Construction of Symmetric Balanced Squares with Blocksize More than One

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Abstract. In this paper we study a generalization of symmetric latin squares. A symmetric balanced square of order v, side s and blocksize k is an $s \times s$ symmetric array of k-element subsets of $\{1, 2, \ldots, v\}$ such that

every element occurs in $\lfloor ks/v \rfloor$ or $\lceil ks/v \rceil$ cells of each row and column.

every element occurs in $\lfloor ks^2/v \rfloor$ or $\lceil ks^2/v \rceil$ cells of the array.

Depending on the values s, k and v, the problem naturally divides into three subproblems: (1) $v \geq ks$ (2) $s < v < ks$ (3) $v \leq s$. We completely solve the first problem and we recursively reduce the third problem to the first two. For $s \leq 4$ we provide direct constructions for the second problem. Moreover, we provide a general construction method for the second problem utilizing flows in a network. We have been able to show the correctness of this construction for $k \leq 3$. For $k \geq 4$, the problem remains open.

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1. Introduction

By a symmetric balanced square with blocksize k, order v and side s we mean an $s \times s$ array in which every cell contains a subset of cardinality k from a set of elements V of cardinality v satisfying the following properties:

1. Every element occurs in $\lfloor ks/v \rfloor$ or $\lceil ks/v \rceil$ cells of each row or column.
2. Every element occurs in $\lfloor ks^2/v \rfloor$ or $\lceil ks^2/v \rceil$ cells of the array.
3. The array is symmetric.

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Note that it is inherent in our definition that $k \leq v$. Let, $m = \lfloor ks/v \rfloor$ (i.e., the integer part of ks/v) and $n = \lfloor ks^2/v \rfloor$. We shall use the notation $SBS_k(s, v)$ to denote such a symmetric balanced square. Observe that an $SBS(s, s)$ is a symmetric Latin square of order s. We will use the notation $SBS(s, v)$ to denote an $SBS_k(s, v)$ when $k = 1$. Dutta and Roy [1] have completely resolved the existence problem when $k = 1$. (The case $k = 1$ is also a special case of Theorems 25 and 55, which we prove later.)

Clearly there is an $SBS_k(1, v)$ for every positive integer k and every integer $v \geq k$. Suppose A is an $SBS_k(s, v)$. Dividing ks^2 by v, we obtain unique nonnegative integers n and r such that

$$ks^2 = vn + r, \quad \text{where } 0 \leq r < v,$$

or equivalently,

$$ks^2 = r(n+1) + (v-r)(n).$$

This implies that A has r elements of frequency $n+1$ and $v-r$ elements of frequency n. Let d, e, δ and ε be integers such that

$$ks^2 = r(n+1) + (v-r)(n) = \delta(d) + \varepsilon(e), \tag{1}$$

where e is an even integer, $\{d, e\} = \{n, n+1\}$, and $\{\delta, \varepsilon\} = \{r, v-r\}$. Then A has δ elements of odd frequency d and ε elements of even frequency e. An element of odd frequency d is defined to be an odd element, and an element of even frequency e is defined to be an even element. Since A is symmetric, every odd element is contained in an odd number of cells of the main diagonal. Thus, the number of odd elements cannot exceed ks; that is, $\delta \leq ks$. This observation is recorded in the following lemma.

LEMMA 1. *A necessary condition for the existence of an $SBS_k(s, v)$, where $k \leq v$, is that the number of odd frequency elements in the array is at most ks.*

LEMMA 2. *There is an $SBS_k(s, v)$ if and only if there is an $SBS_{v-k}(s, v)$.*

Proof. Let A be an $SBS_k(s, v)$. If we replace the k-subset $A_{i,j}$ in row i and column j of A, for $1 \leq i, j \leq s$, by its complement, the result is an $SBS_{v-k}(s, v)$. ■

Lemmas 1 and 2 motivate the following definition.

Definition 3. We say that an $SBS_k(s, v)$ is feasible if $k \leq v/2$ and there exist nonnegative integers $d, e, \delta, \varepsilon$ satisfying equation (1), such that d is odd, $\{d, e\} = \{n, n+1\}, \{\delta, \varepsilon\} = \{r, v-r\}$ and $\delta \leq ks$.

Remark 1. In light of Lemma 2, we assume throughout this paper that $k \leq \lfloor v/2 \rfloor$.

The following result is an immediate application of Lemma 1.

LEMMA 4. *If* $1 < s$, $1 \le k \le v$ *and* $SBS_k(s,v)$ *is feasible, then* $v \le ks(s+1)/2$.

Proof. Suppose $SBS_k(s,v)$ is feasible. We use Definition 3 to show that this necessarily implies $v \le ks(s+1)/2$. Let the parameters r, δ and n be as defined in Definition 3. By the feasibility condition, we must have $\delta \le ks$.

If $\delta = r$, then n is even and $ks^2 - vn = r \le ks$. Since $s > 1$, $0 < ks^2 - ks \le vn$ and hence $n > 0$. Since n is even,

$$2 \le n = \left\lfloor \frac{ks^2}{v} \right\rfloor \le \frac{ks^2}{v}.$$

This gives $v \le ks^2/2 < ks(s+1)/2$.

If $\delta = v - r$, then n is odd and $v(n+1) - ks^2 = v - r = \delta \le ks$. Since n is odd, $n \ge 1$ and

$$2v \le (n+1)v \le ks^2 + ks = ks(s+1).$$

Therefore, $v \le ks(s+1)/2$.

This completes the proof. ■

It is possible to prove Lemma 4 directly by counting the maximum number of distinct elements possible in a symmetric $s \times s$ square where each cell can accommodate at most k elements. However, the proof we have provided shows that Lemma 4 is dependent on Lemma 1. Thus Lemma 1 is an independent necessary condition and along with Lemma 2 provides the definition of feasibility for an $SBS_k(s,v)$. The rest of the paper is devoted to providing evidence that it is possible to construct an $SBS_k(s,v)$ whenever it is feasible.

2. Basic Results

LEMMA 5. *For any positive integer* s *and for* $t = 0, 1$ *or* 2, *there exists an* $SBS(s, s+t)$ *whenever the necessary condition of Lemma 1 is satisfied.*

CONSTRUCTION 6. *Suppose there exists an* $SBS_k(s,v)$ *and an* $SBS_\ell(s,u)$ *such that* $ks \le v$, $\ell s \le u$ *and every element occurs with frequency* $n-1$ *or* n. *Then there exists an* $SBS_{k+\ell}(s, v+u)$.

Proof. Let A be an $SBS_k(s,v)$ and let B be an $SBS_\ell(s,u)$. Without loss of generality, we may assume the elements of A are distinct from those of B. Since $ks \le v$ and $\ell s \le u$, any element occurs at most once in any line of A or B. Since each element occurs with frequency $n-1$ or n, it follows that the superposition of A and B is an $SBS_{k+\ell}(s, v+u)$. ■

CONSTRUCTION 7. *If there is an $SBS_k(s, v)$, then, for any positive integer t, there is an $SBS_{kt}(s, vt)$.*

Proof. Let A be an $SBS_k(s, v)$. For $1 \leq i \leq t$, let A_i be the $SBS_k(s, v)$ obtained by replacing each element a in A by the ordered pair (a, i). Let B be the array obtained by superimposing $A_1, A_2, \ldots, A_t$. Then B is an $SBS_{kt}(s, vt)$. ■

LEMMA 8. *For positive integers k, s and t, $1 \leq t \leq k$, there exists an $SBS_k(s, sk + t)$.*

Proof. Since there exists an $SBS(s, s+1)$, there exists an $SBS_t(s, ts + t)$. Since there exists an $SBS(s, s)$, there exists an $SBS_{k-t}(s, (k-t)s)$. Since every element of an $SBS(s, s)$ has frequency s and every element of an $SBS(s, s+1)$ has frequency $s - 1$ or s, it follows that every element of the $SBS_t(s, ts + t)$ and the $SBS_{k-t}(s, (k-t)s)$ has frequency $s - 1$ or s. Therefore, by Construction 6 there is an $SBS_k(s, ks + t)$. ■

We now study the relationship among the parameters s, k and v with respect to the feasibility conditions for the existence of $SBS_k(s, v)$.

LEMMA 9. *If $SBS_k(s, v)$ is feasible and $2k \leq v$, then $SBS_{2k}(s, v)$ is also feasible.*

Proof. We show that the feasibility condition on the number of elements with odd frequency is satisfied. Write

$$ks^2 = nv + r = r(n+1) + (v - r)n \tag{2}$$

and

$$2ks^2 = 2nv + 2r. \tag{3}$$

Now two cases can occur.

Case 1. $2r < v$: There are two subcases to consider. If n is odd, then the number of elements of odd frequency in (2) is $v - r$. Since $SBS_k(s, v)$ is feasible, we must have $v - r \leq ks$. Since $2r < v$, the number of elements with odd frequency in (3) is $2r$. So it is enough to show that $2r \leq 2ks$. Since $2r < v$, we have $r < v - r \leq ks$ and hence $2r < 2ks$.

If n is even, then the number of elements of odd frequency in (2) is r and by the feasibility condition of $SBS_k(s, v)$ we must have $r \leq ks$. Again since $2r < v$, the number of elements with odd frequency in (3) is $2r \leq 2sk$.

Thus in this case $SBS_{2k}(s, v)$ is feasible.

Case 2. $2r \geq v$: This is similar to Case 1 and hence we do not provide the details. ■

COROLLARY 10. *If $SBS(s,v)$ is feasible and $2^i \leq v$, then so is $SBS_{2^i}(s,v)$.*

Using Corollary 10 we know that whenever $SBS(s,v)$ is feasible $SBS_2(s,v)$ must also be feasible. The following example shows that the converse does not hold.

EXAMPLE.. *Let $s=5, v=9$. Then $s^2=25=7(3)+2(2)$, i.e., 7 elements have frequency 3 and 2 elements have frequency 2. Since the number of odd frequency elements is greater than 5, we infer that $SBS(5,9)$ is not feasible and hence does not exist.*

On the other hand $SBS_2(5,9)$ does exist and one such square is given below.

4,8	3,7	1,5	8,9	2,6
3,7	5,9	3,8	2,6	1,4
1,5	3,8	6,9	1,4	2,7
8,9	2,6	1,4	7,9	3,5
2,6	1,4	2,7	3,5	8,9

LEMMA 11. *For any $s,v,k>0$, we have that $SBS_k(s,v)$ is feasible if and only if $SBS_{2k}(s,2v)$ is feasible.*

Proof. If an $SBS_k(s,v)$ is feasible, then $k \leq v$ and there exist nonnegative integers d,e,δ,ε such that

$$ks^2 = \delta d + \varepsilon e,$$

where $\{d,e\} = \{n,n+1\}$, e is even, $\delta+\varepsilon = v$ and $\delta \leq ks$. Therefore, $2k \leq 2v$ and there exist nonnegative integers $d,e,2\delta,2\varepsilon$ such that

$$2ks^2 = (2\delta)d + (2\varepsilon)e,$$

where $\{d,e\} = \{n,n+1\}$, e is even, $2\delta+2\varepsilon = 2v$ and $2\delta \leq 2ks$. Hence, an $SBS_{2k} \times (s,2v)$ is feasible.

In a similar manner we can establish the converse result. ■

For $k=1$, we know that, whenever $SBS(s,v)$ is feasible, it exists [1]. Hence we get the following corollary.

COROLLARY 12. *For any $s>0, SBS_2(s,2v)$ exists if and only if $SBS(s,v)$ exists.*

Proof.

$$\begin{aligned}\exists\ SBS_2(s,2v) &\Rightarrow SBS_2(s,2v) \text{ is feasible (by definition)}\\ &\Rightarrow SBS(s,v) \text{ is feasible (by Lemma 11)}\\ &\Rightarrow \exists\ SBS(s,v)\ [1]\\ &\Rightarrow \exists\ SBS_2(s,2v) \text{ (by Construction 7)}\end{aligned}$$ ■

Later we show that for $k = 2, 3$, whenever $SBS_k(s, v)$ is feasible it can be constructed (Theorems 25, 46, 50, 55).

3. Case $ks \leq v \leq ks(s+1)/2$

When $v = ks$, we have to construct $SBS_k(s, ks)$. Since an $SBS(s, s)$ exists, by Construction 7, an $SBS_k(s, ks)$ exists. Thus we consider the case $v > ks$.

We approach the construction problem in parts. The three main parts are the following.

1. $\lfloor ks^2/v \rfloor = s - 1$. This holds iff $ks < v \leq ks + ks/(s-1)$.
2. $\lfloor ks^2/v \rfloor = s - 2$. This holds iff $ks + ks/(s-1) < v \leq ks + 2ks/(s-2)$.
3. $\lfloor ks^2/v \rfloor < s - 2$. This holds iff $ks + 2ks/(s-2) < v \leq ks(s+1)/2$.

The last two cases are further subdivided into different cases which will become clear as we go into the details of the construction. The case $ks < v \leq ks + k$ is settled by Lemma 8 above. So for the rest of this section we only consider the case $v > k(s+1)$.

3.1. *Subcase* $\lfloor ks^2/v \rfloor = s - 1$

First we prove the following result which we require later.

LEMMA 13. *For any odd positive integer s, there exists an $SBS_{s-1}(s, s^2)$ in which every element has frequency $s - 1$.*

Proof. Let A be the addition table for $\mathbf{Z}_s$, and let B be the $SBS_{s-1}(s, (s-1)s)$ obtained from A by Construction 7. Observe that the cells of the main diagonal of B partition the $s(s-1)$ elements of B since s is odd. Now consider s new elements, $\infty_i, 1 \leq i \leq s$. For $1 \leq i \leq s$, let C_i be the set of elements

$$C_i = \{\infty_1, \infty_2, \ldots, \infty_s\} \setminus \{\infty_i\}.$$

Replace the $s-1$ elements in cell (i,i) of B by C_i. The result is an $SBS_{s-1}(s,s^2)$. It can be shown that every element of this SBS has frequency $s-1$. ■

LEMMA 14. *If s, k and v are positive integers such that $v > k(s+1)$ and $n = \lfloor ks^2/v \rfloor = s-1$, then there exists an $SBS_k(s,v)$ whenever the necessary condition of Lemma 1 is satisfied.*

Proof. Since $n = s-1$,

$$ks^2 = v(s-1) + r = r(s) + (v-r)(s-1), \quad \text{where } 0 \le r < v. \tag{4}$$

When s is an even integer, the necessary condition of Lemma 1 for the existence of an $SBS_k(s,v)$ implies that $v - r \le ks$, or equivalently, that $r \ge v - ks$. Therefore,

$$ks^2 = v(s-1) + r \ge v(s-1) + v - ks = vs - ks$$

from which it follows that $v \le k(s+1)$. Hence, when s is even, there are no positive integers s, k and v that satisfy the necessary condition of Lemma 1 for the existence of an $SBS_k(s,v)$ and the hypotheses of this Lemma.

