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THE SAMPLING DISTRIBUTION OF p-STATISTICS AND CERTAIN ALLIED STATISTICS ON THE NON-NULL HYPOTHESIS

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Introduction

In an earlier paper written[1] by the author jointly with Mr. R. C. Bose, an attempt was made to solve the problem of discrimination between and classification of different multivariate normal populations in terms of means, on the hypothesis of equivariances and equi-co-variances. In a later paper[4] by the author the comparatively harder problem of a similar discrimination and classification in terms of variances and co-variances (now supposed to be different for the different multivariate normal populations) was sought to be tackled in the following manner. Given two samples S' and S'' of sizes n' and n'' supposed to have been drawn at random from two multivariate normal populations Σ' and Σ'', a linear compound of the variates was taken as a sort of new character and the compounding coefficients were sought to be so chosen as to maximise the ratio of the variance of the first sample (for the compound character) and the variance of the second sample for the same character. This led in the case of p-variate populations, to as many as p studentised statistics of which one in some sense could be said to correspond to a maximum, one to a minimum and the others to what might be called stationary values. These statistics, which came as the p roots of a certain determinantal equation involving the variances and covariances of the two samples (and of course the sample sizes) were shown to be invariant under general linear transformations of the p-variates to p new variates. There would be similar p-statistics for the two populations, given as the roots of a similar determinantal equation involving the population variances and covariances. These also would be invariant under similar linear transformations. The p-statistics for the samples were next geometrically interpreted as simple trigonometric functions of the p critical angles between two definite Euclidean flats, one of p dimensions and the other of $(n''-1)$ dimensions associated with sample S''; the flat of $(n'-1)$ dimensions associated with sample S' might of course have been taken as well. The joint sampling distribution of the p-statistics (for the sample pair) was then obtained in the last paper on the null hypothesis, i.e. on the assumption that the two populations sampled were identical in the sets of variances and covariances. This marked a significant step in advance so far as the problem of discrimination is concerned. Suggestions were made at the end of the paper in question as to how these p-statistics (with their sampling distributions) could be possibly applied to the practical task of discrimination. About the same time came out a paper[3] by Prof. Fisher on practically the same problem which was sought to be solved by a similar set of statistics leading to a joint sampling distribution similar to the one obtained by the author. In the same journal and in the same issue[3] an elegant proof of the derivation of the distribution was given by Dr. Hsu. But the problem of classification yet remained to be tackled. For that was needed the sampling distribution of these p-statistics on the non-null hypothesis, i.e. when the populations sampled differ in the sets of variances and co-variances. This latter distribution was also worked out sometime after the earlier one and the results were announced elsewhere early this year.[5] The solution to the problem of discrimination also was carried some way further and this, too, was published in a summary form at about the same time.[6]

The derivation on the non-null hypothesis, of the joint sampling distribution of the p-statistics, worked out earlier, is set forth in detail in the present paper. The distribution on the null hypothesis being only a special case of it, is directly obtained by introducing the proper simplifications in the general scheme, and is seen to agree as it should, with the earlier form obtained in the last paper. The mathematical technique used here is entirely different from the one in the last paper, which was frankly and deliberately pedestrian for reasons explained there. The present technique is based on geometrical reasoning of a fairly general and rigorous character and is really the method promised in the last paper. Certain functions of the p-statistics are proposed as being suitable for purposes of both discrimination and classification and their sampling distributions are obtained both on the null as well as on the non-null hypothesis. Some suggestions are made in conclusion as to how to use these functions to test for agreement with or departure from different kinds of null hypothesis, the agreement giving the clue to discrimination and the departure the clue to classification.

Detailed discrimination and classification will form the subject of the next paper (which will come out in a subsequent issue of this Journal) where furthermore suitable invariants for more than two populations and samples will also be discussed and their sampling distributions obtained just as in the last and present papers joint invariants—and p-statistics are after all nothing but that—suitable for certain specific purposes have been discussed and their sampling distributions obtained. This will mean further progress in the collective study groups or populations.

1. DEFINITIONS AND MATHEMATICAL PRELIMINARIES

With slight modifications certain definitions and results from the earlier paper[1] will be taken over here.

S' and S'' are samples of sizes n' and n'' drawn at random from 2 p-variate normal populations Σ' and Σ''.

α'_{ij}, α''_{ij} $(i, j=1, 2, \ldots, p)$ denote the sets of variances and covariances of Σ' and Σ'' respectively so that $\alpha'_{ij}=\rho'_{ij}\,\sigma'_i\,\sigma'_j$, $\alpha''_{ij}=\rho''_{ij}\,\sigma''_i\,\sigma''_j$, σ'_i and σ''_i being the standard deviations for the i-th character, σ'_j and σ''_j being the standard deviations for the j-th character, and ρ_{ij}' and ρ_{ij}'' the correlation coefficients between the i-th and the j-th character for the populations Σ' and Σ'' respectively. $\|\alpha'_{ij}\|$, $\|\alpha''_{ij}\|$ are called the dispersion matrices for Σ' and Σ'' respectively.

a'_{ij}, a''_{ij} $(i, j=1, 2, \ldots, p)$ denote the sets of variances and covariances for the samples S' and S'' respectively, so that $\|a'_{ij}\|$, $\|a''_{ij}\|$ are the dispersion matrices for the samples.

α'_i, $\alpha''_i (i=1, 2, \ldots . p)$ denote the means of the different characters for the populations Σ' and Σ'' respectively.

a'_i, $a''_i (i=1, 2, \ldots . p)$ denote the means of the different characters for the samples S' and S''.

$x'_{i\nu'}$, $x''_{i\nu''}$ $(i=1, 2, \ldots . p;\ \nu'=1, 2, \ldots . n',\ \nu''=1, 2, \ldots . n'')$ denote the sample readings for the samples S and S' respectively, the first suffix referring to the character and the second to the individual.

$$\left.\begin{aligned} a'_i &= \frac{1}{n'} \sum_{\nu'=1}^{n'} x'_{i\nu'}\ (i=1, 2, \ldots . p) \\ a''_i &= \frac{1}{n''} \sum_{\nu''=1}^{n''} x''_{i\nu''}\ (i=1, 2, \ldots . p) \\ a'_{ij} &= \frac{1}{n'-1} \sum_{\nu'=1}^{n'} (x'_{i\nu'}-a'_i)\ (x'_{j\nu'}-a'_j),\ (i, j=1, 2, \ldots . p) \\ a''_{ij} &= \frac{1}{n''-1} \sum_{\nu''=1}^{n''} (x''_{i\nu''}-a''_i)\ (x''_{j\nu''}-a''_j),\ (i, j=1, 2, \ldots . p) \end{aligned}\right\} \quad \ldots (1\cdot1)$$

A compound character based on a linear compound of the variates is taken for both the samples S' and S'' which are now represented respectively by readings (for the different individuals).

$$\sum_{i=1}^{p} \lambda_i\, x'_{i\nu'} \text{ and } \sum_{i=1}^{p} \lambda_i\, x''_{i\nu''}\ (\nu'=1, 2, \ldots . n';\ \nu''=1, 2, \ldots . n'')$$

If now we denote by v' and v'' the variances of samples of S' and S'' for the compound character then from (1·1)

$$v' = \sum_{i,\ j=1}^{p} \lambda_i\ \lambda_j\ a'_{ij},\quad v'' = \sum_{i,\ j=1}^{p} \lambda_i\ \lambda_j\ a''_{ij} \quad \ldots (1\cdot2)$$

Setting $v'/v''=k^2$ and maximising k^2 with regard to the arbitrary coefficients λ_i ($i=1, 2, \ldots . p$) we get the desired stationary values as the roots of the p—fold determinantal equation in k^2

$$|\, a'_{ij}-k^2\, a''_{ij}\, | = 0 \quad \ldots (1\cdot3)$$

Similarly for the populations Σ' and Σ'', if we start from two similar compound characters (based on a linear compound of the p—variates) and denote by V' and V'' the population variances for the compound character, then stationary values of $\kappa^2=\text{V}'/\text{V}''$ (obtained by maximising V'/V'' with regard to the compounding coefficients μ_i's) come out as the roots of the p—fold determinantal equation in κ^2

$$|\, \alpha'_{ij}-\kappa^2\, \alpha''_{ij}\, | = 0 \quad (1\cdot4)$$

It can be easily proved that all the k_i's in (1·3) are unity when and only when $a'_{ij}=a''_{ij}$ ($i, j=1, 2, \ldots . p$), that is, when the two samples happen to have identical variances and covariances. Similarly κ_i^2s in (1.4) are all unity when and only when $\alpha'_{ij}=\alpha''_{ij}$ ($i, j=1, 2, \ldots . p$) that is, when the two populations happen to be identical in the variances and covariances.

