{z}$. By Campbell-Hausdorff formula, the group law is given by

$$(X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']). \tag{2.2}$$

The Haar measure on N is given by $dX\ dZ$ where dX and dZ are Lebesgue measures on $\mathfrak{v}$ and $\mathbf{z}$.

The best known example of a H-type group is the Heisenberg group H^n. The Lie algebra of H^n is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the Lie bracket

$$[(x, y, t), (x', y', t')] = (0, 0, (y \cdot x' - x \cdot y'))$$

where $x \cdot y$ stands for the standard inner product on $\mathbb{R}^n$. For $Z = (0, 0, t)$ in the center we define J_Z by $J_Z(x, y) = t(-y, x)$. Then

$$\langle J_Z(x, y), (x', y') \rangle = t(-y, x) \cdot (x', y') = \langle [(x, y, 0), (x', y', 0)], t \rangle$$

and $J_Z^2 = -|Z|^2 I$. The group law on H^n takes the form

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x)).$$

Identifying $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$ and writing $z = x + iy$ we can identify H^n with $\mathbb{C}^n \times \mathbb{R}$ and the group law takes the form

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} Im(z \cdot \bar{z}')).$$

We return to the Heisenberg group in Section 3.

Given a H-type group N let $S = NA$ be the semidirect product of N with $A = \mathbb{R}^+$ with respect to the action of A on N given by the dilation $(X, Z) \to$

$(a^{\frac{1}{2}}X, aZ)$, $a \in A$. We write (X, Z, a) to denote the element $\exp(X+Z)a$. The product law on NA is given by

$$(X, Z, a)(X', Z', a') = (X + a^{\frac{1}{2}}X', Z + aZ' + \frac{1}{2}a^{\frac{1}{2}}[X, X'], aa'). \tag{2.3}$$

For any $Z \in \mathbf{z}$ with $|Z| = 1$, $J_Z^2 = -I$ and hence J_Z defines a complex structure on $\mathfrak{v}$. Consequently $\mathfrak{v}$ is even dimensional. Let $2m$ be the dimension of $\mathfrak{v}$ and k the dimension of $\mathbf{z}$. Then $Q = m+k$ is called the homogeneous dimension of S. The left Haar measure on S is (up to a multiplicative constant) given by $a^{-Q-1}dX\,dZ\,da$.

The Lie algebra s of S is simply $\mathfrak{n} \oplus \mathbb{R}$ equipped with the inner product

$$\langle (X, Z, t), (X', Z', t') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + tt'.$$

This makes S into a Riemannian manifold which is a harmonic space. Rank one symmetric spaces of noncompact type constitute a subclass of NA harmonic spaces. As proved in Cowling et al [5], the geodesic distance between $x = (X, Z, a)$ and the identity $e = (0, 0, 1)$ is given by

$$d(x) = d(x, e) = \log \frac{1 + r(x)}{1 - r(x)} \tag{2.4}$$

where $r(x)$ is given by the expression

$$1 - r(x)^2 = 4a\{(1 + a + \frac{1}{4}|X|^2)^2 + |Z|^2\}^{-1}. \tag{2.5}$$

A function f on S is called radial if $f(x)$ depends only on $d(x)$. In [8] Damek and Ricci proved that the subalgebra of $L^1(S)$ consisting of radial functions is commutative. They also have studied the spherical Fourier transform on S.

An analogue of Helgason Fourier transform on S was introduced and studied by Astengo et al in [2]. This was done in terms of the Poisson kernel associated to the Laplace-Beltrami operator Δ on S. If f is a bounded harmonic function on S, i.e. $\Delta f = 0$ then as proved by Damek [7] f can be represented as

$$f(x) = \int_N \mathcal{P}(x, n)F(n)dn, \quad x \in S$$

where F is the restriction of f to N. Here $\mathcal{P}(x, n)$ is the Poisson kernel which is defined as follows. For $a \in \mathbb{R}^+$ and $n = (X, Z)$ define

$$P_a(n) = P_a(X, Z) = c_{mk}\, a^Q((a + \frac{1}{4}|X|^2)^2 + |Z|^2)^{-Q} \tag{2.6}$$

where $c_{mk} = P_1(0, 0)$. Then $\mathcal{P}(x, n) = P_a(n_1^{-1}n)$ if $x = n_1 a \in S$. For a complex number λ define

$$\mathcal{P}_\lambda(x, n) = (\mathcal{P}(x, n))^{\frac{1}{2} - \frac{i\lambda}{Q}} = (P_a(n_1^{-1}n))^{\frac{1}{2} - \frac{i\lambda}{Q}}. \tag{2.7}$$

The Helgason Fourier transform is defined using this kernel.

Given a C_0^∞ function f on S, its Helgason Fourier transform $\hat{f}$ is the function on $\mathbb{C} \times N$ given by

$$\hat{f}(\lambda, n) = \int_S f(x)\mathcal{P}_\lambda(x, n)dx. \tag{2.8}$$

In [2] the authors have proved the following inversion formula for $f \in C_0^\infty(S)$:

$$f(x) = \frac{c_{mk}}{4\pi} \int_{\mathbb{R}} \int_N \mathcal{P}_{-\lambda}(x, n)\, \hat{f}(\lambda, n)|c(\lambda)|^{-2} dn\, d\lambda$$

where the c-function is given by

$$c(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)\Gamma((2m+k+1)/2)}{\Gamma(\frac{Q}{2}+i\lambda)\Gamma(\frac{m+1}{2}+i\lambda)}. \tag{2.9}$$

They have also proved the Plancherel formula

$$\int_S f(x)\bar{g}(x)dx = \frac{c_{mk}}{4\pi} \int_R \int_N \hat{f}(\lambda, n)\overline{\hat{g}(\lambda, n)}|c(\lambda)|^{-2} dn\, d\lambda$$

valid for all $f, g \in C_0^\infty(S)$.