Now let us consider odd integers s. Equation (4) implies that $ks^2 \ge v(s-1)$, from whence it follows that

$$v \le \frac{ks^2}{(s-1)} = k(s+1) + \frac{k}{s-1}.$$

Since $k(s+1) < v \le k(s+1) + k/(s-1)$, there exists an integer p such that

$$v = k(s+1) + p, \quad \text{where } 0 < p \le \frac{k}{s-1}.$$

Let $q = k - p(s-1)$. Then $k = p(s-1) + q$ and $v = ps^2 + q(s+1)$.

Since there exists an $SBS_{s-1}(s,s^2)$ (from Lemma 13), there exists an $SBS_{p(s-1)}(s,ps^2)$. Since there exists an $SBS(s,s+1)$, there exists an $SBS_q(s,q(s+1))$. In these two SBSs, each element has frequency $s-1$ or s. Therefore there exists an $SBS_k(s,ps^2+q(s+1)) = SBS_k(s,v)$. ■

3.2. Subcase $\lfloor ks^2/v \rfloor = s-2$

LEMMA 15. *If s, k and v are positive integers such that s is even and $n = \lfloor ks^2/v \rfloor = s-2$, then there exists an $SBS_k(s,v)$ whenever the necessary condition of Lemma 1 is satisfied.*

Proof. Since $n = s-2$,

$$ks^2 = v(s-2) + r = r(s-1) + (v-r)(s-2), \quad \text{where } 0 \le r < v. \tag{5}$$

Since s is even, the necessary condition of Lemma 1 for the existence of an $SBS_k(s, v)$ implies that $r \le ks$. Therefore, $ks^2 = v(s-2) + r \le v(s-2) + ks$; that is,

$$v \ge \frac{ks(s-1)}{(s-2)} = ks + \frac{ks}{s-2}.$$

From equation (5) it follows that $ks^2 \ge v(s-2)$ which is equivalent to

$$v \le \frac{ks^2}{(s-2)} = ks + \frac{2ks}{s-2}.$$

Therefore

$$ks + \frac{ks}{s-2} \le v \le ks + \frac{2ks}{s-2}.$$

We now construct an $SBS_k(s, v)$ for all v in this range.

Let $v = ks + p$ where $ks/(s-2) \le p \le 2ks/(s-2)$. Let A be the addition table of $\mathbf{Z}_{s+1}$ where the rows and columns are labeled with the symbols $0, 1, \ldots, s$ in their natural order. For $1 \le i \le k$, let A_i be the array obtained by replacing each element x in A by the ordered pair (x, i), and let B be the superposition of $A_1, A_2, \ldots, A_k$. Let C be the array obtained by deleting row and column 0 from B. The result is an $SBS_k(s, k(s+1))$ in which every pair of $\{1, 2, \ldots, s\} \times \{1, 2, \ldots, k\}$ occurs with frequency $s-1$ and every pair of $\{0\} \times \{1, 2, \ldots, k\}$ occurs with frequency s. Observe that every pair of $\{1, 2, \ldots, s\} \times \{1, 2, \ldots, k\}$ occurs in precisely one cell of the main diagonal; furthermore, each cell of the main backward diagonal (i.e., the cells $(i, s+1-i)$ for $i = 1, 2, \ldots, s$) contains the set of the pairs $\{0\} \times \{1, 2, \ldots, k\}$. Remove all the pairs of $\{0\} \times \{1, 2, \ldots, k\}$ from C, leaving all cells of the main backward diagonal empty. Call this array D. We now add p new elements to D, namely the elements ∞_i for $1 \le i \le p$.

First we place k elements into each cell of the main backward diagonal so that the following conditions are satisfied:

1. For $2 \le i \le p$, element ∞_i is not to be placed in any cell of the main backward diagonal of D until ∞_{i-1} has been placed in $s-2$ distinct cells.
2. Whenever ∞_i, $1 \le i \le p$, is placed in cell $(j, s+1-j)$, it is also to be placed in cell $(s+1-j, j)$.
3. Never place an element ∞_i in a cell $(j, s+1-j)$ having t elements if there is a cell $(\ell, s+1-\ell)$ with fewer than t elements.

It is easy to see that this requires precisely $u = \lceil ks/(s-2) \rceil \le p$ elements, and each element $\infty_i, 1 \le i < u$, has frequency $s-2$. The element ∞_u has frequency $ks - (u-1)(s-2) \le s-2$.

The second stage of this construction involves placing elements ∞_i, $u \le i \le p$, in cells of the main diagonal of D. Since these cells already contain k pairs of

$\{1,2,\ldots,s\}\times\{1,2,\ldots,k\}$, they will now contain more than k elements. This problem is readily corrected as the last step in the construction.

Place ∞_u in $u(s-2)-ks$ cells of the main diagonal of D so that it is not contained in more than one cell of any row or column. Continue to add elements to the cells of the main diagonal according to the following procedure:

1. For $u<i\leq p$, element ∞_i is not to be placed in any cell of D until ∞_{i-1} has been placed in $s-2$ distinct cells.
2. Never place an element ∞_i in a cell (j,j) having t elements if there is a cell (ℓ,ℓ) with fewer than t elements.

Finally, discard pairs of $\{1,2,\ldots,s\}\times\{1,2,\ldots,k\}$ from cells of the main diagonal, as necessary, so that each cell again has cardinality k.

The maximum number of new elements, having frequency $s-2$, that can be accommodated in the main backward diagonal and the main forward diagonal of D is $\lfloor 2ks/(s-2)\rfloor \geq p$. Hence, the resulting array is an $SBS_k(s,ks+p)=SBS_k(s,v)$, and each element has frequency $s-2$ or $s-1$. ■

We now prove the corresponding result for s an odd integer.

LEMMA 16. *If s, k and v are positive integers such that s is odd and $n=\lfloor ks^2/v\rfloor = s-2$, then there exists an $SBS_k(s,v)$ whenever the necessary condition of Lemma 1 is satisfied.*

The proof of this Lemma is more complicated than the previous one. For this reason, we approach it by way of several intermediate lemmas.

LEMMA 17. *For s an odd positive integer and k an integer in the range $1\leq k\leq s-1$, there exists an $SBS_k(s,k(s+1)+1)$.*

Proof. This result follows for $k=1$ by Lemma 5.

Now consider the integers k in the range $2\leq k\leq s-1$. Let $A=(a_{ij})$ be the addition table of $\mathbf{Z}_{s+1}$, the integers modulo $s+1$. Let A_i, $i=1,2$, be the array obtained by replacing every element x of A by the pair (x,i). Let E_1 be the array obtained by deleting row and column 0 from matrix A_1. Let E_2 be the array obtained by deleting row and column $(s+1)/2$ from matrix A_2. Label the rows and columns of E_1 and E_2 with the integers $1,2,\ldots,s$ in their natural order.

Let S_i be the set consisting of the elements in cell (i,i) of E_1 and E_2. Since each S_i has cardinality 2, and since each element of $\{0,1,2,\ldots,s\}\times\{1,2\}$ is in at most 2 of the S_i's, the S_i's have a system of distinct representatives, say R; furthermore, we may assume that $\{0_1,0_2\}\subset R$ since 0_2 is only in S_1 and 0_1 is only in $S_{(s+1)/2}$.

Let $B=(b_{ij})$ be the addition table of $\mathbf{Z}_s$, the integers modulo s. Let C be the array obtained by replacing each element on the main diagonal of B by an element y that is not in $\mathbf{Z}_s$ nor $\mathbf{Z}_{s+1}$. For $i=3,4,\ldots,k$, let E_i be the array obtained by replacing every element x of C by the pair (x, i). Let F be the array obtained by superimposing the arrays $E_1, E_2, \ldots, E_k$. This array F is an $SBS_k(s, k(s+1))$. The elements $0_1, 0_2, y_3, y_4, \ldots, y_k$ all have frequency s and all the other elements have frequency $s-1$.

Introduce a new element, ∞, to each cell of the main diagonal of F. Remove a single occurrence of each of the elements $\infty, 0_1, 0_2, y_3, y_4, \ldots, y_k$ from the cells of the main diagonal, so that at most one element is removed from any single cell. (Recall that $k \leq s-1$.) From the remaining $s-k-1$ cells of the main diagonal (those which still contain $k+1$ elements), delete the element of the system of distinct representatives R. Since every element now has frequency $s-1$ or $s-2$, the result is an $SBS_k(s, k(s+1)+1)$. ■

LEMMA 18. *Let s, k and v be positive integers such that s is odd and $ks^2/(s-1) < v \leq k(s+2)$. Then there exists an $SBS_k(s, v)$ whenever the necessary condition of Lemma 1 is satisfied.*

Proof. Since $ks^2/(s-1) < v \leq k(s+2)$,

$$k(s+1) + \frac{k}{(s-1)} < v \leq k(s+1) + k.$$

Therefore, $v = k(s+1) + t$ where $k/(s-1) < t \leq k$. Let p and q be the unique nonnegative integers such that $k = pt + q$ where $0 \leq q < t$. Then $k = q(p+1) + (t-q)(p)$. Since $k/(s-1) < t$ and $k = pt + q \geq pt$, it follows that

$$p \leq \frac{k}{t} < s - 1.$$

Since p and s are both integers, it follows that $p + 1 \leq s - 1$. Then, by Lemma 17, there exists an $SBS_{p+1}(s, (p+1)(s+1)+1)$ and an $SBS_p(s, p(s+1)+1)$. Therefore, there exists an $SBS_{q(p+1)}(s, q(p+1)(s+1)+q)$ and an $SBS_{(t-q)p}(s, (t-q)p(s+1)+t-q)$. Superimposing these two SBSs gives an $SBS_k(s, k(s+1)+t) = SBS_k(s, v)$. ■

LEMMA 19. *If s is any odd positive integer, then there exists an $SBS_{(s-1)/2}(s, s \times (s+1)/2)$ in which s elements have frequency $s-1$ and the remaining $s(s-1)/2$ have frequency $s-2$.*

Proof. Let A be the addition table of $\mathbf{Z}_s$. Let B be the $SBS_{(s-1)/2}(s, s(s-1)/2)$ obtained by Construction 7. Let C be the array obtained by deleting all the elements from cells $(2i, 2i+1)$ and cells $(2i+1, 2i)$ of B for all $i \in \mathbf{Z}_s$. From the structure of matrix A, it follows that each of the $s(s-1)/2$ elements in C has frequency $s-2$ and C is symmetric.

For every $i \in \mathbf{Z}_s$, introduce a new element ∞_i. For every $i \in \mathbf{Z}_s$, define

$$D_i = \{\infty_{i+1}, \infty_{i+2}, \ldots, \infty_{i+(s-1)/2}\},$$

where the subscripts are determined by addition over $\mathbf{Z}_s$. For every $i \in \mathbf{Z}_s$, place the set of elements D_i in cell $(2i+1, 2i)$ and in cell $(2i, 2i+1)$, where arithmetic is over $\mathbf{Z}_s$. It can be shown that each element ∞_i, for $i \in \mathbf{Z}_s$, is absent from row and column $2i$, but is contained in each other row and column precisely once. Hence, the new elements all have frequency $s-1$ in C.

It follows that the result is an $SBS_{(s-1)/2}(s, s(s+1)/2)$, as required. ■

LEMMA 20. *Let s, k and v be positive integers such that s is odd and $k(s+2) < v \leq k(s+2) + 2k/(s-1)$. Then there exists an $SBS_k(s, v)$ whenever the necessary condition of Lemma 1 is satisfied.*

Proof. For any integer v in the range $k(s+2) < v \leq k(s+2) + 2k/(s-1)$, there exists an integer p such that $v = k(s+2) + p$ where $0 < p \leq 2k/(s-1)$. Therefore, there exists a nonnegative integer q such that

$$k = p\left(\frac{s-1}{2}\right) + q.$$

By the previous Lemma, there exists an $SBS_{(s-1)/2}(s, s(s+1)/2)$. By Construction 7, there exists an $SBS_{p(s-1)/2}(s, ps(s+1)/2)$. By Lemma 5 and Construction 7, there exists an $SBS_q(s, q(s+2))$. Every element in either of these SBSs has frequency $s-1$ or $s-2$. Hence, by Construction 6, there exists an $SBS_k(s, q(s+2) + ps(s+1)/2) = SBS_k(s, v)$. ■

Proof of Lemma 16. Since $n = s-2$, there exists a unique integer r such that

$$ks^2 = v(s-2) + r = r(s-1) + (v-r)(s-2), \quad \text{where } 0 \leq r < v.$$

Since s is odd, the necessary condition of Lemma 1 for the existence of an $SBS_k(s, v)$ implies that $v - r \leq ks$, or equivalently, that $r \geq v - ks$. Therefore,

$$ks^2 = v(s-2) + r \geq v(s-2) + v - ks,$$

from whence it follows that

$$v \leq \frac{ks(s+1)}{s-1} = k(s+2) + \frac{2k}{s-1}.$$

We also have that $ks^2 = v(s-2) + r < v(s-1)$, which implies that

$$\frac{ks^2}{s-1} < v.$$

By Lemmas 18 and 20, there exists an $SBS_k(s, v)$ for all integers v in the range

$$\frac{ks^2}{s-1} < v \leq k(s+2) + \frac{2k}{s-1},$$

whenever the necessary condition of Lemma 1 is satisfied. This establishes the result. ■

3.3. Subcase $\lfloor ks^2/v \rfloor < s-2$

We are now in a position to establish the existence of an $SBS_k(s, v)$ whenever $n = \lfloor ks^2/v \rfloor < s-2$. However, before proceeding with this result, we digress to consider a generalization of *SBS*s, called near symmetric balanced squares. This digression will allow us to introduce the construction used to prove this result.

By a near symmetric balanced square with blocksize k, order v and side s, we mean an $s \times s$ array in which every cell contains a subset of cardinality k from a set V of cardinality v satisfying the following properties:

1. Every element occurs in m or $m+1$ cells of each row or column.
2. Every element occurs in $f-1$, f or $f+1$ cells of the array.
3. The array is symmetric.

Again, we will assume throughout that $k \leq v/2$. We refer to such a design as an $NSBS_k(s, v)$. Clearly, every $SBS_k(s, v)$ is an $NSBS_k(s, v)$.

LEMMA 21. *If s, k and v are positive integers such that $v > ks$, then there exists an $NSBS_k(s, v)$.*

Before we proceed with a proof of this fact, we need to introduce some terminology. For any $s \times s$ array, say S, whose rows and columns are labeled with the elements $g \in \mathbf{Z}_s$, the g-th back-diagonal consists of the set of cells

$$\{(i, g-i) \ : \ i \in \mathbf{Z}_s\}.$$

However, we wish to consider the cells of this back-diagonal in a specific order, as prescribed in the following definitions.