The p roots of (1·3) and also of (1·4) are invariant under any general linear transformation of the variates, i.e. for any arbitrary values of λ_i's$(i=1, 2, \ldots. p)$ for the sample pair and of μ_i's$(i=1, 2,\ldots. p)$ for the population pair. The sample S' with readings $x'_{i\nu'}$ $(i=1, 2, \ldots. p;\ \nu'=1, 2, \ldots. n')$ is represented in the usual Fisherian flat sample space $f'_{n'}$ of n' dimensions by the p points with co-ordinates $(x'_{i1}, x'_{i2}, \ldots. x'_{in'})$, $(i=1, 2, \ldots. p)$ or by p vectors x'_i joining the points to the origin. We may take another flat space $f''_{n''}$ of n'' dimensions orthogonal to $f'_{n'}$ and represent in it the sample S" by p other similar vectors x''_i. Let

$$y'_{i\nu'}=x'_{i\nu'}-a'_i,\ y''_{i\nu''}=x''_{i\nu''}-a''_i \qquad \ldots \quad (1\cdot5)$$

where $i=1, 2, \ldots. p$; $\nu'=1, 2, \ldots. n'$; $\nu''=1, 2, \ldots. n''$ and a'_i and a''_i have been already defined in (1·1)

y'_i and y''_i denote vectors with components$(y'_{i1}, y'_{i2}, \ldots. y'_{in'})$ and $(y''_{i1}, y''_{i2}, \ldots. y''_{in''})$ lying in the flats $f'_{n'}$ and $f''_{n''}$ respectively. The end points of y'_i and y''_i we shall denote by Q'_i, Q''_i. Then vectors y'_i are easily seen from (1·5) to lie in a flat $f'_{n'-1}$ of $n'-1$ dimensions orthogonal to the equiangular line in $f'_{n'}$ and immersed in it. Similar considerations apply to the vectors y''_i. The p vectors y'_i will by themselves constitute a flat f'_p of p dimensions included in $f'_{n'-1}$ and similarly $y''_i(i=1, 2,\ldots. p)$ will consitute a flat f''_p of p dimensions included in $f''_{n''-1}$. Furthermore refer the vectors y'_i's to $n'-1$ new orthogonal axes in the $(n'-1)$ flat $f'_{n'-1}$ and let the components of y'_i along those axes be $z'_{i\nu'}(i=1, 2, \ldots. p;\ \nu'=1, 2, \ldots. n'-1)$ and likewise refer y''_i' to $n''-1$ new orthogonal axes in the $(n''-1)$ flat $f''_{n''-1}$, the components being similarly denoted by $z''_{i\nu''}(\nu''=1, 2,\ldots. n''-1)$. The vectors y'_i, $y''_i(i=1, 2, \ldots. p)$ will be called the variation vectors and will characterise the samples S' and S" so far as the variances and covariances are concerned. Any linear transformation of the variates applied to samples S' and S" would now mean changing over from y'_i and y''_i $(i=1, 2, \ldots. p)$ to a new set of vectors $\sum_{j=1}^{p} \lambda_{ij}\, y'_j$ and $\sum_{j=1}^{p} \lambda_{ij}\, y''_j$ $(i=1, 2, \ldots. p)$, the flats f'_p and f''_p remaining, however, invariant under the transformation. Denoting the scalar product of y'_i and y'_j and by $y'_i.\ y'_j$ we have from the foregoing considerations

$$\left.\begin{aligned} a'_{ij}&=(y'_i.\ y'_j)\,/\,(n'-1)=\sum_{\nu'=1}^{n'-1} z'_{i\nu'}\, z'_{j\nu'}\,/\,(n'-1) \\ a''_{ij}&=(y''_i.\ y''_j)\,/\,(n''-1)=\sum_{\nu''=1}^{n''-1} z''_{i\nu''}\, z''_{j\nu''}/(n''-1) \end{aligned}\right\} \qquad \ldots \quad (1\cdot6)$$

Form the resultant of the vectors y'_i and y''_i and call it y_i $(i-1, 2, \ldots. p)$. The p vectors y_i whose end points we shall call Q_i form a flat f_p of p dimensions which makes with the flat $f''_{n''-1}$ (containing the p vectors y''_i) and also with f''_p, p critical angles which we call ϕ_i $(i=1, 2, \ldots. p)$. It should be noticed that the flat f_p constituted by the p-vectors y_i $(i=1, 2, \ldots. p)$ makes with $f'_{n'-1}$ or with f'_p critical angles $(\frac{\pi}{2}-\phi_i)$ $(i=1, 2,\ldots. p)$. it is also well known that associated with each critical angle ϕ_i $(i=1, 2, \ldots. p)$ there are three co-planar lines (called critical lines) l'_i, l''_i and l_i lying respectively in f'_p, f''_p, f_p with the following properties :

(*a*) The critical angle ϕ_i (already defined) is the angle between l_i and l''_i and $\frac{\pi}{2}-\phi_i$ is the angle between l_i and l'_i $(i=1, 2, \ldots. p)$.

(*b*) The two-plane formed by any set l'_i, l''_i and $l_i (i=1, 2, \ldots. p)$ is absolutely perpendicular to the (two) plane formed by any other set l'_j, l''_j, $l_j, (j\neq i, j=1, 2, \ldots. p)$ This means that any member of the triad l'_i, l''_i, l_i is perpendicular to any member of any other triad l'_j, l''_j, l_j $(i, \neq j, i, j=1, 2, \ldots. p)$. Then it was shown in the previous paper[4] that

$$k_i=\tan\ \phi_i\sqrt{(n''-1)/(n'-1)},\ (i=1, 2, \ldots. p) \qquad \text{.. (1·7)}$$

where k_i is defined by (1·3).

Take new *p*-vectors $\eta'_i (i=1, 2, \ldots. p)$ for the population Σ' such that $\alpha'_{ij}=\eta'_i$. η'_j the scalar product of η'_i and η'_j; these p vectors would constitute a *p*-flat F'_p; similarly take η''_i $(i=1, 2, \ldots. p)$ for Σ'' such that $\alpha''_{ij}=\eta''_i\ \eta''_j$, and so manage matters that η''_i's constitute a *p*-flat F''_p absolutely orthogonal to F'_p. Here again a linear transformation of the variates would mean changing over from η'_i and η''_i $(i=1, 2, \ldots. p)$ to a new set of vectors

$$\sum_{j=1}^{p} \mu_{ij}\ \eta'_j \text{ and } \sum_{j=1}^{p} \mu_{ij}\ \eta''_j (i=1, 2, \ldots. p),$$

the flats F'_p and F''_p remaining invariant. If we form now the resultant of η'_i and η''_i and call it $\eta_i (i=1, 2, \ldots. p)$ then the p vectors η_i form a *p*-flat F_p which makes with the *p*-flat F''_p, p critical angles, which let us call $\Phi_i (i=1, 2, \ldots. p)$. As in the case of the samples S' and S'', it is easily seen here that corresponding to any Φ_i we have a triad L'_i, L''_i $L_i (i=1, 2, \ldots. p)$ of which L'_i lies in F'_p, L''_i in F''_p and L_i in F_p. The properties of this p-set of triads are exactly similar to those discussed for the p-set $l'_i, l''_i, l_i (i=1, 2, \ldots. p)$. It could be shown by a procedure exactly similar to the one adopted in the last paper that

$$\kappa_i=\tan\ \Phi_i (i=1, 2, \ldots. p) \qquad \text{.. (1·8)}$$

2. The Reduction of the Distribution Problem

The joint probability for the two samples S' and S'' coming as random samples from Σ' and Σ'' or the probability of the raw sample readings $x'_{i\nu'}$ and $x''_{i\nu''}$ lying between $x'_{i\nu'}$ and $x'_{i\nu'}+dx'_{i\nu'}$ and $x''_{i\nu''}$ and $x''_{i\nu''}+dx''_{i\nu''}$ $(i=1, 2, \ldots. p;\ \nu'=1, 2, \ldots. n'\ \nu''=1, 2, \ldots. n'')$ is given by

$$\text{const. } e^{-\frac{1}{2}\left[\sum_{i,j=1}^{p} \alpha'^{ij}\{n'(a'_i-\alpha'_i)(a'_j-\alpha'_j)+(n'-1)a_{ij}\}+\alpha''^{ij}\{n''(a''_i-\alpha''_i)(a''_j-\alpha''_j)+(n''-1)a''_{ij}\right]}$$

$$\times \prod_{\nu'=1}^{n'} \prod_{i=1}^{p} dx'_{i\nu'} \prod_{\nu''=1}^{n''} \prod_{i=1}^{p} dx''_{i\nu''} \qquad \text{.. (2·1)}$$

where α'^{ij} is the adjusted minor of α'_{ij} in the determinant $|\alpha'_{ij}|$ divided by the determinant itself, α''^{ij} is similarly defined, and all other quantities a'_i, α'_i, a''_i, α''_i, a'_{ij}, α'_{ij}, a''_{ij}, α''_{ij} have been already defined in section 1. From $x'_{i\nu'}$ and $x''_{i\nu''}$ $(i=1, 2, \ldots. p;$ $\nu'=1, 2, \ldots. n';\ \nu''=1, 2, \ldots. n'')$ transform to a'_i, $z'_{i\nu'}$ and a''_i, $z''_{i\nu''}$ $(i=1, 2, \ldots. p;$ $\nu'=1, 2, \ldots. n'-1;\ \nu''=1, 2, \ldots. n''-1)$ already defined in (1·1), (1·6) and in the lines immediately preceding (1·6). Since $z'_{ir'}$'s and $z''_{ir''}$'s are in directions absolutely orthogonal to those in which a'_i's and a''_i's are taken and furthermore, since in general all elements are in perpendicular directions, the volume element in (2·1) is easily seen to transform to

$$\text{Const.} \prod_{i=1}^{p} da'_i \prod_{i=1}^{p} da''_i \prod_{r'=1}^{n'-1} \prod_{i=1}^{p} dz'_{ir'} \prod_{r''=1}^{n''-1} \prod_{i=1}^{p} dz''_{ir''} \quad .. \quad (2{\cdot}2)$$