We fix an orthonormal basis $\{H, E_1, E_2, \dots E_{2m}, T_1, T_2, \dots T_k\}$ adapted to the decomposition of $\mathcal{S}$ as $\mathfrak{v} \oplus \mathbf{z} \oplus \mathbb{R}$. Then the left invariant vector fields on S extending the vectors $H, E_1, \dots E_{2m}, T_1, T_2, \dots T_k$ are given by $a\partial_a$, $a^{\frac{1}{2}}E_1, \dots$, $a^{\frac{1}{2}}E_{2m}, aT_1, aT_2, \dots, aT_k$ respectively. It was shown by Damek [7] that the Laplace-Beltrami operator Δ on S is given by

$$\Delta = a\sum_{j=1}^{2m} E_j^2 + a^2 \sum_{j=1}^{k} T_j^2 + (a\partial_a)^2 - Qa\partial_a. \tag{2.10}$$

We denote by $h_t(x)$ the heat kernel associated to Δ which is a radial function, i.e. it depends only on $d(x)$. This kernel is characterised by the requirement that

$$\hat{h}_t(\lambda, n) = \mathcal{P}_\lambda(e, n)e^{-t(\lambda^2+\frac{1}{4}Q^2)}. \tag{2.11}$$

By abuse of notation we sometimes write $h_t(r)$ in place of $h_t(x)$ when $d(x) = r$. In [1] Anker et al have obtained good estimates on the heat kernel $h_t(r)$.

3 Complex hyperbolic spaces

As we have already remarked, if $G = NAK$ is the Iwasawa decomposition of a semisimple Lie group of real rank one, then N becomes a H-type group and the symmetric space G/K is naturally identified with the solvable group NA. The Helgason Fourier transform on G/K can be written in terms of the Helgason Fourier transform on NA. In this section, our aim is to restate Theorem 1.1 which will serve as a motivation for Theorem 1.2. As we need Theorem 1.1 for the group $G = SU(m+1, 1)$ we restrict ourselves to this case though whatever we say in this section is true for all rank one symmetric spaces.

The Heisenberg group H^m is the most well-known example of a H-type group. Consider now the solvable extension $S_0 = H^m A$ where $A = \mathbb{R}^+$ as before. Let $Q_0 = (m+1)$ be the homogeneous dimension of S_0. The objects on S_0 such as Poisson kernel, heat kernel etc. will be denoted by $\mathcal{P}^0_\lambda$, h^0_t etc. For example, if $(\zeta, u) \in H^m = \mathbb{C}^m \times \mathbb{R}$,

$$P^0_a(\zeta, u) = c_{m1} a^{Q_0} \left((a + \frac{1}{4}|\zeta|^2)^2 + u^2 \right)^{-Q_0} \tag{3.1}$$

and $\mathcal{P}^0_\lambda$ is defined in terms of P^0_a. In particular, we note that

$$\mathcal{P}^0_\lambda(e, (\zeta, u)) = c_{m1} \left((1 + \frac{1}{4}|\zeta|^2)^2 + u^2 \right)^{i\lambda - \frac{Q_0}{2}}.$$

The Helgason Fourier transform on S_0 is defined in terms of this kernel.

To bring out the connection between S_0 and the group $SU(m+1, 1)$ consider the Siegel's upper half space

$$\mathcal{D}_{m+1} = \left\{ (\zeta, t + is) \in \mathbb{C}^{m+1} : s > \frac{1}{4}|\zeta|^2 \right\}.$$

Then S_0 acts on $\mathcal{D}_{m+1}$ as follows. Let

$$h(\zeta, t, s) = \left(\zeta, t + is + \frac{i}{4}|\zeta|^2 \right) \tag{3.2}$$

so that $\mathcal{D}_{m+1}$ is the image of S_0 under h. The action of S_0 on $\mathcal{D}_{m+1}$ is given by

$$L_x(y) = h(x \cdot h^{-1}(y)), \quad x \in S_0, \quad y \in \mathcal{D}_{m+1}.$$

More explicitly, if $x = (z, t, s)$ and $y = (\zeta, u + iv)$ then

$$L_x(y) = h\left((z, t, s)(\zeta, u, v - \frac{1}{4}|\zeta|^2) \right).$$

From this it is clear that $L_x(0, i) = h(x)$. Note that when $y = (\zeta, u + \frac{i}{4}|\zeta|^2) \in \partial\mathcal{D}_{m+1}$ and $x = (z, t, 1) \in H^n$,

$$L_x(y) = \left(z + \zeta, t + u + \frac{i}{4}|z + \zeta|^2 \right) \in \partial\mathcal{D}_{m+1}.$$

Thus $(z, t, 1) \to \left(z, t + \frac{i}{4}|z|^2\right)$ identifies H^m with $\partial\mathcal{D}_{m+1}$.

Let $\mathcal{B}_{m+1}$ be the unit ball in $\mathbb{C}^{m+1}$ defined by

$$\mathcal{B}_{m+1} = \{(w, w_{m+1}) \in \mathbb{C}^{m+1} : |w|^2 + |w_{m+1}|^2 < 1\}.$$

We define the generalised Cayley transform $C : \mathcal{B}_{m+1} \to \mathcal{D}_{m+1}$ by

$$C(w, w_{m+1}) = \left(\frac{2iw}{1 - w_{m+1}}, i\frac{1 + w_{m+1}}{1 - w_{m+1}}\right). \tag{3.3}$$

The Cayley transform maps the origin in $\mathcal{B}_{m+1}$ into the base point $(0, i) \in \mathcal{D}_{m+1}$. We can identify functions on $\mathcal{B}_{m+1}$ with functions on $\mathcal{D}_{m+1}$ via Cayley transform.