Definition 22. Consider an $s \times s$ array, where s is an odd positive integer, say $s = 2t+1$. For any $g \in \mathbf{Z}_s$, let h be such that $2h = g$. Then the g-th back-diagonal is defined to be the t pairs of symmetrically situated cells $\{(h-i, h+i), (h+i, h-i)\}$, where i assumes the values $t, t-1, \ldots, 2, 1$ in that order, followed by the main diagonal cell (h, h). We denote this back-diagonal by D_h.

The situation is more complex when s is a positive even integer.

Definition 23. Consider an $s \times s$ array, where s is an even positive integer, say $s = 2t$. For any $g \in \mathbf{Z}_{2t}$, define back-diagonal E_g to be the ordered set of cells beginning with cell (g, g), followed by the pairs of symmetrically situated cells $\{(g+i, g-i), (g-i, g+i)\}$ for $i = 1, 2, \ldots, t-1$ in that order, and ending with the cell $(g+t, g+t)$. Define back-diagonal F_g to be the sequence of t pairs of symmetrically situated cells $\{(g+i, g+1-i), (g+1-i, g+i)\}$ for $i = 1, 2, \ldots, t$ in that order. Observe that this defines $4t$ distinct diagonals; furthermore, diagonals E_g and E_{g+t} consist of the same cells, but in the reverse order, as do diagonals F_g and F_{g+t}.

We are now in a position to prove Lemma 21.

Proof of Lemma 21. Let A be an $NSBS_k(s, v)$ with $v > ks$, and let c_i be the number of elements in A of frequency i for $i = f-1, f, f+1$. An even element is an element of A whose frequency is an even integer: an odd element is an element of odd frequency. Since A is an $NSBS$ it follows that

$$v = c_{f-1} + c_f + c_{f+1},$$
$$ks^2 = c_{f-1}(f-1) + c_f(f) + c_{f+1}(f+1).$$

We now show that, given any positive integers s, k and v such that $v > ks$, these equations always have a solution in nonnegative integers for f, c_{f-1}, c_f and c_{f+1}; furthermore, there exists a solution in which the number of odd elements is precisely ks.

By the division algorithm, there exist unique nonnegative integers n and r such that

$$ks^2 = vn + r, \quad \text{where } 0 \leq r < v.$$

As before, let d, e, δ and ε be integers such that

$$ks^2 = r(n+1) + (v-r)(n) = \varepsilon(e) + \delta(d),$$

where e is an even integer, $\{e, d\} = \{n, n+1\}$, and $\{\varepsilon, \delta\} = \{r, v-r\}$. If $\delta \geq ks$, then we can write

$$\begin{aligned} ks^2 &= \varepsilon(e) + \delta(d) \\ &= \varepsilon(e) + ks(d) + \left(\frac{\delta - ks}{2}\right)(d-1) + \left(\frac{\delta - ks}{2}\right)(d+1) \\ &= c_{d-1}(d-1) + c_d(d) + c_{d+1}(d+1), \end{aligned}$$

where $c_{d-1} + c_d + c_{d+1} = v$ and $c_d = ks$. If $\delta < ks$, then we can write

$$
\begin{aligned}
ks^2 &= \varepsilon(e) + \delta(d) \\
&= (v - ks)(e) + \left(\frac{ks - \delta}{2}\right)(e-1) + \delta(d) + \left(\frac{ks - \delta}{2}\right)(e+1) \\
&= c_{e-1}(e-1) + c_e(e) + c_{e+1}(e+1),
\end{aligned}
$$

where $c_{e-1} + c_e + c_{e+1} = v$ and $c_{e-1} + c_{e+1} = ks$.

We now consider the case when s is an odd integer. Suppose there exist nonnegative integers f, c_{f-1}, c_f and c_{f+1} such that f is odd and

$$
\begin{aligned}
ks^2 &= c_{f-1}(f-1) + c_f(f) + c_{f+1}(f+1), \\
v &= c_{f-1} + c_f + c_{f+1}, \\
ks &= c_f.
\end{aligned}
$$

To construct an $NSBS_k(s, v)$, we place elements in the cells of an $s \times s$ array A in the following order

$$D_0, D_1, \ldots, D_{s-1}, D_0, D_1, \ldots, D_{s-1}, \ldots, D_0, D_1, \ldots, D_{s-1}. \tag{6}$$

It is to be understood that each diagonal D_i appears in this sequence exactly k times. Symbols are placed in the cells of this sequence one at a time according to the following procedure:

1. Whenever an element is placed in a cell $(i, j), i \neq j$, it is also placed in cell (j, i).
2. Once an element x is placed in a cell, we continue to place the element x in cells until it has been placed in $f-1$, f or $f+1$ cells, as appropriate.
3. • If the next f cells of the Sequence (6) include a cell $(i, i), i \in \mathbf{Z}_s$, of the main diagonal, then

   • place an element in f consecutive cells of the Sequence (6),

   • else, if the number of elements of frequency $f-1$ already placed in the array is less than c_{f-1}, then

   • place an element in $f-1$ consecutive cells of the Sequence (6),

   • else,

   • place an element in $f+1$ consecutive cells of the Sequence (6).

Can anything prevent this strategy from successfully constructing an $NSBS_k(s, v)$?

Since s and f are both odd, and since $v > ks$, it follows that $f \leq s - 2$. It can be shown that any f consecutive cells in the sequence span f distinct rows and f distinct columns of A. Therefore, any odd element is in at most one cell of any row or column. Since even elements lie in the cells of a single diagonal, each even element is in at most one cell of any row or column. Therefore, any element lies in at most one cell of any row or any column.

By construction, any element contained in a cell of the main diagonal is an odd element and no element is contained in more than one cell of the main diagonal. Thus there are ks odd elements, as required. By construction, there are c_{f-1} elements of frequency $f - 1$. Since

$$ks^2 = c_{f-1}(f - 1) + c_f(f) + c_{f+1}(f + 1),$$

it follows that there are c_{f+1} elements of frequency $f + 1$, as required.

Since the construction fills cells in symmetrically situated pairs, the array is symmetric.

Thus, the result is an $NSBS_k(s, v)$.

Next, suppose there exist nonnegative integers f, c_{f-1}, c_f and c_{f+1} such that f is even and

$$ks^2 = c_{f-1}(f - 1) + c_f(f) + c_{f+1}(f + 1),$$

$$v = c_{f-1} + c_f + c_{f+1},$$

$$ks = c_{f-1} + c_{f+1}.$$

Apply the above construction, with Step 3 replaced by

3′ (a) If the number of elements of frequency $f + 1$ already placed in the array is less than c_{f+1}, then an element x is placed in an odd number of cells $f + 1$ whenever the next $f + 1$ cells of the sequence include a cell $(i, i), i \in \mathbf{Z}_s$, of the main diagonal.

(b) If c_{f+1} elements of frequency $f + 1$ have already been placed in the array, then an element x is placed in an odd number of cells $f - 1$ whenever the next $f - 1$ cells of the sequence include a cell (i, i), $i \in \mathbf{Z}_s$, of the main diagonal.

(c) Otherwise, x is placed in an even number of cells f.

Again, it can be shown that the result is an $NSBS_k(s, v)$.

The construction when s is an even integer proceeds in essentially the same way. Symbols are placed in the cells of A in the order determined by the sequence

$$E_0, F_0, E_1, F_1, \ldots, E_{s-1}, F_{s-1}, E_0, F_0, E_1, F_1, \ldots, E_{s-1}, F_{s-1}, \ldots$$

where it must be understood that this sequence consists of exactly sk diagonals. (Note that every diagonal is there $k/2$ times if k is an even integer; otherwise, s

diagonals are there $(k+1)/2$ times and the other s are there $(k-1)/2$ times.) This establishes the result. ■

In the above construction of $NSBS$s, the cells of the main diagonal contain ks distinct elements, each of which is an odd element. This is not, in general, the case for SBSs. The main idea behind the construction of SBSs described below, is to place an appropriate number of even elements in cells of the main diagonal, and then proceed as with $NSBS$s.

LEMMA 24. *If s, k and v are positive integers such that* $n = \lfloor ks^2/v \rfloor < s-2$, *then there is an* $SBS_k(s, v)$ *whenever the necessary condition of Lemma 1 is satisfied.*

Proof. The division algorithm implies there exist unique nonnegative integers n and r such that $ks^2 = vn + r$ where $0 \leq r < v$. As before, define integers $d,\ e,\ \delta$ and ε so that

$$ks^2 = r(n+1) + (v-r)(n) = \varepsilon(e) + \delta(d),$$

where e is an even integer, $\{d, e\} = \{n, n+1\}$, and $\{\delta, \varepsilon\} = \{r, v-r\}$. The necessary condition of Lemma 1 for the existence of an $SBS_k(s, v)$ implies that $\delta \leq ks$. Divide $ks - \delta$ by e to obtain nonnegative integers p and q such that

$$ks - \delta = pe + q, \quad \text{where } 0 \leq q < e.$$

If s is odd, place the element "1" in the first $e-q$ cells of the sequence

$$D_0, D_1, \ldots, D_{s-1}, \ldots, D_0, D_1, \ldots, D_{s-1}. \tag{7}$$

If s is even, place the element "1" in the first $e-q$ cells of the sequence

$$E_0, F_0, E_1, F_1, \ldots, E_{s-1}, F_{s-1}, \ldots \tag{8}$$

(Again, it is to be understood that each of these sequences consists of ks diagonals.) Then place "1" in q distinct cells of the main diagonal in such a way that "1" occurs in at most one cell of any row or column. This is always possible since $e \leq n+1 \leq s-2$. Now for $x \in \{2, 3, \ldots, p+1\}$, place element x in e distinct cells of the main diagonal according to the following procedure:

1. For $x = 3, 4, \ldots, p+1$, element x is not placed in any cell of the main diagonal until element $x-1$ has been placed in e distinct cells.
2. Never place an element x in a cell (i, i) with t elements if there exists a cell (j, j) with fewer than t elements.

This procedure places $ks - \delta$ elements in the cells of the main diagonal.

Complete the $SBS_k(s, v)$ by placing elements in the remaining cells of the Sequence 7 if s is odd, and by placing elements in the remaining cells of the Sequence 8 if s is even, according to the following procedure:

1. If the next d cells in the sequence include a cell (i,i), $i \in \mathbf{Z}_s$, of the main diagonal and this cell (i,i) has fewer than k elements, then place an element x in the next d cells, including cell (i,i).
2. If the next d cells in the sequence include a cell (i,i), $i \in \mathbf{Z}_s$, of the main diagonal and this cell (i,i) contains k elements, then place an element x in the next e cells of the sequence, avoiding the cell (i,i) itself.
3. Otherwise, place an element x in the next e cells of the sequence.

Since $n < s-2$, $d < s-1$. Therefore, every odd element is contained in precisely one cell of the main diagonal and, hence, there are δ elements of odd frequency d. Since $ks^2 = \varepsilon(e) + \delta(d)$, there are ε elements of even frequency e. The method of construction guarantees symmetry. The only property that remains to be verified is that every element occurs in at most one cell of any row or column.

By construction, every occurrence of element x, for any $x > p+1$, is contained in some sequence of $e+1$ or fewer consecutive cells. Now, if s is odd, then $e \leq n+1 \leq s-2$. Since e is even, $e \leq s-3$ and $e+1 \leq s-2$. It can be shown that any $s-2$ consecutive cells in Sequence 7 span $s-2$ distinct rows and $s-2$ distinct columns. If s is even, then $e \leq n+1 \leq s-2$ and $e+1 \leq s-1$. It can be shown that any $s-1$ consecutive cells in Sequence 8 span $s-1$ distinct rows and $s-1$ distinct columns. Thus, no element x is contained in 2 cells of any row or column. This establishes the result. ■

The following theorem summarizes the results of this section.

THEOREM 25. *If s, k and v are nonnegative integers such that $v \geq ks$, then there exists an $SBS_k(s,v)$ whenever the necessary condition of Lemma 1 is satisfied.*

4. Case $s < v < ks$

In this section, we present a method for constructing an $SBS_k(s,v)$ when $\max\{s, 2k\} \leq v < sk$. In Sections 4.5 and 4.6, we prove the correctness of the construction for the cases $k=2$ and $k=3$ respectively. Though the construction has never failed for any particular case we have examined, a general proof that it can always be made to succeed still eludes us. Thus for $k \geq 4$, the construction remains a heuristic. Before describing the general method of attack, we show that the necessary condition for the existence of an $SBS_k(s,v)$ is sufficient when $1 \leq s \leq 4$.

4.1. *Cases when* $1 \leq s \leq 4$

Here we state the result concerning the construction problem for $SBS_k(s,v)$ when $1 \leq s \leq 4$ and for any $1 \leq k \leq v/2$. The constructions are direct and are obtained by assigning a suitably chosen subset of $\{1, \dots, v\}$ to each cell of the $s \times s$ array. The details of the construction and proofs can be found in Sarkar and Schellenberg [2].

LEMMA 26. *For k a positive integer, $v \geq 2k$, and $1 \leq s \leq 4$, there is an $SBS_k(s, v)$ whenever the necessary condition of Lemma 1 is satisfied.*

4.2. General Construction Method

Recall that in an $SBS_k(s, v)$, every element appears with a frequency n or $n + 1$ where $n = \lfloor ks^2/v \rfloor$, and every element appears in each row and column with frequency m or $m + 1$ where $m = \lfloor ks/v \rfloor$. Therefore, there exist integers r and p such that

$$ks^2 = nv + r, \quad \text{where } 0 \leq r < v$$

and

$$ks = mv + p, \quad \text{where } 0 \leq p < v.$$

Since $nv = ks^2 - r = mvs + (ps - r)$, v divides $ps - r$. Hence, we let $a = (ps - r)/v$ and obtain that

$$ps = av + r, \quad \text{where } 0 \leq r < v.$$

This implies that $a = \lfloor ps/v \rfloor$. The equation

$$nv + r = ks^2 = mvs + ps = mvs + av + r,$$

gives one more relationship between these parameters; namely,

$$n = ms + a, \quad \text{where } 0 \leq a < s \text{ (since } p < v).$$

In summary, we have the following relationships among the eight parameters v, s, k, n, r, m, p, a:

$$ks^2 = nv + r, \quad \text{where } 0 \leq r < v,$$

$$ks = mv + p, \quad \text{where } 0 < p < v,$$

$$ps = av + r, \quad \text{where } 0 \leq r < v,$$

$$n = ms + a, \quad \text{where } 0 \leq a < s.$$

Since we are considering v in the range $\max\{s, 2k-1\} < v < ks$, we get that

$$s < n = \left\lfloor \frac{ks^2}{v} \right\rfloor < ks,$$

$$0 < m = \left\lfloor \frac{ks}{v} \right\rfloor = \left\lfloor \frac{n}{s} \right\rfloor \leq \min \{k-1, s/2\},$$

$$0 \leq a = \left\lfloor \frac{ps}{v} \right\rfloor < \min\{p, s\}.$$

We approach the construction problem in the following way. Each element has to be placed in the array either n or $n+1$ times. We complete this allocation in two phases. Suppose an element is to be placed n times. We know that $n = ms + a$. In the first phase the element is allocated either a or $s+a$ times. In the second phase the element is placed either ms or $(m-1)s$ times respectively. Thus the allocation in the second phase can be completed by allocating the element to either m or $m-1$ complete back-diagonals. The only constraint in the second phase is to ensure that the element is not placed in any cell more than once.