From (1·6) it is easily seen that the a'_{ij}'s and a''_{ij}'s occurring in the density factor in (2·1) involve only the $z'_{ir'}$'s and $z''_{ir''}$'s and not the a'_i's and a''_i's. Therefore integrating out (2·4) over a'_i's and a''_i's between the proper limits ($-\infty$ to $+\infty$) and absorbing the integrated thing within the constant we have the joint distribution of the $z'_{ir'}$'s and $z''_{ir''}$'s in the form

$$\text{const.}\; e^{-\frac{1}{2}\sum_{i,j=1}^{p}\{(n'-1)\alpha'^{ij}a'_{ij}+(n''-1)\alpha''^{ij}a''_{ij}\}}$$

$$\times \prod_{r'=1}^{n'-1} \cdot \prod_{i=1}^{p} dz'_{ir'} \prod_{r''=1}^{n''-1} \prod_{i=1}^{p} dz''_{ir''} \quad .. \quad (2{\cdot}3)$$

It is required to obtain the sampling distribution of the k_i's defined by (1·3) where the a'_{ij}'s and a''_{ij}'s are defined in terms of $z'_{ir'}$'s and $z''_{ir''}$'s by (1·6). These k_i's as observed in section 1 and proved in the previous paper, are invariant under any general linear transformation of the p-variates into p new ones—the same transformation being of course applied to the variates of both S′ and S″. Similarly we have the invariance of κ_i's under linear tranformation of the variates of Σ' and Σ''. From this follows the invariance of the p symmetric functions (in the sense of the theory of equations) of k_i's($i=1, 2, \ldots. p$) which are the p roots of the determinantal equation (1·3). The first symmetric function is evidently $\Sigma\, a''^{ij} a'_{ij}$ and this is invariant. Likewise the invariance of the p symmetric functions (in the sense of the theory of equations) of κ_i's ($i=1, 2, \ldots. p$) which are the p root of the determinantal equation (1·4). The first symmetric function is evidently $\Sigma\, \alpha''^{ij} \alpha'_{ij}$ and this is invariant. Similarly starting from the equation

$$|\, a'_{ij} - \lambda^2 \alpha'_{ij}\,| = 0 \quad .. \quad (2{\cdot}4)$$

and applying a linear transformation to the p-variates of S′ and Σ' we could prove the invariance of $\Sigma\alpha'^{ij} a'_{ij}$ and in a similar manner of $\Sigma\alpha''^{ij} a''_{ij}$.

Starting from the determinantal ratios

$$|\,(a'_i-\alpha'_i)(a'_j-\alpha'_j)+\alpha'_{ij}\,| \,/\, |\,\alpha'_{ij}\,| \text{ and } |\,(a''_i-\alpha''_i)(a''_j-\alpha''_j)+\alpha''_{ij}\,| \,/\, |\,\alpha''_{ij}\,|$$

we could similarly prove the invariance of

$$\sum_{i,j=1}^{p} \alpha'^{ij}(a'_i-\alpha'_i)(a'_j-\alpha'_j) \text{ and } \sum_{i,j=1}^{p} \alpha''^{ij}(a''_i-\alpha''_i)(a''_j-\alpha''_j)$$

Assume now that a linear transformation has been applied to the variates of S′, S″ and of course of Σ' and Σ'' such that there has been a change over from vectors η'_i's, η''_i's and η_i's defined in section 1 to new vectors of lengths κ_i' along L'_i's, unit vectors along L''_i's and, of course, vectors of lengths $(1+\kappa^2_i)^{1/2}$ along L_i's where L'_i's, L''_i's and L_i's are defined in lines preceding (1·8) and $i=1, 2, \ldots. p$. This will of course leave the κ_i's unchanged and the density factor in (2·1) invariant. Now in such a case the new $\alpha'_{ii}=\kappa^2_i$

$(i=1, 2, \ldots. p)$ and $a''_{ij}=0 (i \neq j)$ and the new $a''_{ii}=1$ and $a''_{ij}=0$. Hence the new $a'^{ii}=1/\kappa^2_i$, $a''^{ii}=1$, $a'_{ij}=a''_{ij}=0 (i \neq j)$. We can now assume without any loss of generality that $x'_{i\nu'}$, $x''_{i\nu''}$, a'_i, a''_i, $z'_{i\nu'}$, $z''_{i\nu''}$ etc. are all quantities which we get after the transformation.

That is, we would assume without any loss of generality that in (2·1) and (2·3)

$$a'^{ii}=1/\kappa^2_i, \; a''^{ii}=1, \; a'^{ij}=a''^{ij}=0 (i \neq j)$$

Then (2·3) simplifies to

$$\text{Const. } e^{-\frac{1}{2}\sum_{i=1}^{p}\left\{\frac{(n'-1)\,a'_{ii}}{\kappa^2_i}+(n''-1)\,a''_{ii}\right\}}$$

$$\times \prod_{\nu'=1}^{n'-1} \prod_{i=1}^{p} dz'_{i\nu'} \prod_{\nu''=1}^{n''-1} \prod_{i=1}^{p} dz''_{i\nu''} \qquad (2·5)$$

Now $a'_{ii}=y'^2_i$, $a''_{ii}=y''^2_i$, $(i=1, 2, \ldots. p)$ where y'_i and y''_i are supposed to be the magnitudes of vectors y'_i and y''_i defined in section 1. Similarly let y_i's be magnitude of the vectors y_i's defined at the same place. Now resolve each y_i along the p orthogonal axes l_j's, each y'_i along p orthogonal axes l'_j's and each y''_i along l''_j's $(j=1, 2, \ldots. p)$. Let these component parts of y_i, y'_i, y''_i, regarded vectorially be called y_{ij}, y'_{ij}, y''_{ij}, $(j=1, 2, \ldots. p)$ and let the magnitudes be called y_{ij}, y'_{ij}, y''_{ij}.

It easily follows from the definition of critical angles in section 1, immediately before (1·7) that

$$y'_{ij}=y_{ij} \sin \phi_j, \; y''_{ij}=y_{ij} \cos \phi_j \; (i=1, 2, \ldots. p; \; j=1, 2, \ldots. p) \qquad .. \quad (2·6)$$

and further that

$$\left.\begin{aligned}(n'-1)a'_{ii}&=\sum_{j=1}^{p} y'^2_{ij}=\sum_{j=1}^{p} y^2_{ij} \sin^2\phi_j \quad \text{from} \quad (2·6)\\ (n''-1)a''_{ii}&=\sum_{j=1}^{p} y''^2_{ij}=\sum_{j=1}^{p} y^2_{ij} \cos^2\phi_j \text{ also from } (2·6)\end{aligned}\right\} (i=1, 2, \ldots. p) \qquad .. \quad (2·7)$$

The distribution (2·5) now transforms to

$$\text{Const. } e^{-\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} y^2_{ij}\left(\frac{\sin^2\phi_j}{\kappa^2_i}+\cos^2\phi_j\right)}$$

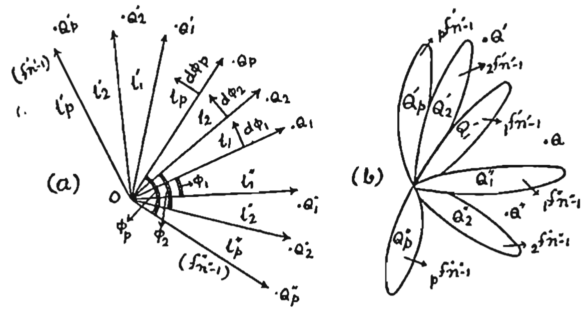
$$\times \prod_{\nu'=1}^{n'-1} \prod_{i=1}^{p} dz'_{i\nu'} \prod_{\nu''=1}^{n''-1} \prod_{i=1}^{p} dz''_{i\nu''} \qquad .. \quad (2·8)$$

where the ϕ_j's are connected with k_j's by the relation (1.7).

We should now seek to change over from the volume element in (2·8) expressed in terms of the $p(n'+n''-2)$ variables $z'_{i\nu'}$'s and $z''_{i\nu''}$'s to a new volume element expressed in terms of the p^2 quantities y_{ij}'s$(i, j=1, 2, \ldots. p)$ and the p quantities ϕ_i $(i=1, 2, \ldots. p)$ which are the only quantities occurring in the density factor (2·8). After that, integrating out out over y_{ij}'s between proper limits we shall obtain the distribution of ϕ_i's, that is, of the k_i's connected with ϕ_i's by (1·7).

