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A NOTE ON THE KHATRI-INVERSE

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SUMMARY. We give necessary and sufficient conditions for the existence of Khatri-inverse, pointing out an error in a condition given by Khatri. We also establish a generalized Cramer's rule to find minimum $\boldsymbol{N}$-norm $\boldsymbol{M}$-least-squares solution.

1. Introduction

We consider matrices over the complex field, unless indicated otherwise. Let $\boldsymbol{A}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ be matrices of order $m \times n$, $m \times m$ and $n \times n$ respectively, where $\boldsymbol{M}$, $\boldsymbol{N}$ are nonsingular (not necessarily hermitian). An $n \times m$ matrix $\boldsymbol{G}$ is said to be the Khatri-inverse of $\boldsymbol{A}$ with respect to $\boldsymbol{M}$, $\boldsymbol{N}$ if the conditions

(1) $\boldsymbol{AGA} = \boldsymbol{A}$ (2) $\boldsymbol{GAG} = \boldsymbol{G}$

(3) $(\boldsymbol{AG})^*\boldsymbol{M} = \boldsymbol{MAG}$ (4) $(\boldsymbol{GA})^*\boldsymbol{N} = \boldsymbol{NGA}$

are satisfied, where * denotes conjugate transpose Khatri [3]. If $\boldsymbol{M}$, $\boldsymbol{N}$ positive are definite, then such a $\boldsymbol{G}$ always exists, is unique, and is called the minimum $\boldsymbol{N}$-norm $\boldsymbol{M}$-least-squares g-inverse of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^+_{\boldsymbol{MN}}$ (Rao and Mitra [4], p. 52). The solution (or the approximate solution) of the system $\boldsymbol{Ax} = \boldsymbol{y}$ given by $\boldsymbol{x} = \boldsymbol{A}^+_{\boldsymbol{MN}}\boldsymbol{y}$ is termed as the minimum $\boldsymbol{N}$-norm $\boldsymbol{M}$-least-squares solution. We have followed the notation of Rao and Mitra [4] rather than that of Khatri [3].

The Khatri-inverse is unique whenever it exists ; this can be seen by suitably manipulating equations (1)–(4). In Khatri [3] it has been claimed that the Khatri-inverse of $\boldsymbol{A}$ exists if and only if $\rho(\boldsymbol{A}^*\boldsymbol{MA}) = \rho(\boldsymbol{AN}^{-1}\boldsymbol{A}^*) = \rho(\boldsymbol{A})$, where ρ denotes rank. The condition can be seen to be necessary from equations (1)–(4). However, the condition is not sufficient as can be seen from the example given below. The error persists in Rao and Mitra [4], p. 69–70 where the result is given as an exercise.

Example. Let $\boldsymbol{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{M} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{N} = [1]$.

Then A has no Khatri-inverse with respect to $\boldsymbol{M}$, $\boldsymbol{N}$ although $\rho(\boldsymbol{A}^*\boldsymbol{MA}) = \rho(\boldsymbol{AN}^{-1}\boldsymbol{A}^*) = \rho(\boldsymbol{A})$. In Khatri [3], $\boldsymbol{G} = \boldsymbol{N}^{-1}\boldsymbol{A}^*(\boldsymbol{AN}^{-1}\boldsymbol{A}^*))^-\boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{MA})^-\boldsymbol{A}^*\boldsymbol{M}$

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is given as the formula for the Khatri-inverse. In this example the matrix G determined using the formula is (1/9) [1, 4], which does not satisfy (3) and (4).