Suppose an element has to be placed $n+1$ times, where $n+1 = ms + a + 1$. In the first phase we place the element either $a+1$ or $s+a+1$ times and in the second phase we place the element either ms or $(m-1)s$ times respectively. This can be achieved by allocating the element to m or $(m-1)$ complete back-diagonals. Again the only constraint in the second phase is that the element is not placed in any cell more than once.

If an element is placed a or $a+1$ times in the first phase, then we will call it a short element. On the other hand, if the element is placed $s+a$ or $s+a+1$ times in the first phase, then we will call it a long element. The number of short and long elements placed in the first phase depends on the values of the parameters. We now describe the different cases that arise. The actual first phase allocation strategy is described in Subsection 4.3 and the second phase allocation strategy is described in Subsection 4.4.

Construction Strategy 1 (s and a both odd). Let v, s, k be integers such that s is odd, $a = \lfloor ps/v \rfloor$ is odd, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a)$$

we get

$$(v - r + p)s = r(a+1) + (v-r)(s+a). \tag{9}$$

First Phase. Apply Construction 30 (see below) to fill $(v-r+p)$ complete back-diagonals with r short elements of frequency $a+1$ and $v-r$ long elements of frequency $s+a$.

Second Phase. Fill up $ks - (v - r + p)$ complete back-diagonals using Construction 39.

Construction Strategy 2 (s and $a+1$ both odd). Let v, s, k be integers such that s is odd, $a = \lfloor ps/v \rfloor$ is even, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a)$$

we get

$$(p+r)s = r(s+a+1) + (v-r)(a). \tag{10}$$

First Phase. Apply Construction 30 to fill $(p+r)$ complete back-diagonals with r long elements of frequency $s+a+1$ and $v-r$ short elements of frequency a.

Second Phase. Fill up $ks - (p+r)$ complete back-diagonals using Construction 39.

Construction Strategy 3 (s, $a+1$ both even and $v - r \geq p$): Let v, s, k be integers such that s is even, $a = \lfloor ps/v \rfloor$ is odd, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a)$$

we get

$$(v-r)s = r(a+1) + (v-r-p)(s+a) + p(a). \tag{11}$$

First Phase. Apply Construction 34 to fill $v - r$ complete back-diagonals with r short elements of frequency $a+1$, p short elements of frequency a and $v-r-p$ long elements of frequency $s+a$.

Second Phase. Fill up $ks - (v-r)$ complete back-diagonals using Construction 39.

Construction Strategy 4 (s, $a+1$ both even and $v - r < p$). Let v, s, k be integers such that s is even, $a = \lfloor ps/v \rfloor$ is odd, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a)$$

we get

$$(2p - v + r)s = (p - v + r)(s + a + 1) + (v-p)(a+1) + (v-r)(a). \tag{12}$$

First Phase. Apply Construction 34 to fill $2p - v + r$ complete back-diagonals with $v - p$ short elements of frequency $a+1, v-r$ short elements of frequency a and $p - v + r$ long elements of frequency $s + a + 1$.

Second Phase. Fill up $ks - (2p - v + r)$ complete back-diagonals using Construction 39.

Construction Strategy 5 (s, a both even and $r \geq p$). Let v, s, k be integers such that s is even, $a = \lfloor ps/v \rfloor$ is even, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a)$$

we get

$$rs = (r-p)(s+a+1) + p(a+1) + (v-r)(a). \tag{13}$$

First Phase. Apply Construction 34 to fill r complete back-diagonals with p short elements of frequency $a+1$, $v-r$ short elements of frequency a and $r-p$ long elements of frequency $s+a+1$.

Second Phase. Fill up $ks-r$ complete back-diagonals using Construction 39.

Construction Strategy 6 (s, a both even and $r < p$). Let v, s, k be integers such that s is even, $a = \lfloor ps/v \rfloor$ is even, and $\max\{s+1, 2k\} \leq v < sk$. From

$$ps = r(a+1) + (v-r)(a) \text{ we get}$$

$$(2p-r)s = r(a+1) + (p-r)(s+a) + (v-p)(a). \tag{14}$$

First Phase. Apply Construction 34 to fill $2p-r$ complete back-diagonals with r short elements of frequency $a+1$, $v-p$ short elements of frequency a and $p-r$ long elements of frequency $s+a$.

Second Phase. Fill up $ks-r$ complete back-diagonals using Construction 39.
In the first phase allocation, an odd frequency element, whether short or long, will be placed in exactly one of the main diagonal cells. An even frequency element is placed in either 0 or 2 main diagonal cells according as it is a short or a long element.

When s is odd, any complete back-diagonal has exactly one main diagonal cell, whereas, when s is even, any complete back-diagonal has either 0 or 2 main diagonal cells. In Subsection 4.3, we will describe an enumeration of the cells of the back-diagonals of an $s \times s$ array and an ordering of the back-diagonals.

The intuition behind equations (9) to (14) is the following. Let b be the number of complete back-diagonals on the left hand side of any of the equations (9) to (14). The first b back-diagonals in the enumeration in Subsection 4.3 include precisely b cells (j, j) of the main-diagonal. Let d be the number of main diagonal cells required in the first phase to properly place the v elements. Then the equations (9) to (14) are designed so that $b = d$.

4.3. First Phase Allocation

We describe the allocation separately for s odd and s even.

4.3.1. Case s Odd

When s is odd, there exists an integer t such that $s = 2t + 1$. Recall from Definition 22, that D_h consists of the symmetrically situated pairs of cells

$$(h - i, h + i) \quad \text{and} \quad (h + i, h - i),$$

where i assumes the values $t, t - 1, \ldots, 2, 1$ in that order, followed by the cell (h, h) on the main diagonal. Define $\overline{D}_h$ to be the same set of cells, but in the opposite order; that is, $\overline{D}_h$ begins with the main diagonal cell (h, h) and then continues with the symmetrically situated pairs of cells

$$(h - i, h + i) \quad \text{and} \quad (h + i, h - i),$$

where i assumes the values $1, 2, \ldots, t$ in that order.

We define the following order of the cells of an $s \times s$ array where each cell is repeated exactly k times.

$$\mathscr{S} = (D_0, \overline{D_{t+1}}, D_1, \overline{D_{t+2}}, \ldots, D_{s-1}, \overline{D}_t, D_0, \overline{D_{t+1}}, \ldots, D_{s-1}, \overline{D}_t, D_0, \overline{D_{t+1}}, \ldots), \tag{15}$$

where it must be understood that this sequence consists of exactly sk back-diagonals. Furthermore, whenever an element x is placed in cell (i, j) of $\mathscr{S}$, $i \neq j$, then x is also placed in cell (j, i).

EXAMPLE.. *We provide an example to illustrate the enumeration of the cells for $s = 9$. For convenience, we list the cells in a 2D form. Observe that each row, after the first row, is obtained by adding 1 modulo 9 to the entries of the preceding row. It is to be understood that this 2D array is extended row wise until it consists of ks back-diagonals.*

$\{5,4\}$	$\{6,3\}$	$\{7,2\}$	$\{8,1\}$	$(0,0)$	$(5,5)$	$\{4,6\}$	$\{3,7\}$	$\{2,8\}$	$\{1,0\}$
$\{6,5\}$	$\{7,4\}$	$\{8,3\}$	$\{0,2\}$	$(1,1)$	$(6,6)$	$\{5,7\}$	$\{4,8\}$	$\{3,0\}$	$\{2,1\}$
$\{7,6\}$	$\{8,5\}$	$\{0,4\}$	$\{1,3\}$	$(2,2)$	$(7,7)$	$\{6,8\}$	$\{5,0\}$	$\{4,1\}$	$\{3,2\}$
$\{8,7\}$	$\{0,6\}$	$\{1,5\}$	$\{2,4\}$	$(3,3)$	$(8,8)$	$\{7,0\}$	$\{6,1\}$	$\{5,2\}$	$\{4,3\}$
$\{0,8\}$	$\{1,7\}$	$\{2,6\}$	$\{3,5\}$	$(4,4)$					

The sequence $\mathscr{S}$ is obtained by scanning the rows from left to right starting from the first row. In this example, the notation $\{5,4\}$ represents the two cells $(4,5)$ and $(5,4)$, while the notation $(1,1)$ represents the single cell $(1,1)$ which lies on the main diagonal of the 9×9 array. If an element is to be placed four times, then it has to placed in the cells $\{5,4\}$ and $\{6,3\}$. On the other hand, to place an element 10 times, it has to be placed in the cells $\{5,4\}, \{6,3\}, \{7,2\}, \{8,1\}, (0,0)$ and $(5,5)$.

The following three observations are critical to the success of the construction.

LEMMA 27. *The main diagonal cell (h, h) of diagonal D_h is immediately succeeded by the main diagonal cell $(h+t+1, h+t+1)$ of diagonal $\overline{D_{h+t+1}}$.*

LEMMA 28. *If $2b < s$, then any $2b$ consecutive cells of this sequence span $2b$ distinct rows of A and $2b$ distinct columns.*

LEMMA 29. *If $s < 2b < 2s$, then any $2b$ consecutive cells of this sequence cover all rows and columns of A at least once, but at most two cells lie in any one row or column.*

CONSTRUCTION 30 (First Phase Allocation: s odd). *The construction is based on equations (9) and (10). Refer to Section 4.2 for the definition of the various parameters.*

1. *Let $\mathscr{S}$ be the ordering of the cells of an $s \times s$ array as defined in equation (15).*
2. *If a is odd, then*
   - *Fill the first $r(a+1)$ cells of the sequence $\mathscr{S}$ by placing each of r elements in $(a+1)$ consecutive cells of $\mathscr{S}$.*
   - *Fill the next $(v-r)(s+a)$ cells of the sequence $\mathscr{S}$ by placing each of $(v-r)$ cells in $(s+a)$ consecutive cells of $\mathscr{S}$.*
3. *If a is even, then*
   - *Fill the first $(v-r)a$ cells of the sequence $\mathscr{S}$ by placing each of $(v-r)$ elements in a consecutive cells of the sequence $\mathscr{S}$.*
   - *Fill the next $r(s+a+1)$ cells of the sequence $\mathscr{S}$ by placing each of r elements in $(s+a+1)$ consecutive cells of the sequence $\mathscr{S}$.*

LEMMA 31. (Correctness of Construction 30). *Let v, s, k be integers such that s is odd and $\max\{s+1, 2k\} \leq v < sk$. Construction 30 ensures that*

1. *Each short element has frequency 0 or 1 in each row and column.*
2. *Each long element has frequency 1 or 2 in each row and column.*
3. *If a is odd, then $v-r+p$ back-diagonals are completely filled, else $r+p$ back-diagonals are completely filled.*

Proof. Consider the case when a is odd. Since a and s are both odd, the integers $a+1$ and $s+a$ are both even. In the sequence $\mathscr{S}$ defined in (15), two main diagonal cells are always consecutive. This ensures that Construction 30 completes successfully. Since

$$(v-r+p)s = r(a+1) + (v-r)(s+a),$$

these elements completely fill up $v-r+p$ back-diagonals of sequence $\mathscr{S}$. By Lemma

28, each short element has frequency 0 or 1 in each row and column. By Lemma 29, each long element has frequency 1 or 2 in each row and column. This establishes the result when a is odd.

In a similar fashion we can establish the corresponding result when a is even and

$$(p+r)s = r(s+a+1) + (v-r)(a).$$

This completes the proof. ■

4.3.2. *Case s Even*

Let the back-diagonals E_g and F_g be as defined in Definition 23. We define the following order of the cells of an $s \times s$ array where each cell is repeated exactly k times.

$$\mathscr{T} = (E_0, F_0, E_1, F_1, \ldots, E_{s-1}, F_{s-1}, E_0, F_0, E_1, F_1, \ldots, E_{s-1}, F_{s-1}, \ldots), \tag{16}$$

where it must be understood that this sequence consists of exactly sk back-diagonals. Furthermore, whenever an element x is placed in a cell (i,j) with $i \neq j$, then x must also be placed in the cell (j,i) in order to achieve symmetry.

EXAMPLE. *We provide an example to illustrate the enumeration of the cells for* $s = 10$. *Again for convenience, we list the cells in a 2D form. Observe that each row, after the first row, is obtained by adding 1 modulo 10 to the entries of the preceding row. It is to be understood that this 2D array is extended rowwise until it consists of ks back-diagonals.*

$(0,0)$	$\{1,9\}$	$\{2,8\}$	$\{3,7\}$	$\{4,6\}$	$(5,5)$	$\{1,0\}$	$\{2,9\}$	$\{3,8\}$	$\{4,7\}$	$\{5,6\}$
$(1,1)$	$\{2,0\}$	$\{3,9\}$	$\{4,8\}$	$\{5,7\}$	$(6,6)$	$\{2,1\}$	$\{3,0\}$	$\{4,9\}$	$\{5,8\}$	$\{6,7\}$
$(2,2)$	$\{3,1\}$	$\{4,0\}$	$\{5,9\}$	$\{6,8\}$	$(7,7)$	$\{3,2\}$	$\{4,1\}$	$\{5,0\}$	$\{6,9\}$	$\{7,8\}$
$(3,3)$	$\{4,2\}$	$\{5,1\}$	$\{6,0\}$	$\{7,9\}$	$(8,8)$	$\{4,3\}$	$\{5,2\}$	$\{6,1\}$	$\{7,0\}$	$\{8,9\}$
$(4,4)$	$\{5,3\}$	$\{6,2\}$	$\{7,1\}$	$\{8,0\}$	$(9,9)$	$\{5,4\}$	$\{6,3\}$	$\{7,2\}$	$\{8,1\}$	$\{9,0\}$

The sequence $\mathscr{T}$ *is obtained by scanning the rows from left to right starting with the first row. As in the case of* s *odd, placing an element in cell* (i,j) *means that it is also placed in the cell* (j,i). *An element placed in the cells* $(0,0), \{1,9\}$ *is placed three times. On the other hand, an element placed in the cells* $\{2,8\}$ *and* $\{3,7\}$ *is placed four times.*

The following two observations are critical to this first phase allocation.