Let G be the group of all biholomorphic automorphisms of $\mathcal{B}_{m+1}$. Then $G_0 = CGC^{-1}$ gives all biholomorphic automorphisms of $\mathcal{D}_{m+1}$. There is a natural identification of $S_0 = H^m A$ with G_0. Moreover, every element of G arises as a fractional linear transformation defined by an element of the semisimple group $G_1 = SU(m+1, 1)$. Identifying G and G_1, let $G = NAK$ be the Iwasawa decomposition of G and let M be the centraliser of A in K. Then the usual Helgason Fourier transform of a function g on G/K is $\tilde{g}(\lambda, b)$ where $\lambda \in \mathbb{R}$ and $b \in K/M$. If we identify G/K with $\mathcal{B}_{m+1}$, K/M is identified with the unit sphere S^{2m+1}.

Let f be a function on S_0 whose (nonnormalised) Helgason transform is $\hat{f}(\lambda, \zeta, t)$. Then $g = f \circ h^{-1} \circ C$ is a function on $\mathcal{B}_{m+1}$ and $\tilde{g}(\lambda, b)$ is related to $\hat{f}(\lambda, \zeta, t)$. In fact,

$$\mathcal{P}^0_\lambda(\zeta, t)^{-1} \hat{f}(\lambda, \zeta, t) = \tilde{g}(\lambda, b) \tag{3.4}$$

if $(\zeta, t) = h^{-1} \circ C(b)$. Thus we can restate Theorem 1.1 in the noncompact picture as follows. Recall that $\hat{K}_M$ is the set of all class-1 representations of K and for each $\delta \in \hat{K}_M$ we have certain spherical harmonics $Y_{\delta,j}$, $1 \le j \le d_\delta$. The family $\{Y_{\delta,j} : 1 \le j \le d_\delta, \delta \in \hat{K}_M\}$ forms an orthonormal basis for $L^2(K/M) = L^2(S^{2m+1})$.

Using Cayley transform we can define $S_{\delta,j}(\zeta, t) = Y_{\delta,j} \circ C^{-1} \circ h(\zeta, t)$. Then $\{S_{\delta,j} : 1 \le j \le d_\delta, \delta \in \hat{K}_M\}$ forms an orthonormal basis for $L^2(H^m, P^0_1(\zeta, t)d\zeta\, dt)$ where P^0_1 is given in (1.3). Therefore, Theorem 1.1 takes the following form.

Theorem 3.1. *Let f be a function on $S_0 = H^m A$ which satisfies $|f(x)| \le c h^0_\alpha(x)$. Further assume that for every $\delta \in \hat{K}_M$, $1 \le j \le d_\delta$ the function*

$$F^0_{\delta,j}(\lambda) = Q_\delta(\lambda)^{-1} \int_{H^m} \hat{f}(\lambda, \zeta, t) S_{\delta,j}(\zeta, t) \mathcal{P}^0_{-\lambda}(\zeta, t) d\zeta\, dt$$

satisfies the estimate $|F_{\delta,j}(\lambda)| \le c_{\delta,j}\, e^{-\beta\lambda^2}$ for all $\lambda \in \mathbb{R}$. Then $f = 0$ or $f = c h^0_\alpha$ according as $\alpha < \beta$ or $\alpha = \beta$.

The proof of Theorem 1.1 given in [20] uses the fact that $F^0_{\delta,j}(\lambda)$ defined in (1.2) reduces to the Jacobi transform of a function related to f. To be more precise, each $\delta \in \hat{K}_M$ is associated with a pair of integers (p, q) (see Johnson-Wallach [14]) so that

$$F^0_{\delta,j}(\lambda) = \int_0^\infty f_{\delta,j}(r) \varphi^{(m+p,q)}_\lambda(r) W_{m+p,q}(r) dr \tag{3.5}$$

where $f_{\delta,j}(r)$ is the spherical harmonic coefficient of f associated to $Y_{\delta,j}$ and $\varphi_\lambda^{(\alpha,\beta)}$ is the Jacobi function of type (α,β). We refer to Helgason [13], Koornwinder [16] and [20] for details.

We conclude this section by indicating a proof of Theorem 1.4 which we promised in the introduction. On the one hand the definition of $F^0_{\delta,j}(\lambda)$ and (3.5) shows that

$$\int_{K/M} \tilde{f}(\lambda,b)Y_{\delta,j}(b)db = Q_\delta(\lambda)F_{\delta,j}(\lambda)$$

is divisible by the polynomial Q_δ. On the other hand if $\hat{f}(\lambda)$ denotes the group Fourier transform of the right K-invariant function f on G corresponding to the spherical principal series representations π_λ then it is well known that $\tilde{f}(\lambda,b) = \hat{f}(\lambda)Y_0(b)$ where $Y_0(b)=1$. Thus

$$(\hat{f}(\lambda)Y_0, Y_{\delta,j}) = Q_\delta(\lambda)F_{\delta,j}(\lambda). \tag{3.6}$$

Writing down the definition of $\hat{f}(\lambda)$ in terms of π_λ and using the estimate on f one can show that $(\hat{f}(\lambda)Y_0, Y_{\delta,j})$ extends to an entire function of order 2. Under the hypothesis on $\hat{f}(\lambda,b)$ we apply a suitable complex analytic lemma to conclude that

$$(\hat{f}(\lambda)Y_0, Y_{\delta,j}) = P_{\delta,j}(\lambda)e^{-\beta\lambda^2} \tag{3.7}$$

where $P_{\delta,j}(\lambda)$ is a polynomial of degree $\leq n$.

Now, as the parameter p associated to δ tends to infinity, degree of Q_δ also goes to infinity. Therefore (3.6) and (3.7) are not compatible unless $f_{\delta,j}=0$ for all but finitely many δ. This means that f is a finite linear combination of functions of the form

$$Y_{\delta,j}(k)(\sinh r)^p(\cosh r)^q P_{\delta,j}(\Delta_\delta)h_\alpha^\delta(r)$$

where Δ_δ is a Jacobi differential operator and h_α^δ is the associated heat kernel. We can now use the method of Anker et al [1] to get a lower bound for $P_{\delta,j}(\Delta_\delta)h_\alpha^\delta$. We can show that the estimate $|f(x)| \leq ch_\alpha(x)$ is compatible with the expression for $f_{\delta,j}$ only if $f_{\delta,j}=0$ for all δ other than the trivial representation. This simply means that $f = ch_\alpha$ proving the theorem.