3. Reduction of the Volume Element

In section 1 we have represented the samples S′ and S″ after reducing for the means, by $2p$ points Q'_i and Q''_i ($i=1, 2, \ldots, p$) in the flats $f'_{n'-1}$, and $f''_{n''-1}$ or by p points Q_i ($i=1, 2, \ldots p$) in the compound flat $f_{n'+n''-2}$ constituted by the flats $f'_{n'-1}$ and $f''_{n''-1}$. A second representation of S′ and S″ is also fruitful. In it we associate with any i-th character for S′ a sample space ${}_if'_{(n'-1)}$ of $(n'-1)$ dimensions and take p such orthogonal spaces for the p characters constituting a flat $f'_{p(n'-1)}$ of $p(n'-1)$ dimensions. We do the same for S″, the p sets for S″ being orthogonal to the p sets for S′, and constituting again a flat $f''_{p(n''-1)}$ of $p(n''-1)$ dimensions. The samples S′ and S″ can now be represented by points Q' and Q'' in the flats $f'_{p(n'-1)}$, and $f''_{p(n''-1)}$, or by a point Q in the resultant flat $f_{p(n'+n''-2)}$, of $p(n'+n''-2)$ dimensions. The volume element in (2·8) can be represented either as the product of the volume elements at Q'_i ($i=1, 2, \ldots p$) in ${}_if'_{n'-1}$, and of those at Q''_i ($i=1,2,\ldots p$) in ${}_if''_{n''-1}$, or alternatively by the product of the volume element at Q' in $f'_{p(n'-1)}$, and of that at Q'' in $f''_{p(n''-1)}$, which ultimately means simply the volume element at Q in $f_{p(n'+n''-2)}$. Integration over the total range of all the variates for all the individuals of the two samples would either mean sweeping out the spaces $f'_{n'-1}$ and $f''_{n''-1}$, or $f_{n'+n''-2}$ p times, or would mean sweeping out the spaces $f'_{p(n'-1)}$, and $f''_{p(n''-1)}$, or $f_{p(n'+n''-2)}$, once. Both these representations are really complementary and for a clear notion of the essence of the geometrical technique both should be kept in mind together. See now Fig.(1) in which the parallel representations are given.



(a) (b)

The left-hand diagram (a) gives the first representation and the right-hand one (b) gives the second representation. In the left-hand diagram (in which $l'_1, l'_2, \ldots l'_p$, the critical lines discussed in Section 1 constitute the flat f'_p immersed in $f'_{n'-1}$, similar critical lines

$l''_1, l''_2, \ldots . l''_p$ the flat f''_p in $f''_{n''-1}$ and $l_1, l_2, \ldots . l_p$ the flat f_p in $f_{(n'+n''-2)}$) the coordinates of Q_i $(i=1, 2, \ldots . p)$ along l_j's $(j=1, \ldots . p)$ are y_{ij} $(j=1, 2, \ldots . p)$, coordinates of Q'_i along l'_j's are y'_{ij} or $y_{ij} \sin \phi_j$ $(j=1, 2, \ldots . p)$ from section 1; similarly coordinates of Q''_i along l''_i's, are y''_{ij} or $y_{ij} \cos \phi$ $(j=1, 2, \ldots . p)$.

The right-hand diagram (b) speaks for itself. We want now to find out in the first instance the locus described by $(Q_1, Q_2, \ldots . Q_p)$ jointly in diagram (a) or—what ultimately comes to the same thing—the locus described by Q in diagram (b), subject to the critical angles ϕ_i's$(i=1, 2, \ldots . p)$ and the coordiantes y_{ij}'s $(i, j=1, 2, \ldots . p)$ of diagram (a) being kept constant. We want next to obtain the volume element described jointly by Q_i's $(i=1, 2, \ldots . p)$ of (a) or by Q of (b) subject to ϕ_i's lying between ϕ_i and $\phi_i+d\phi_i$, and y_{ij}'s between y_{ij} and $y_{ij}+dy_{ij}$. The whole thing is managed in four different stages.

In (a) one could take $(n'-1-p)$ orthogonal directions $e'_{\nu'}$ $(\nu'=1, 2, \ldots . n'-1-p)$ in $f'_{n'-1}$ all perpendicular to the flat of the critical lines l'_j's, $(j=1, 2, \ldots . p)$ and for any of these $e'_{\nu'}$ one could have p directions $e'_{i\nu'}$ $(i=1, 2, \ldots . p)$ one in each of the flats ${}_if'_{n'-1}$ in (b); similarly in (a) one could take $(n''-1-p)$ orthogonal directions $e''_{\nu''}$,$(\nu''=1, 2, \ldots . n''-1-p)$ in $f''_{n''-1}$ perpendicular to the flat of l''_j's$(j=1, 2, \ldots . p)$ for each of which again one could have p directions $e''_{i\nu''}$ $(i=1, 2, \ldots . p)$ one in each of the flats ${}_if''_{n''-1}$ in (b). Fixing upon any direction $e'_{\nu'}$ in (a) we can rotate any critical line l'_j independently in this direction through a small angle $\theta'_{j\nu'}$ $(j=1, 2, \ldots . p; \nu'=1, 2, \ldots . n'-1-p)$ keeping all the other critical lines $l'_1, l'_2, \ldots . l'_{j-1}, l'_{j+1}, \ldots . l'_p$ fixed in position with regard to the flat $f'_{n'-1}$. We could do this with each of the different critical line by turns. The volume element described by Q' in (b) would now evidently be mod $| y'_{ij} \theta'_{j\nu'} |$
or

$$\text{mod } \{ | y'_{ij} | \prod_{j=1}^{p} \theta'_{j\nu'} \} \tag{3·1}$$

where mod means absolute value, the first suffix refers to the character and the second to the critical line and $i, j=1, 2, \ldots . p$. Considering now $f''_{n''-1}$ and the critical lines l''_j's, small angles $\theta''_{j\nu''}$ $(i=1, 2, \ldots . p; \nu''=1, 2, \ldots . n''-1-p)$ and directions $e''_{\nu''}$ and $e''_{i\nu''}$ we easily find that Q'' in (b) describes a volume element

$$\text{mod } \{ | y''_{ij} | \prod_{j=1}^{p} \theta''_{j\nu''} \} \quad .. \tag{3·2}$$

Therefore from (3·1) and (3·2) it is easily seen that Q in (b) describes a volume element.

$$\text{Const. mod } \{ | y'_{ij} | \; | y''_{ij} | \prod_{j=1}^{p} \theta'_{j\nu'} \, \theta''_{j\nu''} \} \quad .. \tag{3·3}$$

This could happen with any pair of directions $e'_{\nu'}$ and $e''_{\nu''}$ in (a). It is easily seen that considering small rotations along all such possible pairs of directions $e'_{\nu'}$ and $e''_{\nu''}$ in (a), $(\nu'=1, 2, \ldots . n'-1-p; \nu''=1, 2, \ldots . n''-1-p)$ and integrating out for the small rotations the volume element described by Q in (b) would be.

$$\text{Const. mod} \{ | y'_{ij} |^{n'-p-1} \; | y''_{ij} |^{n''-p-1} \} \quad .. \tag{3·31}$$

$$\text{But } y'_{ij} = y_{ij} \sin \phi_j \text{ and } y''_{ij} = y_{ij} \cos \phi_j \quad .. \tag{3·32}$$

Hence the volume element reduces to

$$\text{Const. mod}\left\{ |y_{1j}|^{n'+n''-2p-1} \prod_{j=1}^{p} (\sin \phi_j)^{n'-1-p} (\cos\phi_j)^{n''-1-p} \right\} \quad .. \quad (3{\cdot}33)$$

or to

$$\text{Const. mod}\left\{ |y_{1j}|^{n'+n''-2p-1} \prod_{j=1}^{p} (s_j)^{n'-1-p} (c_j)^{n''-1-p} \right\} \quad .. \quad (3{\cdot}4)$$

where $s_j = \sin \phi_j$ and $c_j = \cos \phi_j$ .. (3·41)

This is the volume element described by Q in (*b*) if in (*a*) we rotate as rigid bodies the flats f'_p and f''_p in $f'_{n'-1}$ and $f''_{n''-1}$ respectively, (where f'_p and f''_p denote the flats formed by the l'_i's and the l''_i's), without permitting any internal rotation of l'_j's or l''_j's within the flats f'_p and f''_p. This is the first stage of reduction mentioned just now.

The second stage is as follows; the volume element described jointly by Q_i's$(i=1, 2, \ldots.p)$ in (*a*) or by Q in (*b*) when we give small latitudes dy_{ij}'s to the y_{ij}'s is evidently

$$\prod_{i,\,j=1}^{p} dy_{ij} \quad .. \quad (3{\cdot}5)$$

The p directions of increment of y_{ij}'s shown in diagram (*a*) will correspond to p^2 directions in (*b*) which will be evidently perpendicular to the volume element (3.4) in (*b*). This is the second stage of the reduction.

Let us now consider the third stage of reduction. Each critical axis $l_j(j=1, 2, \ldots. p)$ can now be rotated through a small angle $d\phi_j$ (in directions shown in diagram (*a*) in the critical plane formed by l'_j, l''_j, l_j. Corresponding to each such direction $d\phi_j$ shown in (*a*) there are p directions one in each of the flats $({}_i f'_{n'-1} + {}_i f''_{n''-1})$, $(i=1, 2, \ldots. p)$, shown in diagram (*b*). This means that in the space $f_{p(n'+n''-2)}$, in (*b*) we have now p^2 orthogonal axes, which are also perpendicular in the first instance to (3.4) and also to (3.5). For any direction of rotation $d\phi_j$ of l_j in (*a*), Q in (*b*) will describe a length whose projections along the p^2 orthogonal axes mentioned just now would be

$$0, 0, 0 \ldots. y_{1j}\, d\phi_j, y_{2j}\, d\phi_j, \ldots. y_{pj}\, d\phi_j, 0, \ldots. 0 \quad .. \quad (3{\cdot}51)$$

that is, zero along all the p^2 axes except those p axes, one in each of the flats $({}_i f'_{n'-1} + {}_i f''_{n''-1})$ in (*b*), that are associated with $d\phi_j$ in (*a*).