LEMMA 32. *If $0 < b < s$, then any b consecutive cells of the sequence $\mathscr{T}$ span b distinct rows and b distinct columns (where it must be understood that these b cells must always include cell (i,j) whenever its symmetric mate, (j,i), is included).*

LEMMA 33. *If $s < b < 2s$, then any b consecutive cells of the sequence $\mathscr{T}$ include at least one cell, and at most two cells, of each row and column (where it must be understood that these b cells must always include cell (i,j) whenever its symmetric mate, (j,i), is included).*

We say an element is properly placed in sequence $\mathscr{T}$ defined above if (a) it occupies a subsequence of consecutive cells and, (b) whenever it occupies cell (i,j), $i \neq j$, then it also occupies cell (j,i).

We now describe the first phase allocation when s is even.

CONSTRUCTION 34 (First Phase Allocation: s even). *The idea behind the construction is based on equations (11) to (14). In each of these cases, elements have to be placed $a, a+1, s+a$ times or $a, a+1, s+a+1$ times. Let $f_1 = a$, $f_2 = a+1$ and $f_3 = s+a$ or $s+a+1$ as the case may be. For $1 \leq i \leq 3$, let n_i be the number of elements which have to be placed with frequency f_i.*

1. *Let $\mathscr{T}$ be the sequence as defined in equation (16).*
2. *While $(n_1 > 0)$ or $(n_2 > 0)$ or $(n_3 > 0)$ do*
   - *If $(n_1 > 0)$ and an element can be placed with frequency f_1 in the next f_1 cells of $\mathscr{T}$, then*
     - *Place an element with frequency f_1.*
     - $n_1 = n_1 - 1$.
   - *else if $(n_2 > 0)$ and an element can be placed with frequency f_2 in the next f_2 cells of $\mathscr{T}$, then*
     - *Place an element with frequency f_2.*
     - $n_2 = n_2 - 1$.

   else if $(n_3 > 0)$, then
     - *Place an element with frequency f_3.*
     - $n_3 = n_3 - 1$.
3. *End do*

LEMMA 35 (Correctness of Construction 34). *Let v, s, k be integers such that s is even and $\max\{s+1, 2k\} \leq v < sk$. Construction 34 ensures that*

1. *Each short element has frequency 0 or 1 in each row and column.*

2. *Each long element has frequency 1 or 2 in each row and column.*

3. *Exactly n_i elements are placed with frequency f_i for $1 \leq i \leq 3$.*

Proof. The first two assertions follow from Lemmas 32 and 33. The last assertion is equivalent to the fact that at each iteration of the while loop exactly one element is properly placed. Note that each iteration places at most one element. So it is sufficient to show that at least one element can be placed at each iteration. We now proceed to do this.

As long as both $n_1, n_2 > 0$, elements will be placed with frequencies f_1 and f_2. At some stage, one of n_1 or n_2 will become 0 and the other will remain positive. Denote by n' the remaining positive value and let f' be the corresponding frequency. From this point onwards, elements will be placed with frequencies f' and f_3 until one of n' or n_3 becomes 0. Denote by n'' the remaining positive value and let f'' be the corresponding frequency. From this point onwards, n'' elements have to be placed with frequency f''.

Thus it is sufficient to show the following three claims.

Claim 1. If $n_1, n_2 > 0$, then either an element of frequency f_1 or an element of frequency f_2 can be properly placed.

Claim 2. If $n', n_3 > 0$, then either an element of frequency f' or an element of frequency f_3 can be properly placed.

Claim 3. If $n'' > 0$, then an element of frequency f'' can be properly placed.

Claim 1 is easy to verify. The proofs of Claims 2 and 3 are more complicated. These two claims have to be proved individually for Construction Strategies 3 to 6 in Section 4.2. We provide a detailed proof for Construction Strategy 3. The proofs for the other Construction Strategies are similar.

Proof of Claim 2 for Construction Strategy 3. In this case, a is odd, $v - r \geq p$: The corresponding equation is

$$(v - r)s = r(a + 1) + (v - r - p)(s + a) + p(a).$$

There are $(v - r)$ back-diagonals to be filled up by r elements of frequency $(a + 1), (v - r - p)$ elements of frequency $(s + a)$ and p elements of frequency a. Thus in this case $f_3 = s + a$. Note that $(v - r)$ is even, whenever s is even. A properly placed element of frequency a or $s + a$ occupies exactly one diagonal cell and a properly placed element of frequency $a + 1$ does not occupy any diagonal cell.

The value of f' is either a or $a + 1$ and hence we have two cases.

Subcase 1. $f' = a$: If the next a cells of the sequence $\mathcal{T}$ contain a main diagonal cell (j,j), then it is possible to properly place an element of frequency a; otherwise, it is possible to properly place an element of frequency $s+a$.

Subcase 2. $f' = a+1$: All p elements of frequency a have been properly placed and $(v-r-p-n_3)$ elements of frequency $s+a$ have been placed. Each of these $(v-r-n_3)$ elements with frequencies a and $s+a$ occur in one main diagonal cell (j,j).

This means that in the sequence $\mathcal{T}$, $(v-r-n_3-1)$ back-diagonals have been completely filled, one is partially full, having $(s-i)$ full cells and i empty ones, and n_3 back-diagonals are yet to be filled by Construction 34.

Therefore, n' elements of frequency $a+1$ and n_3 elements of frequency $s+a$ will exactly fill the remaining $i+n_3s$ cells; that is $i+n_3s = n'(a+1)+n_3(s+a)$ and so $i = (n'+n_3)a+n' \geq 2a+1 > a+1$, since $n', n_3 > 0$. Since $i > a+1$, one can always properly place an element of frequency $a+1$. ■

Proof of Claim 3 for Construction Strategy 3. There are three subcases.

Subcase 1. $f'' = a$: Since the previous elements have been properly allocated, the number of diagonal cells that must be left must also be n''. Since there are a total of $(v-r)$ back-diagonals and $(v-r)$ is even, the number of complete back-diagonals left must also be n'', to provide the n'' diagonal cells required by the n'' elements of frequency a each. Additionally there could be i cells remaining in a partially filled back-diagonal. Further, if n'' is odd, then i must be positive. Thus the total number of cells remaining is $sn''+i$ and the total number of cells required by the n'' elements is $n''a$. Thus we must have $sn''+i = n''a$ which is not possible since $a < s$. Thus this case cannot occur.

Subcase 2. $f'' = a+1$: In this case, all p elements of frequency a and all $(v-r-p)$ elements of frequency $s+a$ have been properly placed. As well, $n_2 - n''$ elements of frequency $a+1$ have been properly placed. Each of the elements of frequencies $a, s+a$ occur in one cell (j,j) of the main diagonal. Hence, the first $(v-r-1)$ back-diagonals of sequence $\mathcal{T}$ are filled and back-diagonal $v-r$ contains $s-i$ filled cells and i empty cells. Since $(v-r)s = r(a+1)+(v-r-p)(s+a)+p(a)$, we must have $i = n''(a+1)$. None of the n'' elements of frequency $a+1$ may occupy a cell (j,j) on the main diagonal. Since $v-r$ is even, back-diagonal $(v-r)$ has no main diagonal cell (j,j). Therefore, the n'' elements of frequency $a+1$ can all be properly placed.

Subcase 3. $f'' = s+a$: Each of these elements require exactly one diagonal cell. So there must be n'' complete back-diagonals left and i cells in a partially filled back-diagonal. Note that any $s+a$ cells always contain a diagonal cell. Thus the only way an element with frequency $(s+a)$ cannot be placed is if the next $(s+a)$ elements contain two diagonal cells. But this can happen only if $i < a$. We show that this is not possible. The total number of cells left is $i+n''s$ and this must be equal to $n''(s+a)$. This gives $i = n''a$. Since n'' is at least 1, we have $i \geq a$.

This completes the proof of Claim 3 for Construction Strategy 3. As mentioned before, the proofs of Claims 2 and 3 for Construction Strategies 4, 5 and 6 are similar to this proof. ■

4.4. Second Phase Allocation

Having filled an integral number of back-diagonals, we now want to complete the $SBS_k(s, v)$ by placing each of the v elements in m or $m-1$ (as the case may be) complete back-diagonals. This placement cannot be done arbitrarily as each cell must contain k distinct elements.

We formulate the constraint in the placement of elements to back-diagonals in the second phase in terms of flow in a network.

CONSTRUCTION 36 (Network Construction). *We describe the construction of a network N as follows. First we construct a bipartite graph.*

1. *Let $\{b_1, b_2, \ldots, b_s\}$ be a set of vertices, one for each back-diagonal.*
2. *Let $\{e_1, e_2, \ldots, e_v\}$ be a set of vertices, one for each element.*
3. *Vertices b_i and e_j are joined by an arc (b_i, e_j) directed from b_i to e_j of capacity 1 if and only if element e_j has not been allocated to any cell of back-diagonal b_i in the first phase, i.e., by Constructions 30 or 34.*

To this bipartite graph we add a source and a sink node and additional arcs.

1. *Introduce a source node S. For each back-diagonal b_i, introduce an arc (S, b_i) of capacity $k-\ell$, where each cell of back-diagonal b_i contains ℓ elements, i.e., the back-diagonal b_i has been filled up ℓ times in the first phase.*
2. *Introduce a sink node T. For each element e_j, introduce an arc (e_j, T)*
   - *of capacity m if element e_j is a short element of frequency a or $a+1$, or*
   - *of capacity $m-1$ if element e_j is a long element of frequency $s+a$ or $s+a+1$.*

This gives us the network N. For any arc λ in N, we denote the capacity of λ by $cap(\lambda)$.

We provide an example to illustrate the bipartite graph and the network N.

EXAMPLE. *Let $k=5, s=9, v=10$. This gives $n=40, p=5, m=4, a=4, r=5$. In the first phase 10 complete back-diagonals are filled up. Five elements have been placed 14 times each and five other elements have each been placed four times each. We now have to complete the construction.*

The adjacency matrix of the bipartite graph is the following. The direction of the arcs is from the back-diagonals to the elements.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
b_1	0	0	0	1	1	1	1	1	1	0
b_2	1	1	0	0	0	1	1	1	1	1
b_3	1	1	1	1	0	0	1	1	1	1
b_4	1	1	1	1	1	0	0	1	1	1
b_5	1	1	1	1	1	1	0	1	1	1
b_6	1	1	1	1	1	1	0	0	1	1
b_7	1	1	1	1	1	1	1	0	0	1
b_8	1	1	1	1	1	1	1	1	0	1
b_9	1	1	1	1	1	1	1	1	0	0

Elements e_1 to e_5 must each be assigned to five complete back-diagonals and elements e_6 to e_{10} must each be assigned to four complete back-diagonals. Thus for $1 \leq j \leq 5$, the capacities of the arcs (e_j, T) are all five and for $6 \leq j \leq 10$, the capacities of the arcs (e_j, T) are all four.

Each cell of back-diagonal b_1 has to be filled up three more times whereas each cell of back-diagonals b_2 to b_9 has to be filled up four more times. Hence the capacity of the arc(S, b_1) is three and for $2 \leq i \leq 9$, the capacities of the arcs (S, b_i) are all four.

LEMMA 37. *In the network N of Construction 36*

$$\sum_{i=1}^{s} \text{cap}(S, b_s) = \sum_{j=1}^{v} \text{cap}(e_j, T).$$

Proof. Let the total number of back-diagonals filled in the first phase be β. There are a total of ks back-diagonals. Thus,

$$\sum_{i=1}^{i=s} \text{cap}(S, b_s) = ks - \beta.$$

In an $SBS_k(s, v)$ each cell is filled k times. So there are a total of ks^2 cells. The total number of cells filled in the first phase is βs. Thus the number of cells filled in the second phase is $ks^2 - \beta s$. In the second phase, the element e_j is to be placed in $s \times cap(e_j, T)$ cells. Thus the total number of cells filled in the second phase is $s \sum_{j=1}^{j=v} cap(e_j, T)$. Equating this to $ks^2 - \beta s$ we get $\sum_{j=1}^{j=v} cap(e_j, T) = ks - \beta$. Thus the result holds. ■

LEMMA 38. *For any back-diagonal b in the network N of Construction 36,*

$$\mathrm{cap}(b) = \left\lfloor \frac{v(m-1)+\sigma}{s} \right\rfloor \quad or \quad \left\lceil \frac{v(m-1)+\sigma}{s} \right\rceil,$$

where $\sigma \leq v$ is the number of short elements placed in the first phase.

Proof. If e is a short element, then $cap(e, T) = m$ and if e is a long element, then $cap(e, T) = m - 1$. Thus using Lemma 37, we get

$$\sum_{j=1}^{v} \mathrm{cap}(e_j, T) = \sigma m + (v - \sigma)(m - 1) = v(m-1) + \sigma. \tag{17}$$

Let $l = \min_{1 \leq i \leq s} \{cap(S, b_i)\}$ and $L = \max_{1 \leq i \leq s} \{cap(S, b_i)\}$. Then any back-diagonal is filled upat least $k - L$ times and at most $k - l$ times.

The first phase allocation uses the enumeration $\mathcal{S}$ or $\mathcal{T}$ of the back-diagonals given by equation (15) or (16) respectively. In both $\mathcal{S}$ and $\mathcal{T}$, all back-diagonals occur exactly once between two successive occurrences of any particular back-diagonal. The first phase allocation ensures that $(k - l) - (k - L) \leq 1$. This gives $(L - l) \leq 1$. This combined with Lemma 37 and equation (17) gives the result. ■

CONSTRUCTION 39 (Algorithm Second Phase)

1. *Construct the network N as in Construction 36.*
2. *Find a maximum capacity flow f in N.*
3. *For each back-diagonal vertex b_i and each element vertex e_j, if $f(b_i, e_j) = 1$, then place the element e_j in all the cells of the back-diagonal b_i.*

We now turn to the correctness of the algorithm. The only way the algorithm can fail is if the maximum flow does not saturate the requirement of all the back-diagonals. We analyze this possibility.

Observe that by Lemma 37, the capacity of the cut $(\{S\}, \{\overline{S}\})$ is the same as the capacity of the cut $(\{\overline{T}\}, \{T\})$. If these are minimum cuts, then a maximum flow in N is equal to the capacity of these minimum cuts,and any such maximum flow provides us with the required assignment of elements to back-diagonals. We record this in the following result.