We now introduce some notation. Let A be an $m \times n$ matrix and let $\alpha = \{i_1, \ldots, i_r\}$, $\beta = \{j_1, \ldots, j_r\}$ be subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively. We denote by A^{α}_{β} the submatrix of A, determined by rows indexed by α and columns indexed by β. The submatrix determined by rows indexed by α and by all columns is denoted by A^{α}. Similarly A_{β} can be defined. The determinant of a square matrix A is denoted by $|A|$, and $\frac{\partial}{\partial a_{ij}} |A|$ denotes the cofactor of a_{ij} in the A. The r-th compound matrix of A is denoted by $C_r(A)$. Recall that the rows of $C_r(A)$ are indexed by the r-element subsets of $\{1, \ldots, m\}$, the columns are indexed by the r-element subsets of $\{1, \ldots, n\}$ and the (α, β) entry of $C_r(A)$ is given by $|A^{\alpha}_{\beta}|$.

We will need the following results from Bapat *et al.* [1], Bhaskara Rao [2].

(i) Let A be an $m \times n$ matrix of rank r. Then $\rho(C_r(A)) = 1$. ... (1.1)

(ii) Let A be an $m \times n$ matrix over the integral domain R. Then A is regular (i.e., has a g-inverse) if and only if there exists $\mathbb{C}^{\alpha}_{\beta}$ in R such that

$$\sum_{\alpha, \beta} \mathbb{C}^{\beta}_{\alpha} \, |A^{\alpha}_{\beta}| = 1 \qquad \text{... (1.2)}$$

where the summation is over all r-element subsets α, β of $\{1, \ldots, m)$ and $\{1, \ldots, n\}$ respectively. Furthermore if $\mathbb{C}^{\beta}_{\alpha}$ satisfy (1.2) then $G = (g_{ij})$ is a g-inverse of A, where

$$g_{ji} = \sum_{\alpha, \beta} \mathbb{C}^{\beta}_{\alpha} \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| ; \qquad \text{... (1.3)}$$

see, Bhaskara Rao [2], Theorem 8.

(iii) Let A be an $m \times n$ matrix of rank r over the integral domain R. Let G be a reflexive g-inverse of A. Then for all i, j

$$g_{ji} = \sum_{\alpha, \beta} |G^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| \qquad \text{... (1.4)}$$

where α, β run over all r-element subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively ; see Bapat *et al.* [1], Theorem 3.

In this paper we give a necessary and sufficient condition for the existence of the Khatri-inverse. We also give a Cramer-type rule for computing the minimum N-norm M-least-squares solution to the linear system $Ax = y$.

2. Existence of Khatri-inverse

It easily follows from the definition that if the Khatri-inverse of $\boldsymbol{A}$ with respect to $\boldsymbol{M}$, $\boldsymbol{N}$ exists, then $\rho(\boldsymbol{A^*MA}) = \rho(\boldsymbol{AN^{-1}A^*}) = \rho(\boldsymbol{A})$.

Theorem 1. *Let* $\boldsymbol{A}$ *be an* $m\times n$ *matrix of rank* r *and let* $\boldsymbol{A} = \boldsymbol{BC}$ *be a rank factorization of* $\boldsymbol{A}$. *Then the following conditions are equivalent :*

(i) $\boldsymbol{A}$ *has Khatri-inverse with respect to* $\boldsymbol{M}$, $\boldsymbol{N}$

(ii) $\boldsymbol{B^*MB}$, $\boldsymbol{CN^{-1}C^*}$ *are nonsingular and*

$$(\boldsymbol{B^*MB})^{-1}\boldsymbol{B^*M} = (\boldsymbol{B^*M^*B})^{-1}\boldsymbol{B^*M^*}, \quad \ldots (2.1)$$

$$\boldsymbol{N^{-1}C^*(CN^{-1}C^*)^{-1}} = \boldsymbol{N^{*-1}C^*(CN^{*-1}\,C^*)^{-1}} \quad \ldots (2.2)$$

Proof. Let $\boldsymbol{G}$ be the Khatri-inverse of $\boldsymbol{A}$. Since $\boldsymbol{G}$ is a reflexive g-inverse it can be written in the form $\boldsymbol{G} = \boldsymbol{UV}$ such that

$$\boldsymbol{VB} = \boldsymbol{I} \text{ and } \boldsymbol{CU} = \boldsymbol{I}. \quad \ldots (2.3)$$