Lastly we have the fourth stage as follows; turning now to the internal rotations of l'_i's and l''_i's in f'_p and f''_p in (*a*) let us consider any pair l'_i and l'_j in f'_p of (*a*) and keeping the others fixed, rotate l'_i and l'_j as a rigid system in the plane (l'_i, l'_j) through a small angle ψ'_{ij}, the direction being from i to $j(i \neq j$ and $i < j)$. As a result $Q_m(m=1, 2, \ldots. p)$ in (*a*) will describe a length with projections along the directions $d\phi_i$ and $d\phi_j$ given by $-y'_{mj}\, c_i\, \psi'_{ij}$, $y'_{mi}\, c_j\, \psi'_{ij}$.

Accordingly Q in (*b*) will describe a length with projections along the p^2 axes of the third stage given by

$$0, 0, \ldots. -y'_{1j}\, c_i\, \psi'_{ij}, -y'_{2j}\, c_i\, \psi'_{ij} \ldots. -y'_{pj}\, c_i\, \psi'_{ij}, 0, 0, \ldots. y'_{1i}\, c_j\, \psi'_{ij} \ldots. y'_{pi}\, c_j\, \psi'_{ij}, 0, 0 \ldots. 0. \quad .. \quad (3{\cdot}52)$$

where the c_i's are given by (3·41).

This happens with any pair of axes l'_i and l'_j in f'_p of (a) ($i \neq j$; $i, j=1, 2, \ldots. p$) leading to a motion of Q of (b) with projections along the p^2 axes, given by elements similar to those in (3·8).

Likewise with l''_i and l''_j in f''_p of (a) and with a small angle of rotation ψ''_{ij} we have an exactly similar state of things.

Moreover due to these internal rotations ψ'_{ij} the lengths described by Q in (b) will have projections also in the p^2 directions in (b) referred to in the second stage of reduction, the projections corresponding to ψ'_{ij} being

$$0, 0, \ldots. 0, -y'_{1j}\, s_i\, \psi'_{ij}, -y'_{2j}\, s_i\, \psi'_{ij}, \ldots. -y'_{pj}\, s_i\, \psi'_{ij}, 0, \ldots. 0, y'_{1i}\, s_j\, \psi'_{ij}, \ldots. y'_{pi}\, s_j\, \psi'_{ij}, 0, \ldots. 0 \quad .. \quad (3·53)$$

and there being similar projections corresponding to ψ''_{ij}. But from (3.32) and (3.41) these could be easily shown to contribute nothing to the volume element given by (3·5). Hence remembering from (2·6) and (3·41) that

$$y'_{ij}=y_{ij}\, s_j \text{ and } y''_{ij}=y_{ij}\, c_j \quad .. \quad (3·54)$$

we easily see that the volume element described by Q in (b) owing to rotations $d\phi_j (j=1, 2, \ldots. p)$ and the internal rotations ψ'_{ij}, ψ''_{ij} in f'_p and f''_p of (a) will be given by

$$\prod_{i=1}^{p} d\phi_i \prod_{j=1}^{p} \prod_{i=1}^{j-1} \psi'_{ij}\, \psi''_{ij} \times \text{modulus of (A)}, \quad .. \quad (3·55)$$

(A) being given on p. 33.

Summing up, that is, integrating over ψ'_{ij}'s and ψ''_{ij}'s and putting

$$t_j=\tan\, \phi_j \quad .. \quad (3·56)$$

we could reduce (3.74) to

$$\text{Const.} \prod_{i=1}^{p} d\phi_i \prod_{i=1}^{p} (c_i)^{2(p-1)} \times \text{modulus of (B)}, \quad .. \quad (3·57)$$

(B) being given on p. 33.

This is a determinant of order p^2 giving a volume element of p^2 dimensions. Manipulating the $(p+1)^{th}$ row with $(2p+1)^{th}$ row we can take out a factor $(t^2_1-t^2_2)$, the $(p+2)^{th}$ row with $(2p+2)^{th}$ row we can take out a factor $(t^2_1-t^2_3)$, and so on till we get to $(t^2_1-t^2_p)$; similarly manipulating the $(2p+1)^{th}$ row with $(3p+1)^{th}$ row we can take out $(t^2_2-t^2_3)$ and so on till we get to $(t^2_2-t^2_p)$; thus we ultimately get on to the factor $(t^2_{p-1}-t^2_p)$ which also we take out. Then we open out the resulting determinant after Laplace and finally we easily reduce (3·76) finally to

$$\text{Const.}\ |\, y_{ij}\, |^{p}.\ \text{mod}\ \{(t^2_1-t^2_2) \ldots. (t^2_1-t^2_p)\ (t^2_2-t^2_3), \ldots. (t^2_2-t^2_p) \ldots. (t_{p-1}-t^2_p)\} \times \prod_{i=1}^{p} d\phi_i\ (c_i)^{2(p-1)} \quad (3·6)$$

The volume elements (3·4), (3·5) and (3·6) described by Q in (b) are all orthogonal and hence Q describes in the space $f_{p(n'+n''-2)}$ of $p(n'+n''-2)$ dimensions a volume element.

$$\text{Const. mod}\ \{|y_{ij}|^{n'+n''-p-2}\} \prod_{i,j=1}^{p} dy_{ij} \prod_{j=1}^{p} (s_j)^{n'-1-p}\ (c_j)^{n''-p-2}$$

$$\times\ \text{mod}\ \{(t^2_1-t^2_2) \ldots. (t^2_1-t^2_p)\ (t^2_2-t^2_3) \ldots. (t^2_2-t^2_p) \ldots. (t^2_{p-1}-t^2_p)\} \prod_{i=1}^{p} d\phi_i \quad .. \quad (3·7)$$

4. THE JOINT DISTRIBUTION OF THE p-STATISTICS

The joint distribution of the y_{ij}'s and ϕ_i's (i, $j=1, 2, \ldots p$) can now be written from (2·8) and (3·7) in the form

$$\text{Const.}\ e^{-\frac{1}{2}\sum\limits_{i,j=1}^{p} y^2_{ij}\left(\frac{s^2_j}{\kappa_i^2}+c^2_j\right)}$$

$$\times \bmod\{|y_{ij}|^{n'+n''-p-2}\}\cdot \prod_{i,j=1}^{p} dy_{ij} \prod_{j=1}^{p} (s_j)^{n'-1-p}(c_j)^{n''-p-2}\, d\phi_j$$

$$\times \bmod\{(t^2_1-t^2_2)\ldots\ldots(t^2_1-t^2_p)\,(t^2_2-t^2_3)\ldots\ldots(t^2_2-t^2_p)\ldots\ldots(t^2_{p-1}-t^2_p)\} \quad \ldots (4\cdot 1)$$

where we have

$$s_i=\sin\phi_i;\ c_i=\cos\phi_i;\ t_i=\tan\phi_i$$

and from (1·7) the p-Statistics $k_i(i=1, 2, \ldots p)$ are given by

$$\left.\begin{aligned} k_i&=t_i\sqrt{\frac{(n''-1)}{(n'-1)}}=t_i/\lambda \\ \text{with}\qquad \lambda^2&=(n'-1)/(n''-1) \end{aligned}\right\} \quad \ldots (4\cdot 2)$$

From (4·2) we have

$$\left.\begin{aligned} s_i&=\frac{\lambda k_i}{(1+\lambda^2 k^2_i)^{1/2}}\ ;\ c_i=\frac{1}{(1+\lambda^2 k^2_i)^{1/2}}\ ;\ t_i=\lambda k_i \\ d\phi_i&=\frac{\lambda k_i}{(1+\lambda^2 k^2_i)} \end{aligned}\right\} \quad \ldots (4\cdot 3)$$

Changing over from ϕ'_is and t'_is to k'_is, (4·1) now reduces to

$$\text{Const.}\ e^{-\frac{1}{2}\sum\limits_{i,j=1}^{p} y^2_{ij}(\kappa^2_i+\lambda^2 k^2_j)/(1+\lambda^2 k^2_j)} \bmod\{|y_{ij}|^{n'+n''-p-2}\} \prod_{i,j=1}^{p} dy_{ij}$$

$$\times \prod_{j=1}^{p} \frac{k_j^{n'-p-1}\, dk_j}{(1+\lambda^2 k^2_j)^{(n'+n''-2)/2}}$$

$$\times \bmod\{(k^2_1-k^2_2)\ldots\ldots(k^2_1-k^2_p)\,(k^2_2-k^2_3)\ldots\ldots(k^2_2-k^2_p)\ldots\ldots(k^2_{p-1}-k^2_p)\} \quad \ldots (4\cdot 4)$$

where the k_i's are roots of the determinantal equation (1·3) and the κ_i's are the roots of the determinantal equation (1·4).

The y_{ij}'s as observed earlier vary from $-\infty$ to $+\infty$. To get the joint distribution of the k'_js we have to intergate out

(4.4) over y_{ij}'s ($i, j=1, 2, \ldots p$) from $-\infty$ to $+\infty$

The integration can be managed comparatively easily if in (4.4) the exponent of $|y_{ij}|$ which is $(n'+n''-p-2)$ happens to be even, say, 2N where N is a positive integer.