THEOREM 40. *The second phase allocation can be properly completed if and only if $(\{S\}, \{\overline{S}\})$ is a minimum cut in the network N of Construction 36.*

Thus, to complete the correctness proof for the second phase allocation strategy, we must show that $(\{S\}, \{\overline{S}\})$ and $(\{\overline{T}\}, \{T\})$ are cuts of minimum capacity in N. Note that the capacity on an arc (S, b_i) is the number of elements that must be added to each cell of back-diagonal b_i in order to fill it. We call the capacity of arc (S, b_i) the requirement of back-diagonal b_i. The maximum flow provides the desired assignment if the requirement on all the back-diagonals (and hence the elements) can be satisfied. The next result simplifies this to show that it is sufficient to satisfy the requirement on any $m+1$ back-diagonals.

THEOREM 41. *Consider the Network N of Construction 36. Then the following are equivalent.*

1. $(\{S\}, \{\overline{S}\})$ *is a cut of minimum capacity.*
2. *There is a flow f such that $f(S, b) = cap(S, b)$ for each back-diagonal b.*
3. *For every set B_0 of $(m+1)$ back-diagonals, there is a flow f_{B_0} such that $f_{B_0}(S, b) = cap(S, b)$, for each $b \in B_0$.*

Proof.

$(1) \Rightarrow (2)$ follows from the Min-cut, Max-flow Theorem.

$(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (2)$: We are given that the requirement on every set of $(m+1)$ back-diagonals can be satisfied by some flow in network N. We show that under this assumption the capacity of any cut cannot be less than the sum of the requirements of all the back-diagonals. For this we have to consider the different possible cuts and argue about their capacities.

Let B be the set of back-diagonals and E be the set of elements. Let X and Y be the sum of the requirements of all the back-diagonals and and all the elements, respectively. We have $X = Y$. There are different cases to consider depending on the nature of the cut $(C, \overline{C})$.

Case 1. $C = \{S\}$. In this case clearly the cut capacity is equal to X.

Case 2. $C = \{S\} \cup B \cup E$. In this case the cut capacity is $Y = X$.

Case 3. $C = \{S\} \cup B$. Thus the cut capacity is equal to the total number of arcs connecting the back-diagonals to the elements. The in-degree of each element is at least $s-3$ and the requirement is at most $m = \lfloor ks/v \rfloor$. Since $k \leq v/2$, we have $m \leq \lfloor s/2 \rfloor$. Since we have completely resolved the existence question for $SBS_k(s, v)$ designs when $s \leq 4$, we need only consider $s > 4$. In this case, $s - 3 \geq \lfloor s/2 \rfloor$; that is, the in-degree of each element is greater than its requirement. Hence the sum of all the in-degrees of the elements is greater than $Y = X$.

Case 4. $C = \{S\} \cup B \cup E_1$ where E_1 is a proper subset of E. In this case, the capacity of the cut is equal to the sum of the requirements of the elements in E_1 plus the sum of the in-degrees of the elements in $E - E_1$. As in Case 3, in this case also it can be argued that capacity is greater than X.

Case 5. $C = \{S\} \cup B_1$ where B_1 is a proper subset of B. For the first time we use the hypothesis that the requirement of any $(m+1)$ back-diagonals can be satisfied. This implies that the requirement of any one back-diagonal can also be satisfied. Thus the out-degree of any back-diagonal is greater than or equal to its requirement. The cut capacity is the sum of the requirements of the back-diagonals in $B - B_1$ and the sum of the out-degrees of the back-diagonals in B_1. Since the out-degree of any back-diagonal is greater than or equal to its requirement, the cut capacity is greater than or equal to the sum of the requirements of all the back-diagonals.

Case 6. $C = \{S\} \cup B_1 \cup E_1$ where B_1 is a proper subset of B and E_1 is a proper subset of E. The capacity of cut $(C, \overline{C})$ is equal to the sum of the capacity of cut $(E_1, \{T\})$ plus the capacity of cut $(\{S\}, B_2)$,where $B_2 = B \setminus B_1$, plus the capacity of cut (B_1, E_2) where $E_2 = E \setminus E_1$.

We first show that if the size of B_1 is more than $m+1$, then the cut capacity is $\geq X$, without using the hypothesis. For each of the elements the requirements are either m or $m-1$. Further, if the requirement is m, then the in-degree can be $s-1$ or $s-2$ and if the requirement is $m-1$, then the in-degree can be $s-2$ or $s-3$. The size of the set B_1 greater than $m+1$ implies that the size of the set B_2 is less than $s-m-1$, i.e., less than or equal to $s-m-2$. Each of the back-diagonals in B_2 can be connected at most once to each of the elements in E_2. Hence for each element e in E_2, the back-diagonals in B_2 can account for at most $s-m-2$ of the arcs coming into e. If the requirement of e is m, then its degree is at least $s-2$ out of which at most $s-m-2$ is contributed by the back-diagonals in B_2. Thus the back-diagonals in B_1 must contribute at least $s-2-(s-m-2) = m$ to its in degree. In a similar fashion, it is possible to argue that if the requirement of e is $m-1$, then also at least $m-1$ of its in-degree must be contributed by the back-diagonals in B_1. Thus for each element $e \in E_2$, the number of arcs coming into it from the back-diagonals in B_1 is at least as great as its requirement. Hence the cut capacity is $\geq Y = X$.

We now turn to the case when the size of B_1 is less than or equal to $m+1$. In this case we need to use the hypothesis. Consider a flow f that meets all the requirements on back-diagonals of B_1. Clearly,

$$\begin{aligned}
\text{cap}(S, B_1) &= f(S, B_1) \\
&\leq f(B_1, E_2) + f(E_1, T) \\
&\leq \text{cap}(B_1, E_2) + \text{cap}(E_1, T).
\end{aligned}$$

Therefore,

$$\begin{aligned} X = cap(S, B) &= cap(S, B_1) + cap(S, B_2) \\ &\leq cap(B_1, E_2) + cap(E_1, T) + cap(S, B_2) \\ &= cap(C, \overline{C}), \end{aligned}$$

as required. ■

In the next two subsections we use Theorem 41 to show the correctness of the construction strategy for the cases $k = 2$ and $k = 3$. We conjecture that in the general case $(\{S\}, \{\overline{S}\})$ is a minimum cut in the network N.

4.5. *Case* $k = 2$, $5 \leq s < v < 2s$

Here we prove that in the case $k = 2$, the requirement of all the back-diagonal in the graph G can be satisfied. For this we use Theorem 41. Since $k = 2$ we have $m = 1$ and hence by Theorem 41, it is sufficient to show that the requirement on any two back-diagonals can be satisfied. First we state some preliminary results.

LEMMA 42. *Let $s > 0$ be an even integer and $k \leq 3$. In the First Phase Allocation of Construction Strategies 3, 4, 5 and 6, there are at least four short elements.*

Proof. We prove the result only for $k = 2$ and Construction Strategy 3. The proofs for the other Construction Strategies and the case $k = 3$ are similar.

In Construction Strategy 3 we have s to be even, a to be odd and $(v - r) \geq p$. The corresponding equation is

$$(v - r)s = r(a + 1) + (v - r - p)(s + a) + p(a).$$

It is easy to check that in this case $(v - r)$ must be even. The number of short elements is $p + r = 2s - v + r = 2s - (v - r)$. Since $(v - r)$ is even it follows that the number of short elements must be even. Since $p > 0$, there must be at least one short element. We show that there cannot be exactly two short elements. Suppose that there are two short elements. Then $2 = r + p = 2s - v + r$ which implies that $v - r = 2s - 2$. Since $2s - 1 \geq v$, this gives $r \leq 1$. Thus $r = 0, 1$ and correspondingly $p = 2, 1$. If $p = 1$, then $v = 2s - 1$ which gives $n = \lfloor ks^2/v \rfloor = \lfloor 2s^2/(2s - 1) \rfloor = s$ and $r = s$. Since $s > 1$, this contradicts $r = 1$. On the other hand, if $p = 2$, then $v = 2s - 2$ which gives $n = s + 1$ and $r = 2$. This contradicts $r = 0$. Thus we get a contradiction and hence there cannot be exactly two short elements. ■

The next result holds for all k.

LEMMA 43. *Let $0 < k$, $1 < s < v < ks$ and "Short" be the set of short elements placed in the first phase of Construction Strategies 1 to 6. Also let N be the network of Construction 36. Then the following holds.*

1. *In Construction Strategies 1 to 5, if N has a back-diagonal b such that there are no arcs (b, e) for any $e \in Short$, then b must be the first back-diagonal of either the sequence $\mathcal{S}$ or the sequence $\mathcal{T}$.*
2. *In Construction Strategy 6, if N has a back-diagonal b such that there are no arcs (b, e) for any $e \in Short$, then b must be either the first back-diagonal or the 2p-th back-diagonal of the sequence $\mathcal{T}$.*

Proof. Item 1 has to be proved separately for s odd (Construction Strategies 1 and 2) and for s even (Construction Strategies 3, 4 and 5). We provide only the proof for s even.

We have $p, v - r > 0$. It is easy to check that this implies that in Construction Strategies 3, 4 and 5, there is at least one short odd element. By Construction 34, it follows that since there is at least one short element and the sequence $\mathcal{T}$ starts with a diagonal cell, the first element placed must be a short odd element. Hence the first back-diagonal completely contains at least one short element. Denote the first back-diagonal by b_1 and the first short element placed by e_1. Then the arc (b_1, e_1) is not present in the network N. Further, for any back-diagonal $b' \neq b_1$, the arc (b', e_1) is present in N. Hence if b is such that there are no arcs (b, e) for all $e \in Short$, then we must have $b = b_1$, i.e., b must be the first back-diagonal.

Now we turn to the proof of Item 2. In Construction Strategy 6, if there is at least one short odd element, then the proof is similar to Item 2. There is no short odd element only if $r = 0$. In this case, the number of back-diagonals filled in the first phase is $2p$. Since $2p$ is even, the last back-diagonal filled in the first phase does not have any diagonal cell. Hence the last element placed in the first phase cannot be a long element and it must be a short even element. Thus the $2p$-th back-diagonal completely contains at least one short element. Now it is possible to argue as before that b is the $2p$-th back-diagonal. ■

COROLLARY 44. *If a back-diagonal b has fewer than k elements in each cell when Phase 1 is complete, then there is a short element that is available for b.*

We are now ready to turn to the case $k = 2$.

LEMMA 45. *Let $k = 2$ and s,v be such that $5 \leq s < v < 2s$. Then in the network N of Construction 36, for any set $\{b_1, b_2\}$ of two back-diagonals, there is a flow f such that $f(S, b_i) = cap(S, b_i)$, for $i = 1, 2$.*

Proof. Since $k = 2$, we necessarily have $m = 1$. Thus for any long element e, the capacity $cap(T, e) = 0$. So any flow that meets the capacities of b_1 and b_2 must use only the short elements to do so. We consider two cases.

Case 1. For each back-diagonal b, there is a short element e such that the arc (b, e) is present in the network N of Construction 36.

If the number of short elements is at most 2, then s must be odd, only the first two elements can be short elements and only the last two back-diagonals can have requirement 1. In this case, it is easy to satisfy the requirement of the last two back-diagonals.

So suppose the number of short elements is at least 3. Let b_1 and b_2 be any two back-diagonals and for $i = 1, 2$, let A_i be the set of short elements which do not occur in b_i. Since b_1 and b_2 can have at most one short element in common and there are at least three short elements, it follows that $|A_1 \cup A_2| \geq 2$. Hence it is possible to pick a system of distinct representatives e_1, e_2 for A_1 and A_2 respectively. Placing e_1 in b_1 and e_2 in b_2 satisfies the requirements of b_1 and b_2.

Case 2. There is a back-diagonal b such that there are no arcs (b, e) for each short element e, in the network N of Construction 36.

If $a = 0$, then in the first phase, short elements corresponding to frequency a have not been placed in any back-diagonal. Thus for any back-diagonal b' the arc (b', e) must be present for each short element e corresponding to $a = 0$. This contradicts the hypothesis of the case. Thus we must have $a > 0$.

In this case, it is sufficient to show that b must have been filled twice in the first phase, i.e., the capacity of the arc (S, b) in the network N of Construction 36 is 0. We prove this by contradiction. Suppose that b has been filled only once in the first phase. Since $a > 0$, the number of short elements is at most $\lceil (s+1)/2 \rceil$.

By Lemmas 42 and 43, in Construction Strategies 1 to 5, b must be the first back-diagonal. If the first back-diagonal has been filled up only once in the first phase, then the sum of requirements of all the back-diagonals is at least s. Using Lemma 37, this means that there should be at least s short elements. This contradicts the fact that there are at most $\lceil (s+1)/2 \rceil$ short elements.

In Construction Strategy 6, if some back-diagonal contains all the elements, then this is either the first back-diagonal or the $2p$-th back-diagonal (Lemma 43). If it is the first back-diagonal, then the argument is similar to the one given above. So suppose the $2p$-th back-diagonal contains all the short elements. Note that in Construction Strategy 6, the $2p$-th back-diagonal is the last back-diagonal that is filled in the first phase. If $2p > s$, then this back-diagonal is completely full. If $2p \leq s$, then the total requirement of the back-diagonals is at least s and hence there must be at least these many short elements. Again this contradicts the fact that the number of short elements is at most $\lceil (s+1)/2 \rceil$. ■

From Lemma 45, and Theorem 41, we get the following result.

THEOREM 46. *If $5 \leq s < v < 2s$, then an $SBS_2(s, v)$ can be constructed.*

4.6. *Case* $k = 3, 5 \leq s < v < 3s$

Here we prove the correctness of Construction 39 for the case $k = 3$. By definition $m = \lfloor ks/v \rfloor = \lfloor 3s/v \rfloor < 3$, since $v > s$. Thus the possible values of m are 1 and 2. We consider these cases separately. Further since $k = 3$, the requirement in any back-diagonal is at most 3.

LEMMA 47. *Let* $5 \leq s < v$, $k = 3$ *and* $m = 1$. *Suppose there is a back-diagonal* b *such that there are no arcs* (b, e) *for all short elements* e *in the network* N *of Construction 36. Then* b *must have been completely filled in the first phase, i.e., capacity of the arc* (S, b) *must be 0.*

Proof. As in the proof of Lemma 45, the hypothesis on b forces $a > 0$. We show that b is completely filled separately for s even and s odd.

If s is odd, then using Lemma 43, b must be the first back-diagonal. In Construction 30, all the short elements are placed before any long element. Hence b can be filled by short elements only once. Thus the total number of short elements can beat most $\lceil s/2 \rceil$. Since there are at most $\lceil s/2 \rceil$ short elements, using Lemma 37 there are at most $\lceil s/2 \rceil$ back-diagonals to fill in the second phase. Therefore the first back-diagonal was filled up three times in Phase 1. This proves the result when s is odd.

We now consider the case s even. There are two subcases, depending on whether there is at least one short odd element or not.