We refer to [20], [21] for more details of this argument. By going through the heat kernel estimates given in Anker et al [1] the reader can write down the precise estimates on the derivatives of the heat kernels also.

Stated in terms of the Helgason Fourier transform on S_0, Theorem 1.4 takes the following form.

Theorem 3.2. *Let f be a function on S_0 which satisfies $|f(x)| \leq ch^0_\alpha(x)$. Further assume that*

$$\int_{H^m} |\hat{f}(\lambda,\zeta,t)|^2 P_1^0(\zeta,t)d\zeta\, dt \leq c(1+|\lambda|)^{2n}e^{-2\beta\lambda^2}$$

for all $\lambda \in \mathbb{R}$. Then the conclusions of Theorem 1.4 hold.

We use this theorem in the proof of our main result viz Theorem 1.5.

4 The hyperbolic reduction

In the next section we prove our main results by reducing them to the case of complex hyperbolic spaces. This is achieved by means of a partial Radon transform. Our aim in this section is to introduce this and list some important lemmas which are very crucial in the proof of Hardy's theorem.

Let f be a function on $\mathbb{R}^n$ and let S^{n-1} stand for the unit sphere in $\mathbb{R}^n$. The Radon transform of f is a function on $\mathbb{R}^+ \times S^{n-1}$ defined by

$$Rf(t,\omega) = \int_{\omega^\perp} f(t\omega + y)dy \tag{4.1}$$

where dy is the Lebesgue measure on $\omega^\perp$. This transform is a very useful tool in reducing problems on $\mathbb{R}^n$ into problems on $\mathbb{R}$. For example, see the solution of the Cauchy problem for the wave equation on $\mathbb{R}^n$ given in Folland [10]. In [19] the same transform has been used to deduce Hardy's theorem for $\mathbb{R}^n$ from the one dimensional result.

Given a function $f(X, Z, a)$ on S define its partial Radon transform $f_\omega(X, t, a)$, $t \in \mathbb{R}$, $\omega \in S^{k-1}$ by

$$f_\omega(X,t,a) = \int_{\omega^\perp} f(X, t\cdot\omega + u, a)du. \tag{4.2}$$

In [18] Ricci introduced this transform and used it to show that the subalgebra $L^1_{rad}(S)$ of $L^1(S)$ consisting of radial functions is commutative. He also used it to deduce the inversion formula for the spherical Fourier transform on S from the corresponding result for rank one symmetric spaces. The following lemmas indicate how this reduction is achieved.

Given $\omega \in S^{k-1}$ let $Z_\omega = \exp\omega^\perp$ which is a subgroup of S. The following lemma has been proved in Ricci [18].

Lemma 4.1. *The quotient group $S_\omega = S/Z_\omega$ with the quotient metric is a symmetric space.*

In fact, as we have already observed, J_ω defines a complex structure on $\mathfrak{v}$ and we can equip it with the Hermitian product

$$\{u, v\}_\omega = \langle u, v\rangle + i\langle J_\omega u, v\rangle, \quad u, v \in \mathfrak{v}.$$

Then $\mathfrak{n}$ becomes isomorphic to the Heisenberg group H^m (if $\mathfrak{v}$ is of dimension $2m$) and S_ω is isomorphic to $S_0 = H^m A$. The map $(X, t, a) \to (X, t\omega, a)Z_\omega$ gives an isomorphism between S_0 and S_ω. Another important property of the Radon transform is given in the next lemma. We let $f *_\omega g$ to stand for the convolution on S_ω.

Lemma 4.2. *Let f and g be continuous functions on S with compact support. Then for every $\omega \in S^{k-1}$, $(f * g)_\omega = f_\omega *_\omega g_\omega$ and $f = g$ if and only if $f_\omega = g_\omega$ for every $\omega \in S^{k-1}$. Moreover, f is radial if and only if f_ω is independent of ω and $a^{-\frac{(k-1)}{2}} f_\omega(X, t, a)$ is radial on S_ω.*

It is instructive to go through the proof of this lemma given in Ricci [18]. The partial Radon transform also gives a useful relation between the spherical functions φ_λ on S and φ_λ^0 on S_ω corresponding to the same parameter λ.

Lemma 4.3. *For $-\frac{Q_0}{2} < Im\lambda < 0$ we have*

$$(\varphi_\lambda)_\omega(X, t, a) = \alpha(\lambda) a^{\frac{k-1}{2}} \varphi_\lambda^0(X, t, a)$$

where $\alpha(\lambda)$ is the meromorphic function

$$\alpha(\lambda) = \frac{\pi^{\frac{k-1}{2}} \Gamma(m + \frac{k+1}{2}) \Gamma(\frac{Q_0}{2} + i\lambda) \Gamma(\frac{Q_0}{2} - i\lambda)}{\Gamma(m+1) \Gamma(\frac{Q}{2} + i\lambda) \Gamma(\frac{Q}{2} - i\lambda)}.$$

The above lemma is also proved in Ricci [18]. Let Δ_0 be the Laplace-Beltrami operator on S_0 which is isomorphic to S_ω. Let h_s^0 be the heat kernel associated to Δ_0. In the next lemma we obtain a relation between h_s and h_s^0.