From (1) and (3) we get

$$\boldsymbol{AGM^{-1}(AG)^*MA} = \boldsymbol{A}$$

i.e.,

$$\boldsymbol{BCGM^{-1}G^*C^*B^*MBC} = \boldsymbol{BC}$$

which implies

$$(\boldsymbol{CGM^{-1}G^*C^*})(\boldsymbol{B^*MB}) = \boldsymbol{I}$$

so that $\boldsymbol{B^*MB}$ is nonsingular. By (3) we get $\boldsymbol{MBV} = \boldsymbol{V^*B^*M}$ and hence

$$\boldsymbol{B^*MBV} = \boldsymbol{B^*M} \text{ (since } \boldsymbol{B^*V^*} = \boldsymbol{I}). \quad \ldots (2.4)$$

Also,

$$\boldsymbol{MB} = \boldsymbol{V^*B^*MB} \text{ (since } \boldsymbol{VB} = \boldsymbol{I}). \quad \ldots (2.5)$$

From (2.4), (2.5) we get

$$\boldsymbol{V} = (\boldsymbol{B^*MB^{-1}B^*M} = (\boldsymbol{B^*M^*B})^{-1}\boldsymbol{B^*M^*}$$

since $\boldsymbol{B^*MB}$ is nonsingular. Similarly we can prove that $\boldsymbol{CN^{-1}C^*}$ is non-singular and

$$\boldsymbol{U} = \boldsymbol{N^{-1}C^*(CN^{-1}C^*)^{-1}} = \boldsymbol{N^{*-1}C^*(CN^{*-1}C^*)^{-1}}.$$

Conversely if (ii) of the theorem holds then it can be verified that

$$\boldsymbol{G} = \boldsymbol{N^{-1}C^*(CN^{-1}C^*)^{-1}(B^*MB)^{-1}B^*M}$$

is the Khatri-inverse of $\boldsymbol{A}$ with respect to $\boldsymbol{M}$, $\boldsymbol{N}$. □

Corollary 2. *Let* $\boldsymbol{A}$ *be an* $m\times n$ *matrix of rank* 1. *Then* $\boldsymbol{A}$ *has Kharti-inverse with respect to* $\boldsymbol{M}$, $\boldsymbol{N}$ *if and only if* $tr(\boldsymbol{N^{-1}A^*MA})$ *is nonzero and the matrices* $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{M}$, $\boldsymbol{N}$ *satisfy* (2.1), (2.2) ; *in which case*

$$\boldsymbol{G} = [tr(\boldsymbol{N^{-1}A^*MA})]^{-1}\boldsymbol{N^{-1}A^*M}$$

is the Kharti-inverse.

Proof. Let $A = BC$ be a rank factorization of A. Note that $tr(N^{-1} A^*MA) = |CN^{-1}C^*|\ |B^*MB|$ whereas $N^{-1}\ C^*(CN^{-1}C^*)^{-1}\ (B^*MB)^{-1}B^*M = tr(N^{-1}\ A^*MA)^{-1}N^{-1}\ A^*M$ and the result follows from Theorem 1. □

Corollary 3. *Let A be an $m \times n$ matrix of rank r. Then A has Khartri-inverse with respect to M, N if and only if $\sum_{\alpha,\beta} |(N^{-1}\ A^*M)^{\beta}_{\alpha}|\ |A^{\alpha}_{\beta}|$ is nonzero and* (2.1), (2.2) *are satisfied.*

Proof. By Cauchy-Binet formula we get

$$|CN^{-1}\ C^*|\ |B^*MB| = \sum_{\alpha,\beta,\gamma,\delta} |C_{\beta}|\ |N^{-1}\ \beta|\ |C^{*\gamma}\ |B^*{}_{\delta}|\ |M^{\delta}_{\alpha}|\ |B^{\alpha}|$$

$$= \sum_{\alpha,\beta} |(N^{-1}\ A^*M)^{\beta}_{\alpha}|\ |A^{\beta}_{\alpha}|$$

and hence the result follows from Theorem 1. □

We now prove the main result of this paper.