First suppose there is at least one short odd element. Then by Lemma 43, b must be the first back-diagonal. If b is filled up only once, then there can be at most $\lceil (s+1)/2 \rceil$ short elements. On the other hand, the total requirement of all the back-diagonals must be at least $2s$. Using Lemma 37, this shows that there must be at least $2s$ short elements. This is a contradiction.

So suppose b is filled up twice. In Construction 34, as long as there are both short odd and short even elements remaining, these will continue to be placed. Thus after b is filled up once, there will be short odd or short even elements remaining but not both.

Suppose only short odd elements remain after b is filled up for the first time. Since b is the first back-diagonal, it has two diagonal cells. Thus when b is filled up the second time, at most two short elements can be placed. Thus the total number of short elements is at most $\lceil (s+1)/2 \rceil + 2$. Since b is filled up twice, the total requirement of all the back-diagonals is at least s. This again gives a contradiction.

So suppose that only short even elements are left after b is filled up for the first time. We now distinguish two cases. If $a > 1$, then the total number of short elements in b is at most $s/2 + (s-2)/2 = s - 1$, whereas the total requirement of all the back-diagonals is at least s. Using Lemma 37, this again gives a contradiction. So suppose $a = 1$. Then the short frequencies are 1 and 2 and the number of short elements with frequency 1 is at most 2. If possible, let this number be 2. When b is filled up for the first time, the first and last elements placed in b have frequency 1. The second back-diagonal starts with an off-diagonal cell and an element of frequency 2 will be placed in this cell by Construction 34. Thus all short elements have not been placed in b, which contradicts the hypothesis. So the only possibility is that $a = 1$ and the number of elements with frequency a is exactly 1. It is easy to check from equations (11) to (14) that this cannot happen. Thus b must have been filled up three times.

The other case that we need to consider is when there are no short odd elements. This can happen only in Construction Strategy 6 with $r = 0$. In this case there are only short even and long even elements. The last back-diagonal filled in the first phase is the $2p$-th back-diagonal. Since $2p$ is even, this back-diagonal does not have any diagonal cell. Hence the last element placed in the first phase must be a short even element. Since this element is placed in the $2p$-th back-diagonal, by the hypothesis, the $2p$-th back-diagonal must be b. Since $a > 0$ and in Construction Strategy 6, a is even, we must have a to be at least 2. Thus if b is filled up twice, there can be at most s short elements. If $2p < 2s$, then the total requirement on the back-diagonals must be greater than s, which is a contradiction to the maximum number of short elements. If $2p > 2s$, then b must have been filled three times in the first phase. Thus the only possibility is that $2p = 2s$. Then the number of short elements must be exactly s. If $a > 2$, then the number of short elements present in b is less than s. Thus a must be 2. This combined with $r = 0, 2p = 2s$ and equation (14), gives $s < 5$. Since s is even, this means $s \leq 4$. This contradicts the hypothesis of the lemma, where $s > 4$. Thus b must have been filled up three times in the first phase. ■

LEMMA 48. *Let $k = 3, 5 \leq s < v < 3s$ and $m = 1$. Then in the network N of Construction 36, for any set $\{b_1, b_2\}$ of back-diagonals, there is a flow f such that $f(S, b_i) = cap(S, b_i)$.*

Proof. Since $m = 1$, as in Lemma 45, any such flow must use only the short elements to meet the capacities of b_1, b_2. Further, since $k = 3$, the requirement on any one back-diagonal can be at most 3. Thus the requirements on b_1 and b_2 can respectively be the following values.

(1) 0, 0 (2) 0, 1 (3) 1, 1 (4) 1, 2 (5) 2, 2 (6) 2, 3 (7) 3, 3.

Case (1) is trivial. In case (2), we have to show that there is a short element which can be placed in b_2. If there is no such short element, then using Lemma 47, we know that b_2 must be completely full, which contradicts the fact that requirement of b_2 is 1. Thus there must be a short element which does not occur in b_2 and hence can be placed in b_2. In case (7), using Lemma 37, there must be at least six short elements which have not been placed in either b_1 or b_2 (since no element has been placed in b_1 or b_2). So we can put any three short elements in b_1 and three others in b_2. In case (6), we first show that the requirement on b_1 can be satisfied. Since the requirement on b_1 is 2, it has been filled up only once in the first phase and hence can contain a maximum of $\lceil (s+1)/2 \rceil$ short elements. Since the requirements are 2,3, using Lemma 37, the number of short elements must be at least $2s + 1$. Since $s \geq 5$, there must be two short elements which do not occur in b_1 and can be placed in b_1. Then it can be shown that there are three distinct elements available for back-diagonal b_2.

Thus we have to consider the cases (3) to (5).

In case (3), from Lemma 47, neither of b_1 or b_2 can contain all the short elements. Then proceeding as in the case of $k = 2$, we can show that the requirement of both b_1 and b_2 can be satisfied.

In case (4), again using Lemma 47, there must be a short element e which does not occur in b_1. Then e can be placed in b_1 to satisfy the requirement of b_1. Since b_2 has requirement 2, it must have been filled up only once in the first phase and hence can contain at most $\lceil (s+1)/2 \rceil$ short elements. These elements cannot be placed in b_2 and also e cannot be placed in b_2. Thus a total of $1 + \lceil (s+1)/2 \rceil$ cannot be placed in b_2. Since the requirements are 1 and 2, the total requirement of all the back-diagonals must be at least $s+1$. Using Lemma 37, there must be a total of $s+1$ short elements. Out of these $1 + \lceil (s+1)/2 \rceil$ short elements cannot be placed in b_2. Since $s \geq 5$, there must be two other short elements which can be placed in b_2.

In case (5), we first use the argument of case (6) to show that the requirement of b_1 can be satisfied. Then the number of short elements which cannot be placed in b_2 is $2 + \lceil (s+1)/2 \rceil$. Since the requirements are 2,2, the total requirement of all the back-diagonals must be at least $s+2$. Using Lemma 37, the number of short elements must also be $s+2$. Since $s \geq 5$, this means that there are two short elements which do not occur in b_2 and hence can be placed in b_2. ■

It will be convenient to introduce the notion of gap for the sequel. In the sequences $\mathscr{S}$ and $\mathscr{T}$, we refer to the $s-1$ back-diagonals (consisting of $s(s-1)$ cells) between two consecutive occurrences of back-diagonal b as a gap of b. If b is completely filled up t times in the first phase, then $t-1$ gaps of b have also been filled in Phase 1.

We say back-diagonal b covers an element e, if e occupies one or more cells of b; otherwise we say e is available for b. Thus (b, e) is an arc in the Network N of Construction 36, iff e is available for b. Can all cells of a gap of b be filled by elements covered by b? The answer is no, since

$$\begin{aligned} s(s-1) - 2s - 2a &= s^2 - 3s - 2a \\ &\geq s^2 - 3s - 2s + 2, \quad \text{since } a \leq s-1 \\ &= (s-3)(s-2) - 4 \\ &\geq 2, \qquad\qquad \text{since } s \geq 5. \end{aligned}$$

Therefore, there is at least one element available for b corresponding to each gap of b filled in Phase 1.

LEMMA 49. *Let $k = 3, 5 \leq s < v < 3s$, and $m = 2$. Then in the network N of Construction 36, for every set $\{b_1, b_2, b_3\}$ of back-diagonals, there is a flow f such that $f(S, b_i) = cap(S, b_i)$, for $i = 1, 2, 3$.*

Proof. Without loss of generality we assume that the first occurrence of b_1 is before the first occurrence of b_2 and the first occurrence of b_2 is before the first occurrence of

b_3 in either the sequence $\mathscr{S}$ (equation (15)) or the sequence $\mathscr{T}$ (equation (16)). We have to consider the following possible requirements on the back-diagonals b_1, b_2 and b_3 respectively.

(1) $0,0,0$ (2) $0,0,1$ (3) $0,1,1$ (4) $1,1,1$ (5) $1,1,2$
(6) $1,2,2$ (7) $2,2,2$
(8) $2,2,3$ (9) $2,3,3$ (10)$3,3,3$

Since $m = 2$, in the network N of Construction 36, for each element vertex e, the arc of the form (e, T) has capacity either 1 or 2. From this, using Lemma 37, we get that the requirement of any back-diagonal b (i.e., the capacity of the arc (S, b)) in the second phase must be at least 1. Thus the cases (1) to (3) above cannot arise. We now separately consider the cases (4) to (10). For $i = 1, 2, 3$, let A_i be the set of elements available for back-diagonal b_i in the first phase. Then in the second phase any element from A_i can be placed in b_i. Further, let σ be the total number of short elements. Clearly, $\sigma \leq v$.Then each of these σ short elements have to be assigned to two complete back-diagonals in the second phase.

Case (4) Requirements are 1,1,1. Let e_1 be the first element which is placed in b_1 in the first phase and let e_3 be the last element which is placed in b_3 in the first phase. Then e_1 does not occur in b_3 and e_3 does not occur in b_1. Placing e_1 in b_3 and e_3 in b_1 satisfies the requirements of b_1 and b_3. The back-diagonal b_2 is filled up twice in the first phase and hence one gap of b_2 is also filled. Since $s \geq 5$, there must be an element e_2 in the gap of b_2 which is available for b_2. Also e_2 cannot be the same as either e_1 or e_3. Hence e_2 can be placed in b_2 to satisfy the requirement of b_2.

Case (5) Requirements are 1,1,2. We consider two cases.

b_3 covers at least three elements. Then there is at least one short element x in b_3. This element x can be assigned to two back-diagonals. Hence x is placed in both b_1 and b_2 to satisfy the requirements of both b_1 and b_2. The maximum number of elements occurring in b_3 is $\lceil (s+1)/2 \rceil$. Since the number of elements is at least $s+1$, the number of elements which can be placed in b_3 is at least $\lfloor (s+1)/2 \rfloor \geq 3$, for $s \geq 5$. Hence there must be two elements which can be placed in b_3 to satisfy its requirement.

b_3 covers at most two elements. For $i = 1, 2$, let e_i be an element available for b_i corresponding to the first gap of b_i.

If $e_1 \neq e_2$, then place e_1 in b_1 and e_2 in b_2. This satisfies the requirements of b_1 and b_2. Since b_3 covers at most two elements, $|A_3| \geq v - 2$. Therefore, $|A_3 \setminus \{e_1, e_2\}| \geq v - 4 \geq 2$, since $v \geq 6$. Hence there are at least two elements available for b_3.

If $e_1 = e_2$, then there is only one element in the first gap of b_1 and b_2 which is available for both b_1 and b_2. Let e_3 be the element immediately succeeding $e_1 = e_2$ in Phase 1. Then e_3 is not available for either b_1 or b_2. Hence the element e_4

succeeding e_3 in the Phase 1 must be available for b_1. Therefore we place e_1 in b_2 and e_4 in b_1 to meet the requirements of b_1 and b_2. The requirement of b_3 can be satisfied as above.

Case (6) Requirements are 1,2,2. Let the first element placed in b_1 in the first phase be e_1 and the last element placed in b_1 in the first phase be e_2. Then e_1 (respectively e_2) does not occur in b_3 (respectively b_2). Thus we can place e_1 (respectively e_2) in b_3 (respectively b_2). Since b_1 is filled up twice, it has one gap and since $s \geq 5$, there must be an element e occurring in the gap of b_1 which is not present in b_1. This can be placed in b_1. Thus the requirement of b_1 is satisfied and the requirements of b_2 and b_3 are now 1, 1. Since b_2, b_3 have been filled only once in the first phase, we have $|A_2|, |A_3| \geq s+1-\lceil (s+1)/2 \rceil = \lfloor (s+1)/2 \rfloor$. Also, $|A_2 \setminus \{e, e_2\}|, |A_3 \setminus \{e, e_1\}| \geq \lfloor (s+1)/2 \rfloor - 2 \geq 1$, for $s \geq 5$. Further, $|A_2 \cup A_3| \geq v-1$ and $|A_2 \cup A_3 \setminus \{e, e_1, e_2\}| \geq v-4 \geq 2$ for $v > s \geq 5$. Thus we can find a system of distinct representatives f_2, f_3 respectively for the sets $A_2 \setminus \{e, e_2\}$ and $A_3 \setminus \{e, e_1\}$. Then we can put f_2 in b_2 and f_3 in b_3 to satisfy the requirements of b_2 and b_3.

Case (7) Requirements are 2,2,2. This has three subcases.

b_2 has at least three elements. Then there must be a short element e in b_2. This element is placed in both b_1 and b_3. Let the first element placed in b_2 in the first phase be e_1 and the last element placed in b_2 in the first phase be e_2. Then e_1 does not occur in b_3 and e_2 does not occur in b_1. Thus placing e_1 (respectively e_2) in b_3 (respectively b_1) satisfies the requirements of both b_1 and b_3. Further, since b_2 has been filled only once in the first phase, by the now standard cardinality argument on A_2, there must be two elements in A_2 and hence these can be placed in b_2.

b_2 has exactly two elements. Let the first element placed in b_2 in the first phase be e_1 and the second element placed in b_2 in the first phase be e_2. Then e_1 (respectively e_2) does not occur in b_3 (respectively b_1). So e_1 can be placed in b_3 and e_2 can be placed in b_1. Now $|A_1| \geq v - \lceil (s+1)/2 \rceil \geq s+1-\lceil (s+1)/2 \rceil = \lfloor (s+1)/2 \rfloor \geq 3$. Similarly, $|A_3| \geq 3$. In addition, $|A_1 \cup A_3| = v$. Therefore, $|A_1 \setminus \{e_1, e_2\}| \geq 1$, $|A_3 \setminus \{e_1, e_2\}| \geq 1$ and $|A_1 \cup A_2 \setminus \{e_1, e_2\}| = v-2$. Therefore there are distinct elements f_1, f_3 such that $f_1 \in A_1 \setminus \{e_1, e_2\}$ and $f_3 \in A_3 \setminus \{e_1, e_2\}$. The elements f_1, f_3 can now be used to meet the requirements of the back-diagonals b_1 and b_3 respectively. Now by hypothesis, $|A_2| = v-2$. Therefore, $|A_2 \setminus \{e_1, e_2, f_1, f_3\}| = v-4 \geq 2$, since $v \geq 6$. Hence there are two elements in A_2 distinct from e_1, e_2, f_1, f_3, which can be used to satisfy the requirement of b_2.

b_2 contains exactly one element. Let this element be e. This is necessarily a long element. In this case, there must be an element e_1 (respectively e_3) in b_1 (respectively b_3) which does not occur in b_3 (respectively b_1). Thus e_1 (respectively e_3) can be placed in b_3 (respectively b_1). It is now possible to argue that the sets $A_1 \setminus \{e_3\}$ and $A_3 \setminus \{e_1\}$ has a set of distinct representatives f_1, f_3 respectively. Then f_1 can be placed in b_1 and f_3 can be placed in b_3. Now $|A_2 \setminus \{e, e_1, e_3, f_1, f_3\}| \geq v-5$. If $v \geq 7$, there

must be two elements in $A_2 \setminus \{e, e_1, e_3, f_1, f_3\}$ and hence these can be placed in b_2. So suppose $v = 6$. Then there is at least one element f in $A_2 \setminus \{e, e_1, e_3, f_1, f_3\}$ which can be placed in b_2. Thus we now have to show that there is another element which can be placed in b_2. Since the requirements of the back-diagonals are 2,2,2, we have $v + \sigma \geq s + 3 \geq 8$, for $s \geq 5$. Since $v = 6$, we have $\sigma \geq 2$. Thus there are at least two short elements. The element e is long element. So one of the elements e_1, e_3, f_1, f_3 must be a short element. Let us call this element f'. Then f' has been placed in one of b_1 or b_3 but not both. Since f' is a short element and does not occur in b_2, it can be placed in b_2 to satisfy the requirement of b_2.