Then putting

$$2N=n'+n''-p-2 \quad \ldots (4\cdot 5)$$

we observe that mod $\{|y_{ij}|^{2N}\}$ is the same as $|y_{ij}|^{2N}$. To effect now the integration of (4·4) over y_{ij}'s we notice that if in (4·4) we open out $|y_{ij}|^{2N}$ as a multinomial in y_{ij}'s then any even power of y_{ij} in any term of the expansion, say y^{2m}_{ij} (m being a positive integer) will yield after integration a factor given by

$$2^{\frac{2m+1}{2}}\{(1+\lambda^2 k^2_j)/(\kappa^2_i+\lambda^2 k^2_j)\}^{\frac{2m+1}{2}}\Gamma\left(\frac{2m+1}{2}\right) \quad .. \quad (4{\cdot}6)$$

On the other hand any odd power of y_{ij} in any term of the expansion, say y_{ij}^{2m+1} (m being a positive integer) will contribute a factor zero. This is because

$$\int_{-\infty}^{+\infty} e^{-ax^2} x^{2m}\, dx = a^{-\frac{(2m+1)}{2}}\Gamma\left(\frac{2m+1}{2}\right); \quad \int_{-\infty}^{\infty} e^{-ax^2} x^{2m+1}\, dx = 0 \quad .. \quad (4{\cdot}61)$$

where in both cases m is supposed to be a positive integer. Hence if we now introduce new quantities ζ_{ij} such that

$$\left.\begin{aligned} &\zeta_{ij}^{2m}=0\,;\ \zeta_{ij}^{2m+1}=2^{\frac{2m+1}{2}}\Gamma\left(\frac{2m+1}{2}\right); \\ &\zeta_{ij}^{2m+1}\ \zeta_{i'j'}^{2n+1}=2^{\frac{m+n+1}{2}}\ \Gamma\left(\frac{2m+1}{2}\right)\ \Gamma\left(\frac{2n+1}{2}\right), \text{ when } i\neq i';\ j\neq j' \end{aligned}\right\}$$

and then integrate out (4·4) over y'_{ij}s from $-\infty$ to $+\infty$ (for the case where the exponent $n''+n'-p-2$ of $|y_{ij}|$ is even $=2N$, say) then remembering (4·6), (4·61) and (4·7) we can write the joint distribution of k'_is in the form

$$\text{Const. } D^{2N} \prod_{i,\, j=1}^{p} \frac{\zeta_{ij}}{(\kappa^2_i+\lambda^2 k^2_j)^{1/2}} \prod_{j=1}^{p} k_j^{n-p-1}\, dk_j$$
$$\times \bmod\{(k^2_1-k^2_2)\ldots.(k^2_1-k^2_p)\,(k^2_2-k^2_3)\ldots.(k^2_2-k^2_p)\ldots.(k^2_{p-1}-k^2_p)\} \quad .. \quad (4{\cdot}8)$$

where D is the determinant

$$|\zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}| \quad .. \quad (4{\cdot}81)$$

Except for minor changes in notation and one slip in printing made in the earlier notice this was the form of the distribution on the non-null hypothesis of p-statistics[3] announced early in 1940.

It should be noticed with reference to the form (4·8) that in it we can make the substitutions for ζ_{ij}^{2m}, $\zeta_{i'j'}^{2m+1}$ etc. given by (4·7), only after we have broken up $D^{n''+n'-p-2}$ *i.e.* D^{2N} as a multinomial and have multiplied it with the other factors. We cannot make the substitutions before opening out D^{2N} or $D^{n''+n'-p-2}$. (4·8) can be expressed in several other alternative forms some of which appear to be more convenient than (4·8) itself and will be given in the next paper. One of these forms (which incidentally provides the clue to the solution of the problem when the exponent of $|y_{ij}|$ is an odd integer, say, $(2N+1)$ will be considered here.

It is easily seen from (4·7) that in (4·8)

$$D^{2N} \prod_{i,\, j=1}^{p} \zeta_{ij} \ \textit{i.e. } |\zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}|^{2N}$$

can be written in any of the p alternative forms $(i=1, 2, \ldots p)$

$$2^{\frac{2N-1}{2}}\; 2^{\frac{p-1}{2}}\, \Gamma\left(\frac{2N+1}{2}\right) 1^{p-1}(\tfrac{1}{2}) \left\{ \sum_{j=1}^{p} \frac{D^2_{ij}}{(\kappa^2_i+\lambda^2 k^2_j)} \right\}^N \Pi\, \zeta_{i'j} \qquad .. \quad (4\cdot82)$$

where the product Π is taken over all values of j from 1 to p and over all values of i' from 1 to p except i, and further where D_{ij} is the co-factor of $\zeta_{ij}|(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}$ in the determinant D; in particular putting $i=1$ and absorbing the constant factor in the const. (4·8) can be written in the form

$$\text{Const.} \left\{ \sum_{j=1}^{p} \frac{D^2_{1j}}{(\kappa^2_1+\lambda^2 k^2_j)} \right\}^N \prod_{j=1}^{p} k_j^{n'-p-1}\, dk_j$$
$$\times \operatorname{mod}\{(k^2_1-k^2_2)\ldots.(k^2_1-k^2_p)\ (k^2_2-k^2_3)\ldots.(k^2_2-k^2_p)\ldots.(k^2_{p-1}-k^2_p)\}$$
$$\times \prod_{i'}\prod_{j=1}^{p} (\kappa^2_{i'}+\lambda^2 k^2_j)^{-\frac{1}{2}} \prod_{i=2}^{p}\prod_{j=1}^{p} \zeta_{ij} \qquad .. \quad (4\cdot83)$$

where D_{1j} is the cofactor of $\zeta_{1j}/(\kappa^2_1+\lambda^2 k^2_j)^{1/2}$ in the determinant

$$|\zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}|$$

(*ii*) Let us next consider the case where in (4·4) the exponent $n'+n''-p-2$ of $|y_{ij}|$ is odd and equal to, say, $2N+1$. The integration over the variables y_{ij} in any row of the determinant $|y_{ij}|$ can be effected either by geometrical or by algebraic methods, the former, however, appearing to be much the easier of the two. The geometrical method may be indicated as follows. $|y_{ij}|$ can be also expressed as

$$|y_{ij}(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{1/2}| \times \prod_{j=1}^{p} (1+\lambda^2 k^2_j)^{\frac{1}{2}}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}},$$

by multiplying and dividing the jth column of $|y_{ij}|$ by $(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{\frac{1}{2}}(j=1,2,\ldots p)$. Each row of $|y_{ij}(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{\frac{1}{2}}|$, say, the ith row can be regarded as a vector $\boldsymbol{u}_i$ with components $y_{ij}(\kappa^2+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{\frac{1}{2}}(j=1, 2, \ldots.\, p)$; let u_i denote the magnitude of the vector $\boldsymbol{u}_i$.

We have thus

$$\operatorname{mod}\{|y_{ij}|^{2N+1}\}=\operatorname{mod}|y_{ij}|^{2N+1}= \prod_{j=1}^{p} (1+\lambda^2 k^2_j)^{\frac{2N+1}{2}}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{2N+1}{2}}$$
$$\times\{\operatorname{mod}|y_{ij}(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{\frac{1}{2}}|\}^{2N+1} \qquad .. \quad (4\cdot831)$$

But mod $|y_{ij}\ldots|$ is really the volume of the hyper-par^l formed by the vectors $\boldsymbol{u}_i(i=1, 2, \ldots p)$ and is conveniently written as Vol. $(\boldsymbol{u}_1, \boldsymbol{u}_2\ldots\boldsymbol{u}_p)$. The factor in (4.4) involving only the y_{ij}'s can now be written as

$$\text{Const.}\ e^{-\frac{1}{2}\{u^2_1+ \sum_{i=2}^{p}\sum_{j=1}^{p} y^2_{ij}(\kappa^2_i+\lambda^2 k^2_j)/(1+\lambda^2 k^2_j)\}} \prod_{j=1}^{p} (1+\lambda^2 k^2_j)^{\frac{2N+2}{2}}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{2N+2}{2}}$$

$$\times\{\text{Vol.}\ (\boldsymbol{u}_1, \boldsymbol{u}_2\ \ldots.\ \boldsymbol{u}_p)\}^{2N+1} \prod_{j=1}^{p} d(y_{1j}(k^2_1+\lambda^2 k^2_j)^{\frac{1}{2}}/(1+\lambda^2 k^2_j)^{\frac{1}{2}}) \times \prod_{i=1}^{p}\prod_{j=1}^{p} dy_{ij} \ ..\ (4\cdot84)$$

If θ be the angle made by u_1 with flat formed by the vectors $(u_2 \ldots u_p)$ then Vol. $(u_1, u_2 \ldots u_p)$ can be written as $u_1 \sin\theta$. Vol. $(u_2, u_3 \ldots u_p)$, and the volume element in (4·84) can be written as

$$u_1^{p-1}\, du_1 (\cos\theta)^{p-2}\, d\theta \prod_{i=2}^{p}\prod_{j=1}^{p} dy_{ij} \tag{4·85}$$

Here u_1 varies from 0 to ∞ and θ from $-\pi/_2$ to $+\pi/_2$. Hence integrating out (4·84) over u_1 and θ between the limits mentioned and absorbing the integrated part (involving gamma-functions) within the constant we find that (4·84) reduces to

$$\text{Const.}\ e^{-\frac{1}{2}\sum_{i=2}^{p}\sum_{j=1}^{p} y^2_{ij}(\kappa^2_i+\lambda^2 k^2_j)/(1+\lambda^2 k^2_j)} \prod_{j=1}^{p}(1+\lambda^2 k^2_j)^{\frac{2N-2}{2}}/(\kappa^2_1+\lambda^2 k_j^2)^{\frac{2N,2}{2}}$$

$$\times \{\text{Vol}\ (u_2, u_3 \ldots u_p)\}^{2N'+1} \prod_{i=2}^{p}\prod_{j=1}^{p} dy_{ij} \tag{4·86}$$