Theorem 4. *Let A be an $m \times n$ matrix. Then the following conditions are equivalent* :

(i) *A has Khatri-inverse with respect to M,N*

(ii) $\rho(A^*MA) = \rho(AN^{-1}A^*) = \rho(A)$ *and the equations*

$$A(A^*MA)^{-}A^*M = A(A^*MA)^{-}A^*M^*, \quad \ldots \quad (2.6)$$

$$N^{-1}\ A^*(AN^{-1}A^*)^{-}A = N^{*-1}A^*(AN^{*-1}A^*)^{-}A \quad \ldots \quad (2.7)$$

are satisfied for any choice of g-inverse. Furthermore in such a case

$$G = N^{-1}\ A^*(AN^{-1}A^*)^{-}A(A^*MA)^{-}A^*M \quad \ldots \quad (2.8)$$

is the Khatri-inverse for any choice of g-inverse.

Proof. If (i) is satisfied, then as remarked earlier, $\rho(A^*MA) = \rho(AN^{-1}A^*) = \rho(A)$ holds. Therefore the matrices $A(A^*MA)^{-}A^*$, $A(A^*M^*A)^{-}A^*$, $A^*(AN^{-1}A^*)^{-}\ A$, $A^*(AN^{*-1}\ A^*)^{-}A$ are invariant under the choice of g-inverse. It follows that (2.6) and (2.7) are equivalent to (2.1) and (2.2) respectively and (ii) is satisfied by Theorem 1. Conversely, if (ii) is satisfied then it can be verified that G given in (2.8) is the Khatri-inverse. □

In the next result we give a formula for the Khatri-inverse which is similar to the formula for the Moore-Penrose inverse given in Bapat *et al.* [1].

Theorem 5. *Let A be an $m \times n$ matrix of rank r and let $u = tr[C_r\,(N^{-1}\ A^*MA)]$. If $G = (g_{ij})$, the Khatri-inverse of A with respect to M,N exists, then $u \neq 0$ and*

$$g_{ij} = \sum_{\alpha,\beta} u^{-1}\ |(N^{-1}\ A^*M)^{\beta}_{\alpha}\ |\ \frac{\partial}{\partial a_{ji}}\ |A^{\alpha}_{\beta}| \quad \ldots \quad (2.9)$$

Proof. Let G be the Khatri-inverse of A with respect to M, N. Then it easily follows from the multiplicative property of the compound matrix that $C_r(G)$ is the Khatri-inverse of $C_r(A)$ with respect to $C_r(M)$ and $C_r(N)$. Since $C_r(A)$ has rank one and since the Khatri inverse is unique whenever it exists, we have from Corollary 2,

$$|G_{\alpha}^{\beta}| = u^{-1} \, |(N^{-1} A^{*}M)|$$

for all subsets α, β of $\{1, ..., m\}$, $\{1, ..., n\}$ respectively. It follows from (1.4) that G must be given by (2.9). □

Remark. If M, N are positive definite then it follows from Theorem 2 that the Khatri-inverse of A with respect to M, N exists and as noted earlier it is also refered to as the minimum N-norm M-least-squares g-inverse. Theorem 3 can be used to give a generalized Cramer's rule to obtain the minimum N-norm M-least-squares solution to the linear system $Ax = y$. In fact it is easily verified using (2.9) that

$$(Gy)_i = \sum_{\alpha, \beta : i \in \beta} u^{-1} |(N^{-1}A^{*}M)_{\alpha}^{\beta}| \, |\hat{A}_{\beta}^{\alpha}|$$

where $\hat{A}$ is the matrix obtained by replacing the i-th column of $\hat{A}$ by y.

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