Lemma 4.4. *For every $\omega \in S^{k-1}$*

$$(h_s)_\omega(X, t, a) = c_m h_s^0(X, t, a) a^{\frac{k-1}{2}}.$$

Proof. The heat kernel h_s^0 is given by the inversion formula

$$h_s^0(X, t, a) = c_m' \int_{-\infty}^{\infty} e^{-(\lambda^2 + \frac{Q_0^2}{4})s} \varphi_\lambda^0(X, t, a) |c_0(\lambda)|^{-2} d\lambda$$

where $c_0(\lambda)$ is the c-function for S_0 and is given by

$$c_0(\lambda) = \frac{2^{Q_0 - i\lambda} \Gamma(2i\lambda) \Gamma(Q_0)}{\Gamma(\frac{Q_0}{2} + i\lambda) \Gamma(\frac{Q_0}{2} + i\lambda)}.$$

We can write the above as

$$h_s^0(X, t, a) = c_m' \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-(\lambda - i\epsilon)^2 s - \frac{1}{4} Q_0^2 s} \varphi_{\lambda - i\epsilon}^0(X, t, a) |c_0(\lambda - i\epsilon)|^{-2} d\lambda.$$

As in [18] we can verify that

$$\frac{\Gamma(m+1)}{\pi^{m+1}} \alpha(\lambda)^{-1} |c_0(\lambda)|^{-2} = \frac{2^{2k-2} \Gamma(m + \frac{k+1}{2})}{\pi^{m + \frac{k+1}{2}}} |c(\lambda)|^{-2}.$$

Therefore,

$$h_s^0(X,t,a)$$
$$= c_m'' \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} e^{-(\lambda-i\epsilon)^2 s-\frac{1}{4}Q^2 s}\alpha(\lambda-i\epsilon)\varphi^0_{\lambda-i\epsilon}(X,t,a)|c(\lambda-i\epsilon)|^{-2}d\lambda.$$

Using the result of Lemma 4.3 we obtain

$$h_s^0(X,t,a)$$
$$= c_m'' \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} e^{-(\lambda-i\epsilon)^2 s-\frac{1}{4}Q^2 s}(\varphi_{\lambda-i\epsilon})_\omega(X,t,a)a^{-\frac{(k-1)}{2}}|c(\lambda-i\epsilon)|^{-2}d\lambda.$$

Interchanging the limit and integrating over $\omega^\perp$ we get

$$a^{\frac{k-1}{2}}h_s^0(X,t,a) = c_m(h_s)_\omega(X,t,a)$$

which proves the lemma. □

We conclude this section with the following result known as the support theorem for the Radon transform which is needed in the proof of Paley-Wiener theorem. From the definition (4.1) it is clear that if the function f on $\mathbb{R}^n$ is supported in the ball $|x| \le B$ then for every $\omega \in S^{n-1}$, $Rf(t,\omega)$ is supported on $|t| \le B$. The converse is also true.

Theorem 4.5. *Let f be a continuous function on $\mathbb{R}^n$ such that $|x|^k f(x)$ is bounded for every $k \ge 0$. Assume that for every $\omega \in S^{n-1}$, $Rf(\cdot,\omega)$ is supported in $|t| \le B$. Then f is supported in $|x| \le B$.*

This theorem is due to Helgason and a proof can be found in [12]. We make use of this result in Section 5.

5 Hardy and Paley-Wiener theorems

In this section we prove our main results stated in the introduction. We begin with the following proposition which is crucial in proving both Hardy and Paley-Wiener theorems. The proposition relates the Helgason Fourier transform on S and the (normalised) Helgason Fourier transform on S_ω which we identify with the complex hyperbolic space $S_0 = H^m A$ for all $\omega \in S^{k-1}$. We write $\mathcal{P}_\lambda(Y,Z)$ in place of $\mathcal{P}_\lambda(e,(Y,Z))$ and $\mathcal{P}^0_\lambda(Y,t)$ in place of $\mathcal{P}^0_\lambda(e,(Y,t))$ for the sake of simplicity. Recall that the (normalised) Helgason Fourier transform of a function g on S_0 is denoted by $\tilde{g}(\lambda,Y,t)$.

Proposition 5.1. *For each $\omega \in S^{k-1}$*

$$\int_{\omega^\perp} \hat{f}(\lambda,Y,t\omega+u)du = c_k\, B_{m,k}(\lambda)\mathcal{P}^0_\lambda(Y,t)\tilde{g}_\omega(\lambda,Y,t)$$

where $g_\omega(X,s,a) = a^{-\frac{(k-1)}{2}} f_\omega(X,s,a)$ and $B_{m,k}(\lambda)$ is defined in (1.5).

Proof. Recalling the definition of $P_a(X, Z)$ a simple calculation shows that

$$P_a((X,Z)^{-1}(Y,\zeta)) = c_{mk}\, a^Q \left((a + \frac{1}{4}|X-Y|^2)^2 + |\zeta - Z - \frac{1}{2}[X,Y]|^2\right)^{-Q}.$$

From this we get

$$\begin{aligned}&\mathcal{P}_\lambda((X,Z,a),(Y,\zeta))\\ &= c_{mk} a^{\frac{Q}{2}-i\lambda}\left((a+\frac{1}{4}|X-Y|^2)^2 + |\zeta - Z - \frac{1}{2}[X,Y]|^2\right)^{-\frac{Q}{2}+i\lambda}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\omega^\perp} \hat{f}(\lambda, Y, t\omega+u)du\\ &= \int_S \int_{\omega^\perp} f(X,Z,a)\mathcal{P}_\lambda((X,Z,a),(Y,t\omega+u))a^{-Q-1}du\, da\, dX\, dZ.\end{aligned}$$

We can write

$$[X,Y] = \langle [X,Y],\omega\rangle\omega + v' = \langle J_\omega X, Y\rangle\omega + v'$$

where $v' \in \omega^\perp$ and $Z = s\omega + v$, $v \in \omega^\perp$ so that

$$|\zeta - Z - \frac{1}{2}[X,Y]|^2 = (t - s - \frac{1}{2}\langle J_\omega X, Y\rangle)^2 + |u - v - \frac{1}{2}v'|^2.$$