Now from the definition of the vectors $(u_2, u_3 \ldots u_p)$ it is easily seen that

$$\{\text{Vol}\ (u_2, u_3 \ldots u_p)\}^{2N'+1} \prod_{j=1}^{p}(1+\lambda^2 k^2_j)^{\frac{2N,2}{2}}/(\kappa^2_1+\lambda^2 k^2_j)^{\frac{2N,4}{2}}$$

comes out as

$$\left\{\sum_{j=1}^{p} Y^2_{1j}(1+\lambda^2 k^2_j)/(\kappa^2_1+\lambda^2 k^2_j)\right\}^{\frac{2N+1}{2}} \prod_{j=1}^{p}(1+\lambda^2 k^2_j)^{\frac{1}{2}}/(\kappa^2_1+\lambda^2 k^2_j)^{\frac{1}{2}} \quad \ldots \tag{4·87}$$

where Y_{1j} is the cofactor of y_{1j} in the determinant $|y_{ij}|$. (4·4) after integration over y_{1j}'s $(j=1, 2, \ldots p)$ thus altogether reduces to

$$\text{Const.}\ e^{-\frac{1}{2}\sum_{i=2}^{p}\sum_{j=1}^{p} y^2_{ij}(\kappa^2_i+\lambda^2 k^2_j)/(1+\lambda^2 k^2_j)} \left\{\sum_{j=1}^{p} Y^2_{1j}(1+\lambda^2 k^2_j)\,|\,(\kappa^2_1+\lambda^2 k^2_j)\right\}^{\frac{2N+1}{2}}$$

$$\times \prod_{j=1}^{p}\frac{k_j^{n'-p-1}dk_j}{(1+\lambda^2 k_j^2)^{\frac{n'+n''-2}{2}}} \prod_{j=1}^{p}(1+\lambda^2 k^2_j)^{\frac{1}{2}}/(\kappa^2_1+\lambda^2 k^2_j)^{\frac{1}{2}} \prod_{i=2}^{p}\prod_{j=1}^{p} dy_{ij}$$

$$\times \text{mod}\ \{(k^2_1-k^2_2)\ldots(k^2_1-k^2_p)(k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p)\} \quad \ldots \tag{4·88}$$

We can now straightaway integrate out over y_{ij}'s from $-\infty$ to $+\infty$. After integration of (4·88) over y_{ij}'s $(j=1, 2, \ldots p;\ i=2, 3, \ldots p)$ we obtain the distribution of k_j's which on using the same symbols as were introduced earlier, can be written in the compact form

$$\text{Const.}\ \left\{\sum_{j=1}^{p} D^2_{1j}/(\kappa^2_1+\lambda^2 k^2_j)\right\}^{-\frac{2N+1}{2}} \prod_{j=1}^{p} k_j^{n'-p-1}\, dk_j$$

$$\times \text{mod}\,\{(k^2_1-k^2_2)\ldots(k^2_1-k^2_p)\ (k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p)\}$$

$$\times \prod_{i,j=1}^{p}(\kappa^2_i+\lambda^2 k^2_j)^{-\frac{1}{2}} \prod_{i=2}^{p}\prod_{j=1}^{p}\zeta_{ij} \tag{4·89}$$

where as before D_{1j} is the cofactor of $\zeta_{1j}/(\kappa^2_1+\lambda^2 k^2_j)^{\frac{1}{2}}$ in the determinant, $|\zeta_{ij}|(\kappa^2_1+\lambda^2 k^2_j)^{\frac{1}{2}}|$ and ζ_{ij}'s are given by (4·7).

Let us now look a little more closely into the forms (4·8), (4·83) which correspond to the exponent $n'+n''-p-2$ of (4.4) being an even integer, and into the form (4·89) which corresponds to the exponent being an odd integer. With the explanation of the symbols ζ_{ij} given by (4·7), the joint distribution of p-statistics k_i's for the case of even exponent $(n'+n''-p-2=2N)$ given by (4·8) or (4·83) are perfectly concrete expressions; in fact it is only a matter of algebra to write down the general term in (4·8) in the expansion of

$$|\zeta_{ij}/(\kappa^2+\lambda^2 k^2_j)^{\frac{1}{2}}|^{2N} \prod_{i,j=1}^{p} \zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}} \qquad (4·9)$$

or the general term in (4·83) in the expansion of

$$\left\{ \sum_{j=1}^{p} \frac{D^2_{ij}}{(\kappa^2_i+\lambda^2 k^2_j)} \right\}^N \prod_{i=1}^{p} \prod_{j=1}^{p} \zeta_{ij} \prod_{i,j=1}^{p} (\kappa^2_i+\lambda^2 k^2_j)^{-\frac{1}{2}} \qquad .. \quad (4·91)$$

D_{ij} being the cofactor of $\zeta_{ij}/(\kappa^2+\lambda^2 k^2_j)^{\frac{1}{2}}$ in the determinant

$$|\zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^{\frac{1}{2}}|$$

For the case of odd exponent $(n'+n''-p-2=2N+1)$, however, the distribution of the p-statistics k_i's given by (4·89) with the symbols explained by (4·7) comes out as an expression which is more symbolic and less directly accessible than either (4·8) or (4·83). It is now a case of expanding

$$\left\{ \sum_{j=1}^{p} \frac{D^2_{ij}}{(\kappa^2_i+\lambda^2 k^2_j)} \right\}^{\frac{2N+1}{2}} \prod_{i=1}^{p} \prod_{j=1}^{p} \zeta_{ij} \prod_{i,j=1}^{p} (\kappa^2_i+\lambda^2 k^2_j)^{-\frac{1}{2}} \qquad .. \quad (4·92)$$

not as a straight multinomial but as an infinite series which has to be carefully handled. This is due to the exponent being now $(2N+1)/2$ which involves a half-integer.

As observed earlier the expressions (4·9), (4·91) and (4·92) could be thrown into more convenient forms. As a matter of fact by using the properties of ζ_{ij} 's given by (4·7) we can develop very interesting properties of their functions occurring here and thereby express (4·9), (4·91), (4·92) and hence (4·8), (4·83) and (4·84) in various other familiar forms which for purposes of statistical analysis are more directly useful than (4·8), (4·83) and (4·98) themselves. This will be given in detail in the next paper.

One or two features about the forms (4·9)—(4·92) are, however, worth noticing even at this stage. From (4·7) and from considerations of symmetry it is evident that except for a common factor which for (4·9) and (4·92) can be written respectively in the form

$$\left\{ \prod_{i,j=1}^{p} (\kappa^2_i+\lambda^2 k^2_j) \right\}^{-\frac{1}{2}}$$

or

$$\prod_{i,j=1}^{p} (\kappa^2_i+\lambda^2 k^2_j),$$

(4·9)—(4·92) can each be expressed as a sum of terms involving only integral powers of $(\kappa^2_i+\lambda^2 k^2_j)$; no half-integral powers would occur. This and the nature of the forms

(4·9)—(4·92), where both the sets κ_i's, and k_j's occur in a symmetrical manner, ensure that in the forms (4·8), (4·84) and (4·89) there occur only elementary symmetric functions of κ^2_i's and k^2_is of the forms

$$\left.\begin{array}{l} \sum\limits_{i=1}^{p} \kappa^2_i\,;\ \sum\limits_{i=1}^{p} k^2_i\,;\ \sum\limits_{i,j=1}^{p} \kappa^2_i\,\kappa^2_j\,;\ \sum\limits_{i,j=1}^{p} k^2_i\,k^2_j (i\neq j) \\ \sum\limits_{i,j,l=1}^{p} \kappa^2_i\,\kappa^2_j\,\kappa^2_l\,;\ \sum\limits_{i,j,l=1}^{p} k^2_i\,k^2_j\,k^2_l (i\neq j\neq l) \\ \ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \\ \kappa^2_1\ \kappa^2_2 \ldots\ldots \kappa^2_p\,;\ k^2_1\ k^2_2 \ldots\ldots k^2_p \end{array}\right\} \qquad .. \quad (4·93)$$

This is, of course, apart from the factor

$$\text{mod}\ \{(k^2_1-k^2_2)\ldots(k^2_1-k^2_p)(k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p)\}.$$

In the case where the population variances and covariances are identical, that is $\alpha_{ij}=\alpha'_{ij}$ $(i, j=1, 2, \ldots p)$ it has already been observed in Section 1 that all the κ'_is are unity Now putting $\kappa_i=1 (i=1, 2, \ldots p)$ in the forms (4·9) (4·91) and (4·92) it is easily seen that in (4·9) $|\zeta_{ij}/(\kappa^2_i+\lambda^2 k^2_j)^1|$ becomes

$$\text{Const.} \prod_{i=1}^{p} (1+\lambda^2 k^2_i)^{-1}$$

and hence (4·9) reduces to

$$\text{Const.} \prod_{i=1}^{p} (1+\lambda^2 k^2_i)^{2N'p} \text{ or const.} \prod_{i=1}^{p} (1+\lambda^2 k^2_i)^{\frac{n'+n''-2}{2}}. \qquad .. \quad (4·94)$$

(4·91) and (4·92) also can be easily shown to be reduced to the same form (4·94) and hence all the distributions (4·8), (4·83) and (4·89) reduce to the common form

$$\text{Const.} \prod_{i=1}^{p} \frac{k_i^{n'-p-1}\, dk_i}{(1+\lambda^2 k^2_i)^{\frac{n'+n''-2}{2}}}$$

$$\text{mod}\ \{(k^2_1-k^2_2)\ldots(k^2_1-k^2_p)\ (k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p)\} \qquad .. \quad (4·95)$$

It is clear that apart from the factor mod ( ), (4.95) involves only symmetric functions of k_i's defined in (4·93).