Case (8) Requirements are 2,2,3. In this case, there is either an element e_1 in b_1 which does not occur in b_2 or an element e_2 in b_2 which does not occur in b_1. We assume the first case, the other case being similar. Thus e_1 can be placed in b_2. Then we have, $|A_1| \geq \lfloor (s+1)/2 \rfloor \geq 3, |A_2 \setminus \{e_2\}| \geq \lfloor (s+1)/2 \rfloor - 1 \geq 2$ and $|A_1 \cup A_2 \setminus \{e_2\}| \geq v - 2 \geq 4$, for $s \geq 5$. Thus we can choose three distinct elements f_1, f_2, f_3 such that $f_1, f_2 \in A_1$ and $f_3 \in A_2$. Elements f_1, f_2 are placed in b_1 and f_3 is placed in b_2. This satisfies the requirements of b_1 and b_2. If $v \geq 7$, then there are three other elements which can be placed in b_3. So suppose that $v = 6$. In this case, there are two elements distinct from e_1, f_1, f_2, f_3 which can be placed in b_3. Since the requirements of b_1, b_2, b_3 are 2,2,3 we have $v + \sigma \geq 2s + 1 \geq 11$ for $s \geq 5$. Since $v = 6$, we have $\sigma \geq 5$, i.e., there are at least five short elements. This means that at least one of the elements e_1, f_1, f_2, f_3 must be a short element (say f'). Since none of e_1, f_1, f_2, f_3 occur in b_3, we can place f' in b_3. This satisfies the requirement of b_3.

Case (9) Requirements are 2,3,3. Cardinality argument on A_1 shows that the requirement of b_1 can be satisfied. Suppose the elements that can be placed in b_1 are e_1, e_2. If $v \geq 8$, then there are three elements each for b_2 and b_3. Since the requirements are 2,3,3, we have $v + \sigma \geq 2s + 2 \geq 12$, for $s \geq 5$. If $v = 7$, then $\sigma \geq 5$ and if $v = 6$, then $\sigma \geq 6$. Thus there are at least five short elements. Hence there are at least three short elements distinct from e_1, e_2. Let us call these f_1, f_2, f_3. Then one may place elements f_1, f_2, f_3 in b_2 and in b_3. This satisfies the requirements of b_2 and b_3.

Case (10) Requirements are 3,3,3. If $v \geq 9$, then three distinct elements are placed in each of b_1, b_2, b_3. So suppose $v < 9$. Since the requirements are 3,3,3, we have $v + \sigma \geq 2s + 3 \geq 13$, for $s \geq 5$. Since $\sigma \leq v$, this gives $v \geq 7$. Thus we have to consider the cases $v = 7, 8$. If $v = 7$, then $\sigma \geq 6$ and if $v = 8$, then $\sigma \geq 5$. Thus there are at least five short elements. Let these be e_1, e_2, e_3, e_4, e_5. We place e_1, e_3, e_5 in b_1, e_1, e_2, e_4 in b_2 and e_2, e_3, e_4 in b_3. This satisfies the requirements of b_1, b_2, b_3. ■

Combining Lemmas 48, 49 and Theorem 41, we get the following result.

THEOREM 50. *If $5 \leq s < v < 3s$, then an $SBS_3(s, v)$ can be constructed.*

5. Case $v \leq s$

We describe a construction of $SBS_k(s, v)$ whenever $v \leq s$. Let $s = qv + r$. The case $v \mid s$ (i.e., $r = 0$) is simple and hence we dispose of it first. The construction for the case $v \nmid s$ is based on a re-balancing technique.

Consider the following sequence of numbers.

$$0, 1, \ldots, s-2, s-1.$$

For $0 \leq i \leq s-1$, let W_i be a window of length k, i.e., W_i is the subsequence $i+0, i+1, \ldots, i+(k-1) \bmod s$. Define a new sequence $\mathrm{Seq}(s)$ as follows

$$W_0, W_1, \ldots, W_{s-1}.$$

It is not difficult to verify that each element of $\{0, \ldots, s-1\}$ occurs exactly k times in the sequence $\mathrm{Seq}(s)$.

We start by describing a construction of $SBS_k(s, s)$ based on the sequence $\mathrm{Seq}(s)$. The construction is different for s odd and s even.

Case s odd. Assign the set of elements in the window W_i to the back-diagonal D_i (see Definition 22). Clearly the resulting array is an $SBS_k(s, s)$.

Case s even. For $0 \leq i \leq s/2 - 1$, assign the set of elements in the window W_i to the back-diagonal E_i (see Definition 23). For $s/2 \leq i \leq s-1$, assign the set of elements in the window W_i to the back-diagonal $F_{i-s/2}$. Again it is easy to see that the resulting array is an $SBS_k(s, s)$.

The following important property is achieved by the above construction.

PROPOSITION 51. *In any $SBS_k(s, s)$ constructed by the above method the following holds. For $0 \leq i \leq \lfloor s/2 \rfloor - 1$, the (i, i)-th cell contains the element i and does not contain any of the elements $i+k, i+k+1, \ldots, i+v-1$.*

Proof. This follows immediately since W_i is in cell (i, i) for $0 \leq i < \lfloor s/2 \rfloor$. ■

We now tackle the case $v \mid s$. Let $r = 0$ and $s = qv$. Let M be an $SBS_k(v, v)$ constructed as above. The construction of $SBS_k(qv, v)$ is simply a $q \times q$ block array whose blocks are M. Clearly this construction is correct. Thus we get the following result.

THEOREM 52. *Let k, s, v be positive integers such that $v \mid s$ and $2k \leq v$. Then it is possible to construct $SBS_k(s, v)$.*

When $s = qv + r$, where $0 < r < v$, we use a modification of the above technique. Let L be the $SBS_k((q-1)v, v)$ constructed as described above. Consider the following array D.

$$\begin{bmatrix} L & R & A \\ R^T & M & B \\ A^T & B^T & S \end{bmatrix}.$$

The various components are as follows.

1. The array B is a rectangular $v \times r$ array, where each entry is a k-subset of $\{0, \ldots, v-1\}$. Each column of B has all the v elements exactly k times each. Each element occurs either $\lfloor kr/v \rfloor$ times or $\lceil kr/v \rceil$ times in any row of B. The construction of B is described later.
2. The array A is $[B^T, \ldots, B^T]^T$, where there are exactly $(q-1)$ copies of B^T.
3. The array $R = [M, \ldots, M]^T$, where there are exactly $(q-1)$ copies of M.
4. The array S is a square $r \times r$ array, which is either an $SBS_k(r, v)$ if one exists or else it is an $NSBS_k(r, v)$.

There are two cases to consider.

Case $r < v < rk$. In this case the array B is constructed as follows. Let $kr = q_1 v + r_1$, where $0 \leq r_1 < v$. Consider the following $v \times kr$ matrix E. (All entries are taken modulo v.)

$$\begin{bmatrix} 0 & \ldots & v-1 & 0 & \ldots & v-1 & \ldots & 0 & \ldots & v-1 & 0 & \ldots & r_1-1 \\ 1 & \ldots & v & 1 & \ldots & v & \ldots & 1 & \ldots & v & 1 & \ldots & r_1 \\ . & \ldots & . & \ldots & . & \ldots & . & \ldots & . & \ldots & . & \ldots & . \\ v-1 & \ldots & 2v-2 & v-1 & \ldots & 2v-2 & \ldots & v-1 & \ldots & 2v-2 & v-1 & \ldots & v+r_1 \end{bmatrix}$$

In the first row, the first $q_1 v$ entries are q_1 many repetitions of $0, \ldots, v-1$. The last r_1 entries are $0, \ldots, r_1 - 1$. The array B is constructed from the above matrix. The first column of B is the first k columns of E. In general the i-th column of B consists of the columns $(i-1)k+1$ to ik. Clearly in the above construction in each row any element occurs either $\lfloor kr/v \rfloor$ times or $\lceil kr/v \rceil$ times.

When $k = 1$, the condition for this case is vacuously true. For $k = 2, 3$, using Theorems 46 and 50, we know that $SBS_k(r, v)$ can be constructed. In these cases, we take S to be an $SBS_k(r, v)$.Then it is easy to check that the array D is an $SBS_k(s, v)$ and hence $SBS_k(s, v)$ can be constructed. For $k > 3$, the above technique reduces the problem of constructing $SBS_k(s, v)$ to one of constructing $SBS_k(r, v)$. Thus we get the following.

LEMMA 53. *Let s, v, k be positive integers such that $2k \leq v$, $s = qv + r$, with $q > 0$, $r < v < rk$. Then the following holds.*

1. If $k \leq 3$, then $SBS_k(s, v)$ can be constructed.

2. If $k > 3$, then $SBS_k(s, v)$ can be constructed if $SBS_k(r, v)$ can be constructed.

Case $v \geq rk$. If $SBS_k(r, v)$ exists, then we use the technique described above. If $SBS_k(r, v)$ does not exist, then using Lemma 21, we know that $NSBS_k(r, v)$ exists. We put S to be an $NSBS_k(r, v)$. If $v = rk$, then $SBS_k(r, v)$ exists. If $v \geq rk$ and there is no $SBS_k(r, v)$, then $v > rk$. In this case, there is an $NSBS_k(r, v)$.

Our strategy now is to re-balance the entire array. We replace some of the elements in the diagonal cells of M by other elements. This cannot be done arbitrarily. We describe the conditions that must be satisfied for such replacements to be possible.

In an $NSBS_k(s, v)$, the elements can have three possible frequencies $n-1, n, n+1$. Let v_{-1}, v_0, v_{+1} be the numbers of elements with frequencies $n-1, n, n+1$ respectively. Put $w = \min(v_{-1}, v_{+1})$. Note that $v_{-1} + v_{+1} < v$ and hence $w \leq \lfloor v/2 \rfloor$. Without loss of generality we assume the following. We take the elements $0, 1, \ldots, w-1$ to have frequency $n+1$ and the elements $v-w, v-w+1, \ldots, v-1$ to have frequency $n-1$. The elements $0, 1, \ldots, w-1$ are going to be replaced once by the elements $v-w, v-w+1, \ldots, v-1$ respectively in the array M. By Proposition 51 we have that for $0 \leq i \leq w-1$, the (i, i)-th cell of M contains the element i but not the element $v-w+i$. Thus for $0 \leq i \leq w-1$, we can replace element i in cell (i, i) by element $v-w+i$ provided the frequency of element i in row and column i exceeds the frequency of element $v-w+i$ by 1. After the replacement the row frequencies of the elements will be interchanged. The overall frequencies of the elements i and $v-w+i$ will become equal.

We now describe a construction for B which achieves this property. Note that since $v > kr$, each element must occur either 0 or 1 time in each row of B. Consider the following matrix F. All entries are taken modulo v.

$$\begin{bmatrix} 0 & 1 & \ldots & v-w-1 & v-w+1 & v-w+2 & \ldots & kr & kr+1 \\ 1 & 2 & \ldots & v-w & v-w+2 & v-w+3 & \ldots & kr+1 & kr+2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ v-1 & v & \ldots & 2v-w-2 & 2v-w & 2v-w+1 & \ldots & v+kr-1 & v+kr \end{bmatrix}.$$

Note that in this matrix the column $[v-w, \ldots, 2v-w-1]^T$ is missing.

The construction of B is from F. The first column of B consists of the first k columns of F. In general the j-th column of B consists of the columns $(j-1)k+1$ to jk of F. Since each column of F contains all the v elements exactly once, it follows that any column of B contains each of the v elements exactly k times. Since $v > rk$ in each row of B, any element occurs either 0 or 1 time. Further, for $0 \leq i \leq w-1$:

- The element i certainly occurs in the i-th row of B.
- The element $v-w+i$ does not occur in the i-th row of B.

Thus the substitutions in the diagonal cells of M can be made properly. This gives us the following result.

LEMMA 54. *Let $s, k, v > 0$, such that $s = qv + r$, with $q > 0$, $v \geq rk$. Then it is possible to construct $SBS_k(s, v)$.*

Combining Lemmas 53 and 54, we obtain the following result.

THEOREM 55. *Let $s, k, v > 0$ with $v \leq s$ and $2k \leq v$. Then we have the following.*

- *If $k \leq 3$, $SBS_k(s, v)$ can be constructed.*
- *If $k > 3$, then let $s = qv + r$.*
  - *If $v \geq rk$, then $SBS_k(s, v)$ can be constructed.*
  - *If $r < v < rk$, then $SBS_k(s, v)$ can be constructed if $SBS_k(r, v)$ can be constructed.*

6. Conclusion

We make the following conjecture.

CONJECTURE 1. *Let s, v, k be positive integers. Then an $SBS_k(s, v)$ can be constructed whenever it is feasible (see Definition 3).*

Our work in this paper shows ample positive evidence that Conjecture 1 is indeed true. More precisely, following our work, to prove Conjecture 1, it is sufficient to prove that $SBS_k(s, v)$ can be constructed under the following condition.

- $s < v < ks$, $k > 3$ and $s > 4$.

Our approach to proving Conjecture 1 is to show the correctness of Construction 39. This fact along with Theorem 40 leads us to make the following conjecture.

CONJECTURE 2. *The cut $(\{S\}, \{\bar{S}\})$ in the network N of Construction 36 is a minimum capacity cut.*

We have proved Conjecture 2 and hence the correctness of Construction 39 for the cases $k = 2, 3$. So for $k > 3$, Construction 39 can currently be considered to be only a heuristic. For all the examples that we have investigated with $k > 3$, Construction 39 turned out to be correct. This leads us to believe that Conjecture 2 is true for general k. Proving Conjecture 2 for general k will prove Conjecture 1. However, there may be other methods of proving Conjecture 1.

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