This gives us

$$\begin{aligned}\int_{\omega^\perp} \hat{f}(\lambda, Y, t\omega+u)du = c_k \int_A \int_{\mathfrak{v}} \int_{\omega^\perp}\int_{\omega^\perp} & f(X, s\omega+v, a)a^{-\frac{Q}{2}-i\lambda-1}\\ &\times\left((a+\frac{1}{4}|X-Y|^2)^2 + (t-s-\frac{1}{2}\langle J_\omega X, Y\rangle)^2\right.\\ &\left. + |u-v-\frac{1}{2}v'|^2\right)^{-\frac{Q}{2}+i\lambda} du\, dv\, da\, dX\, ds.\end{aligned}$$

Making a change of variables the inner integral becomes

$$\begin{aligned}&\int_{\omega^\perp}\left((a+\frac{1}{4}|X-Y|^2)^2 + (t-s-\frac{1}{2}\langle J_\omega X, Y\rangle)^2 + |u|^2\right)^{-\frac{Q}{2}+i\lambda}\\ &= \left(\int_{\mathbb{R}^{k-1}} (1+|u|^2)^{-\frac{Q}{2}+i\lambda}du\right)\\ &\quad\times\left((a+\frac{1}{4}|X-Y|^2)^2 + (t-s-\frac{1}{2}\langle J_\omega X, Y\rangle)^2\right)^{-\frac{Q_0}{2}+i\lambda}.\end{aligned}$$

The integral appearing on the right hand side of the above equation can be evaluated. Indeed,

$$\int_{\mathbb{R}^{k-1}} (1+|u|^2)^{-\frac{Q}{2}+i\lambda} du = c_k \int_0^\infty (1+r^2)^{-\frac{Q}{2}+i\lambda} r^{k-2} dr$$
$$= c_k' \int_0^\infty (1+r)^{-\frac{Q}{2}+i\lambda} r^{\frac{k-1}{2}-1} dr.$$

We have the formula (see [6])

$$\int_0^\infty (1+s)^{-b} s^{a-1} ds = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}$$

using which we obtain

$$\int_{\mathbb{R}^{k-1}} (1+|u|^2)^{-\frac{Q}{2}+i\lambda} du = c_k \frac{\Gamma(\frac{k-1}{2})\Gamma(\frac{m+1}{2}-i\lambda)}{\Gamma(\frac{m+k}{2}-i\lambda)}$$
$$= c_k\, B_{mk}(\lambda).$$

We also note that

$$\left((a+\frac{1}{4}|X-Y|^2)^2 + (t-s-\frac{1}{2}\langle J_\omega X, Y\rangle)^2\right)^{-\frac{Q_0}{2}+i\lambda}$$
$$= a^{-\frac{Q_0}{2}+i\lambda}\mathcal{P}_\lambda^0((X,s,a),(Y,t)) \tag{5.1}$$

and therefore,

$$\int_{\omega^\perp} \hat{f}(\lambda, Y, t\omega+u)du = c_k B_{mk}(\lambda) \int_{S_\omega} \left(\int_{\omega^\perp} f(X, s\omega+v)dv \right)$$
$$\times a^{-(\frac{k-1}{2})}\mathcal{P}_\lambda^0((X,s,a),(Y,t))a^{-Q_0-1}da\, dX\, ds.$$

Since

$$\tilde{g}_\omega(\lambda, Y, t) = \mathcal{P}_\lambda^0(Y,t)^{-1} \int_{S_0} g_\omega(X,s,a)\mathcal{P}_\lambda^0((X,s,a),(Y,t))a^{-Q_0-1}da\, dX\, ds$$

we obtain the proposition. □

We are now ready to prove Theorem 1.2. Since we are assuming that $|f(X,Z,a)| \leq ch_\alpha(X,Z,a)$, Lemma 4.4 gives us the estimate

$$|g_\omega(X,s,a)| \leq ch_\alpha^0(X,s,a).$$

Consider now

$$B_{mk}(\lambda)^{-1} Q_\delta(\lambda)^{-1} \int\limits_N \hat{f}(\lambda, Y, Z) S_{\delta,j}(Y, Z\cdot\omega)\mathcal{P}_{-\lambda}(Y, Z\cdot\omega)dY\, dZ$$

$$= B_{mk}(\lambda)^{-1} Q_\delta(\lambda)^{-1} \int\limits_{H^m} \left(\int\limits_{\omega^\perp} \hat{f}(\lambda, Y, t\omega + u)du \right) S_{\delta,j}(Y,t)\mathcal{P}_{-\lambda}(Y,t)dY\, dt.$$

In view of Proposition 5.1 we get

$$B_{m,k}(\lambda)^{-1} Q_\delta(\lambda)^{-1} \int\limits_N \hat{f}(\lambda, Y, Z) S_{\delta,j}(Y, Z\cdot\omega)\mathcal{P}_{-\lambda}(Y, Z\cdot\omega)dY\, dZ$$

$$= Q_\delta(\lambda)^{-1} \int\limits_{H^m} \tilde{g}_\omega(\lambda, Y, t) S_{\delta,j}(Y,t) P_1^0(Y,t)dY\, dt$$

where we have made use of the relation

$$\mathcal{P}_\lambda^0(Y,t)\mathcal{P}_{-\lambda}^0(Y,t) = P_1^0(Y,t).$$

The hypothesis on $\hat{f}(\lambda, Y, Z)$ gives the estimate

$$|Q_\delta(\lambda)^{-1} \int\limits_{H^m} \tilde{g}_\omega(\lambda, Y, t) S_{\delta,j}(Y,t) P_1^0(Y,t)dY\, dt| \le c\, e^{-\beta\lambda^2}.$$

We can now appeal to Theorem 3.1 to conclude that $g_\omega = 0$ whenever $\alpha < \beta$. As this is true for all $\omega \in S^{k-1}$ we get $f = 0$ as desired.