As observed earlier the k_i's can vary from 0 to ∞. If now the following restriction is imposed upon the k_i's

$$\infty \geqslant k_1 \geqslant k_2 \geqslant k_3 \geqslant \ldots \geqslant k_p \geqslant 0 \qquad .. \quad (4·96)$$

then in (4·95) we can drop the mod and write (4·95) in the form

$$\text{Const.} \prod_{i=1}^{p} \frac{k_i^{n'-p-1}\, dk_i}{(1+\lambda^2 k^2_i)^{\frac{n''+n'-2}{2}}}$$

$$\times(k^2_1-k^2_2)\ldots(k^2_1-k^2_p)\ (k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p) \qquad .. \quad (4·97)$$

This agrees as, of course, it should with the distribution of p-statistics on the null-hypothesis given in the earlier paper.[4]

5. Distribution of Certain Allied Statistics on the Null and Non-null Hypotheses

For purposes of discrimination between and classification of multivariate normal populations by means of random samples (supposed to have been drawn from them) symmetric functions (defined in 4·03) of κ^2_i's and k^2_i's appear to be more useful than κ^2_i's and k^2_i's themselves, at any rate from a certain point of view.

The distribution of the symmetric functions of k^2_i's defined in (4·03) can be obtained in the following manner.

Let us take quantities ϖ_i ; w_i defined as follows

$$\left.\begin{array}{ll}
\varpi_1 = \sum\limits_{i=1}^{p} \kappa^2_i \,; & w_1 = \sum\limits_{i=1}^{p} k^2_i \\
\varpi_2 = \sum\limits_{i,\, j=1}^{p} \kappa^2_i\, \kappa^2_j \,; & w_2 = \sum\limits_{i,\, j=1}^{p} k^2_i\, k^2_j (i \neq j) \\
\varpi_3 = \sum\limits_{i,\, j,\, l=1}^{p} \kappa^2_i\, \kappa^2_j\, \kappa^2_l \,; & w_3 = \sum\limits_{i,\, j,\, l=1}^{p} k^2_i\, k^2_j\, k^2_l (i \neq j \neq l) \\
\ldots\ldots & \ldots\ldots \\
\varpi_p = \kappa^2_1\, \kappa^2_2 \ldots\ldots \kappa^2_p \,; & w_p = k^2_1\, k^2_2 \ldots\ldots k^2_p
\end{array}\right\} \qquad \ldots (5\cdot 1)$$

In (4·8), (4·83), (4·80) and (4·05) let us change over from k_i's to the new set of variables w_1, $w_2 \ldots . w_p$ defined in (5·1)

The Jacobian of the transformation

$$\frac{\partial(k_1, k_2, \ldots . k_p)}{\partial(w_1, w_2, \ldots . w_p)} = 1 / \frac{\partial(w_1, w_2, \ldots . w_p)}{\partial(k_1, k_2, \ldots . k_p)}$$

Now

$$\frac{\partial(w_1, w_2, \ldots . w_p)}{\partial(k_1, k_2, \ldots . k_p)}$$

can be easily shown to be

$$\text{mod. } \{(k^2_1 - k^2_2) \ldots . (k^2_1 - k^2_p)\, (k^2_2 - k^2_3) \ldots . (k^2_2 - k^2_p) \ldots . (k^2_{p-1} - k^2_p)\} \qquad \ldots (5\cdot 2)$$

Further, as has been observed towards the end of section 4, (4·01), (4·02) and (4·93) are all really functions of w_i and ϖ_i $(i = 1, 2, \ldots . p)$. Hence the distributions (4·8), (4·83) and (4·89) and (4·05) transform respectively to

$$\text{(i)} \quad \text{Const.}\ \left|\zeta_{ij}/(\kappa^2_i + \lambda^2 k^2_j)^{\frac{1}{2}}\right|^{n'+n''-p-2} \prod_{i,\, j=1}^{p} \zeta_{ij} (\kappa^2_i + \lambda^2 k^2_j)^{-1} \times w_p^{\frac{n'-p-2}{2}} \prod_{i=1}^{p} dw_i \qquad \ldots (5\cdot 3)$$

$$\text{(ii)} \quad \text{Const.} \left\{ \sum_{j=1}^{p} \frac{D^2_{ij}}{(\kappa^2_i + \lambda^2 k^2_j)} \right\}^{\frac{n'+n''-p-2}{2}} \prod_{i=1}^{p} \prod_{j=2}^{p} \zeta_{ij} \prod_{i,\, j=1}^{p} (\kappa^2_i + \lambda^2 k^2_j)^{-1} \times w_p^{\frac{n'-p-2}{2}} \prod_{i=1}^{p} dw_i \qquad \ldots (5\cdot 4)$$

(iii) Const. (Same as 5·4) (5·5)

$$\text{(iv)} \quad \text{Const.}\ w_p^{\frac{n'-p-2}{2}} \prod_{i=1}^{p} \frac{dw_i}{(1 + \lambda^2 k^2_i)^{\frac{n'+n''-2}{2}}} \qquad (5\cdot 6)$$

(A) of (3·55) on p. 25

$$\begin{array}{cccccccccccccccc}
y_{11} & y_{21} & \cdot & y_{p1} & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & 0 & y_{12} & y_{22} & \cdot & y_{p2} & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & y_{1p} & y_{2p} \;\cdot & y_{pp} \\
-y_{11}c_2s_1 & -y_{21}c_2s_1 & \cdot & -y_{p1}c_2s_1 & y_{11}c_2s_1 & y_{21}c_2s_1 & & y_{p1}c_2s_1 & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-y_{1p}c_1s_p & -y_{2p}c_1s_p & \cdot & -y_{pp}c_1s_p & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & y_{11}c_ps_1 & y_{21}c_ps_1 \;\cdot & y_{p1}c_ps_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots 0 & -y_{1p}c_{p-1}s_p & -y_{2p}c_{p-1}s_p & \cdot & -y_{pp}c_{p-1}s_p & y_{1,p-1}c_ps_{p-1} & y_{2,p-1}c_ps_{p-1} \;\cdot & y_{p,p-1}c_ps_{p-1} \\
y_{12}c_2s_1 & y_{22}c_2s_1 & \cdot & y_{p2}c_2s_1 & -y_{11}c_1s_2 & -y_{21}c_1s_2 & \cdot & -y_{p1}c_1s_2 & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{1p}c_ps_1 & y_{2p}c_ps_1 & \cdot & y_{pp}c_ps_1 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & -y_{11}c_1s_p & -y_{21}c_1s_p \;\cdot & -y_{p1}c_1s_p \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & y_{1p}c_ps_{p-1} & y_{2p}c_ps_{p-1} & \cdot & y_{pp}c_ps_{p-1} & -y_{1,p-1}c_{p-1}s_p & -y_{2,p-1}c_{p-1}s_p \;\cdot & -y_{p,p-1}c_{p-1}s_p
\end{array}$$

(B) of (3·57) on p. 25

$$\begin{array}{cccccccccccccccc}
y_{11} & y_{21} & \cdot & y_{p1} & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & 0 & y_{12} & y_{22} & \cdot & y_{p2} & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & y_{1p} & y_{2p} \;\cdot & y_{pp} \\
-y_{11}l_2 & -y_{21}l_2 & \cdot & -y_{p1}l_2 & y_{11}l_1 & y_{21}l_1 & \cdot & y_{p1}l_1 & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-y_{1p}l_p & -y_{2p}l_p & \cdot & -y_{pp}l_p & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & y_{11}l_1 & y_{21}l_1 \;\cdot & y_{p1}l_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots 0 & -y_{1p}l_p & -y_{2p}l_p & \cdot & -y_{pp}l_p & y_{1,p-1}l_{p-1} & y_{2,p-1}l_{p-1} \;\cdot & y_{p,p-1}l_{p-1} \\
y_{12}l_1 & y_{22}l_1 & \cdot & y_{p2}l_1 & -y_{11}l_2 & -y_{21}l_2 & \cdot & -y_{p1}l_2 & 0 \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{1p}l_1 & y_{2p}l_1 & \cdot & y_{pp}l_1 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0 & -y_{11}l_p & -y_{21}l_p \;\cdot & -y_{p1}l_p \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & y_{1p}l_{p-1} & y_{2p}l_{p-1} & \cdot & y_{pp}l_{p-1} & -y_{1,p-1}l_p & -y_{2,p-1}l_p \;\cdot & -y_{p,p-1}l_p
\end{array}$$

It should be noticed that factors in (5·3), (5·4) and (5·5) involving κ^2_i's and k^2_i's are really functions of ω_i, w_i ($i=1, 2, \ldots p$) and the factors in (5·6) invloving k^2_i's are really functions of w_i ($i=1, 2, \ldots p$).

Hence (5·3)—(5·6) are really expressible as explicit functions of ω_i and w_i ($i=1, 2, \ldots p$). Such explicit expressions will be given in the next paper.

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