In the case when $\alpha = \beta$ we obtain the equation

$$g_\omega(X, s, a) = c(\omega) h_\alpha^0(X, s, a)$$

which means that

$$\int\limits_{\omega^\perp} f(X, s\omega + u, a)du = c(\omega) a^{\frac{k-1}{2}} h_\alpha^0(X, s, a).$$

Integrating over S_0 we have

$$c(\omega) \int h_\alpha^0(X, s, a) a^{-Q_0 - 1} da\, dX\, ds$$

$$= \int \left(\int\limits_{\omega^\perp} f(X, s\omega + u, a)du \right) a^{\frac{k-1}{2}} a^{-Q-1} da\, dX\, ds$$

$$= \int\limits_S f(X, Z, a) a^{\frac{k-1}{2}} a^{-Q-1} da\, dX\, dZ.$$

This shows that $c(\omega)$ is a constant. Hence

$$f_\omega(X,s,a)=c\,a^{\frac{k-1}{2}}h_\alpha^0(X,s,a)=c(h_\alpha)_\omega(X,s,a).$$

By the injectivity of the Radon transform we conclude that $f=ch_\alpha$.

Next we prove Theorem 1.5. We are assuming that

$$\int|\hat f(\lambda,Y,Z)|^2(1+|Z|^2)^\gamma dY\,dZ\le c\,e^{-2\beta\lambda^2}$$

for some $\gamma>\frac{k-1}{2}$. Let g_ω be as in Proposition 5.1. We claim that

$$\int\limits_{H^m}|\tilde g_\omega(\lambda,Y,t)|^2P_1^0(Y,t)dY\,dt\le c(1+|\lambda|)^{k-1}e^{-2\beta\lambda^2}.$$

In order to prove this, let φ be a function on $\mathbb{R}^n$ and consider the L^2 norm of φ_ω on $\mathbb{R}$. By Minkowski's inequality

$$\begin{aligned}&\left(\int\limits_{\mathbb{R}}\left|\int\limits_{\omega^\perp}\varphi(t\omega+u)du\right|^2dt\right)^{\frac12}\\&\quad\le\int\limits_{\omega^\perp}\left(\int\limits_{\mathbb{R}}|\varphi(t\omega+u)|^2dt\right)^{\frac12}du\\&\quad\le\left(\int\limits_{\omega^\perp}(1+|u|^2)^{-\gamma}du\right)^{\frac12}\left(\int\limits_{\omega^\perp}\left(\int\limits_{\mathbb{R}}|\varphi(t\omega+u)|^2dt\right)(1+|u|^2)^\gamma du\right)^{\frac12}.\end{aligned}$$

The first integral on the right hand side of the above inequality is finite provided $\gamma>\frac{n-1}{2}$ and hence

$$\begin{aligned}&\left(\int\limits_{\mathbb{R}}\left|\int\limits_{\omega^\perp}\varphi(t\omega+u)du\right|^2dt\right)^{\frac12}\\&\quad\le c\int\limits_{\mathbb{R}}\int\limits_{\omega^\perp}|\varphi(t\omega+u)|^2(1+|u|^2)^\gamma du\,dt\\&\quad\le c\int\limits_{\mathbb{R}^n}|\varphi(z)|^2(1+|z|^2)^\gamma dz.\end{aligned}$$

Applying this argument to $\hat f(\lambda,Y,Z)$ we get

$$\begin{aligned}&\int|\int\limits_{\omega^\perp}\hat f(\lambda,Y,t\omega+u)du|^2dY\,dt\\&\quad\le c\int|\hat f(\lambda,Y,Z)|^2(1+|Z|^2)^\gamma dZ\,dY\le c\,e^{-2\beta\lambda^2}.\end{aligned}$$

In view of Proposition 5.1 the above inequality gives the estimate

$$\int |\tilde{g}_\omega(\lambda, Y, t)|^2 P_1^0(Y, t) dY\, dt \\ \leq c|B_{mk}(\lambda)|^{-2} e^{-2\beta\lambda^2}.$$

Now, recall the definition of $B_{mk}(\lambda)$:

$$B_{mk}(\lambda)^{-1} = \frac{\Gamma(\frac{m+k}{2} - i\lambda)}{\Gamma(\frac{m+1}{2} - i\lambda)\Gamma(\frac{k-1}{2})}.$$

If k is an odd integer, say $k = 2d + 1$,

$$\frac{\Gamma(\frac{m+k}{2} - i\lambda)}{\Gamma(\frac{m+1}{2} - i\lambda)} = (\frac{m+1}{2} - i\lambda + d - 1) \dots (\frac{m+1}{2} - i\lambda)$$

and therefore,

$$|B_{mk}(\lambda)|^{-1} \leq c(1 + |\lambda|)^{\frac{k-1}{2}}.$$

We can prove the same estimate when k is even by using Stirling's formula for the gamma function. Therefore, we get

$$\left(\int |\tilde{g}_\omega(\lambda, Y, t)|^2 P_1^0(Y, t) dY\, dt\right)^{\frac{1}{2}} \leq c(1 + |\lambda|)^{\frac{k-1}{2}} e^{-\beta\lambda^2}.$$

This estimate together with $|g_\omega(X, t, a)| \leq c\, h_\alpha^0(X, t, a)$ allows us to apply Theorem 3.2 to conclude that $g_\omega = 0$ for $\alpha < \beta$ and $g_\omega = c(\omega) h_\alpha^0$ for $\alpha = \beta$. As before, this proves $f = 0$ or $f = ch_\alpha$ according as $\alpha < \beta$ or $\alpha = \beta$.

Finally, we take up Paley-Wiener theorem. For $x \in S$ we have defined $d(x) = d(x, e)$ to be the geodesic distance between x and e. Similarly, we define $d_0(x)$ for $x \in S_0$. Then it is clear that $f_\omega(X, t, a)$ is supported in $d_0(X, t, a) \leq B$ whenever $f(X, Z, a)$ is supported in $d(X, Z, a) \leq B$. The converse is also true: if $f_\omega(X, t, a)$ is supported in $d_0(X, t, a) \leq B$ for all $\omega \in S^{k-1}$ then $f(X, Z, a)$ supported in $d(X, Z, a) \leq B$. To see this, we use the support theorem for the Radon transform.

Let f be a Schwartz class function on S which satisfies the condition

$$|f(x)| \leq c_n\, e^{-nd(x)}, \quad x \in S$$

for all $n \geq 0$. If $r(x)$ is given by (2.5) we have

$$(1 - r(x)^2)^{-1} = \cosh^2 \frac{1}{2} d(x) \sim e^{d(x)}$$

as $d(x) \to \infty$. Therefore, for fixed X and a

$$(4a)^{-1}\left((1 + a + \frac{1}{4}|X|^2)^2 + |Z|^2\right) \sim e^{d(x)}$$

which means $|Z| \sim e^{\frac{1}{2}d(x)}$. Therefore, the hypothesis on f shows that $|Z|^n f(X, Z, a)$ is bounded for every $n \geq 0$. Hence Theorem 4.5 is applicable. Thus in order to prove Theorem 1.3 we only need to show that $f_\omega(X, t, a)$ is supported in $d_0(X, t, a) \leq B$ for every $\omega \in S^{k-1}$.

In view of Proposition 5.1 we have

$$F_{\delta,j}(\lambda, \omega) = Q_\delta(\lambda)^{-1} \int \tilde{g}_\omega(\lambda, Y, t) S_{\delta,j}(Y, t) P_1^0(Y, t) dY\, dt$$

and this is assumed to have an entire extension satisfying the estimate

$$|F_{\delta,j}(\lambda, \omega)| \leq c_n (1 + |\lambda|)^{-n} e^{B|Im\lambda|}.$$

As observed in Section 3, $F_{\delta,j}(\lambda, \omega)$ is the Jacobi transform of a spherical harmonic coefficient of g_ω. Therefore, we can appeal to the Paley-Wiener theorem for Jacobi transforms proved by Koornwinder [16] to conclude that all the spherical harmonic coefficients of g_ω, and hence g_ω itself, vanish outside $d_0(X, t, a) \leq B$. Thus, f_ω is supported in $d_0(X, t, a) \leq B$ and by the previous remark f is supported in $d(X, Z, a) \leq B$ proving Theorem 1.3.

References

1. Anker, J-P., Damek, E., Yacoub, C.: Spherical analysis on harmonic AN groups. Ann. Scoula Norm. Sup. Pisa **23**, 643–679 (1996)
2. Astengo, F., Camporesi, R., Di Blasio, B.: The Helgason Fourier transform on a class of nonsymmetric harmonic spaces. Bull. Aust. Math. Soc. **55**, 405–424 (1997)
3. Astengo, F., Cowling, M., Di Blasio, B., Sundari, M.: Hardy's uncertainty principle on some Lie groups. J. London Math. Soc. **62**, 461–472 (2000)
4. Astengo, F., Di Blasio, B.: A Paley-Wiener theorem on NA harmonic spaces. Colloq. Math. **80**, 211–233 (1999)
5. Cowling, M., Dooley, A.H., Koranyi, A., Ricci, F.: An approach to symmetric spaces of rank one via groups of Heisenberg type. J. Geom. Anal. **8**, 199–237 (1998)
6. Cowling, M., Haagerup, U.: Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math. **96**, 507–549 (1989)
7. Damek, E.: A Poisson kernel on Heisenberg type nilpotent groups. Colloq. Math. **53**, 239–247 (1987)
8. Damek, E., Ricci, F.: Harmonic analysis on solvable extensions of H-type groups. J. Geom. Anal. **2**, 213–248 (1992)
9. Di Blasio, B.: Positive definite spherical functions on harmonic NA spaces. Boll. Unione Mat. Ital. **11**, 759–767 (1997)
10. Folland, G.: Introduction to partial differential equations. Princeton Univ. press, Princeton (1995)
11. Hardy, G.H.: A theorem concerning Fourier transforms. J. London Math. Soc. **8**, 227–231 (1933)
12. Helgason, S.: The Radon transform. Progr. Math. Vol. **5**, Birkhauser (Boston), 1980
13. Helgason, S.: Geometric analysis on symmetric spaces, Math. Surveys and monographs. Amer. Math. Soc. **39**, (1994)
14. Johnson, K.D., Wallach, N.: Composition series and intertwining operators for the spherical principal series I. Trans. Amer. Math. Soc. **229**, 137–174 (1977)

15. Kaplan, A.: On the geometry of groups of Heisenberg type. Bull. London Math. Soc. **15**, 35–42 (1983)
16. Koornwinder, T.H.: A new proof of a Paley-Wiener type theorem for the Jacobi transform. Ark. Mat. **13**, 145–159 (1975)
17. Narayanan, E.K., Ray, S.K.: The heat kernel and Hardy's theorem on symmetric spaces of noncompact type. Proc. Ind. Acad. Sci. **112**, 321–330 (2002)
18. Ricci, F.: The spherical transform on harmonic extensions of H-type groups. Rend. Sem. Mat. Univ. Politec. Torino **50**, 381–392 (1992)
19. Sitaram, A., Sundari, M., Thangavelu, S.: Uncertainty principles on certain Lie groups. Proc. Ind. Acad. Sci. **105**, 135–151 (1995)
20. Thangavelu, S.: Hardy's theorem for the Helgason Fourier transform on rank one symmetric spaces. Colloq. Math. **94**, 263–280 (2002)
21. Thangavelu, S.: On theorems of Hardy, Gelfand-Shilov and Beurling for semisimple Lie groups. To appear in Publications of RIMS Preprint 2002