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A note on linear complementarity problems and multiple objective programming

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Abstract. Kostreva and Wiecek [3] introduced a problem called LCP-related weighted problem in connection with a multiple objective programming problem, and suggested that a given linear complementarity problem (LCP) can be solved by solving the LCP-related weighted problem associated with it. In this note we provide several clarifications of the claims made in [3]. Finally, we feel that solving any LCP by the approach given in [3] may not be as useful as it is claimed.

Key words. linear complementarity problem – lemke's algorithm – LCP-related weighted problem – multiple objective programming problem

1. Introduction

Given a square matrix $M \in R^{n\times n}$ and a vector $q \in R^n$ the linear complementarity problem (denoted by LCP(q, M)) is to find vectors $w,\ z \in R^n$ such that

$$w - Mz = q, \quad w \geq 0,\ z \geq 0, \tag{1.1}$$

$$w^t z = 0. \tag{1.2}$$

If a pair of vectors (w, z) satisfies (1.1), then the problem LCP(q, M) is said to be feasible. A pair (w, z) of vectors satisfying (1.1) and (1.2) is called a solution to the LCP(q, M). This problem is well studied in the literature over the years. This problem arises in some mathematical programming problems, game theory, control theory, economics and some engineering applications. For the recent books on LCP theory and its applications see Cottle, Pang and Stone [2] and Murty [8]. The algorithm presented by Lemke and Howson [6] to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke [5] to solve a LCP(q, M) contributed significantly to the development of the linear complementarity theory. However, this algorithm does not solve every instance of the linear complementarity problem and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists.

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A multiple objective programming problem may be stated as follows:

$$\text{Minimize } f(x), \tag{1.3}$$
$$\text{subject to } x \in \mathcal{X} \tag{1.4}$$

where $f(x) = [f_1(x), f_2(x), \dots, f_m(x)]^t$, $m \geq 2$ is a vector valued function, $f_i : R^n \to R \quad \forall\ i = 1, 2, \dots, m$ and the feasible set $\mathcal{X} = \{x \in R^n \mid g_i(x) \geq 0\ \forall\ i = 1, 2, \dots, m\}$. Multiple objective programming problems (MOP) arise in different branches of science and technology, economics, game theory etc. In [3], Kostreva and Wiecek claim that they prove certain results which may be considered as a bridge between LCP and a class of MOP so that certain ideas of solving MOP can be effectively used for solving LCP and conversely. However, we present examples for clarification of the claims made in [3].

2. Preliminaries

We begin by introducing some basic notations used in this paper. We consider matrices and vectors with real entries. For any matrix $A \in R^{m\times n}$, a_{ij} denotes its i^{th} row and j^{th} column entry. $A_{\cdot j}$ denotes the j^{th} column and $A_{i\cdot}$, the i^{th} row of A. For any set S, $|S|$ denotes its cardinality. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^t denotes the row transpose of x.

In this note we consider a special subclass of MOP, namely

$$\text{Minimize } (y_1x_1, y_2x_2, \dots, y_nx_n)^t \tag{2.1}$$

subject to $x \in X$ where

$$X = \{x \mid x \geq 0,\ y = Mx + q \geq 0\} \tag{2.2}$$

Note that the problem stated above is a quadratic multiobjective problem subject to linear constraints. We denote this problem as QMOP.

We can write down formulations of QMOP as LCP related weighted problem. Note that these formulations of QMOP are equivalent to LCP(q, M) under some assumption. See [3], [4] and references cited there in.

LCP related weighted problem

Given a matrix $M \in R^{n\times n}$ and vectors $v = (v_1, v_2, \dots, v_n)^t \in R_n^+$ such that $\sum_{j=1}^{n} v_j = 1$ and $q \in R^n$, *the LCP related weighted problem*, denoted as $P(v, M, q)$ is as follows:

$$\min \sum_{j=1}^{n} v_j f_j(x)$$
$$\text{subject to } x \in X$$

where $f_j(x) = x_j(M_{j.}x + q_j)$ and $X = \{x \mid x \geq 0,\ Mx + q \geq 0\}$. This problem is a quadratic programming problem subject to linear constraints and may be rewritten as follows:

$$\begin{aligned} \text{Minimize } & \frac{1}{2}x^t(VM + M^tV)x + x^tVq \\ \text{subject to } & -(Mx+q) \leq 0,\ -x \leq 0 \end{aligned}$$

where V is a nonzero diagonal matrix with diagonal entries $v_i \geq 0$.

Karush-Kuhn-Tucker necessary(KKT) conditions of optimality for $P(v, M, q)$ lead to the following LCP

$$\mathcal{M} = \begin{bmatrix} 0 & M \\ -M^t & VM + M^tV \end{bmatrix} \text{ and } \tilde{q} = \begin{bmatrix} q \\ Vq \end{bmatrix}$$

where V is a diagonal matrix with diagonal entries $v_i \geq 0,\ i = 1, 2, \ldots, n$ with $\sum_i v_i = 1$. By solving the LCP($\tilde{q}, \mathcal{M}$) by Lemke's algorithm, it is claimed that a KKT point x^* to the LCP related weighted problem is found which in turn solves the original LCP.

The following LCP(q, M) is solved in [3] as an application of the above result where

$$M = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and } q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

This example is due to Mangasarian[7] and Murty[8] has shown that Lemke's algorithm cannot solve this problem for any $d > 0$. The nonegative vector $v = (v_1\ \ v_2)^t$ used in this example is $v_1 = 0$ and $v_2 = 1$.

It is proposed in [3] that a KKT point $P(v, M, q)$ can be obtained by solving LCP($\tilde{q}, \mathcal{M}$) by Lemke's algorithm and further it is claimed that one can choose any $v \geq 0$ with $\sum_{1=1}^{n} v_i = 1$. In this connection the following questions arise.

(i) Is it true that for any $v \geq 0$, any solution of equivalent LCP($\tilde{q}, \mathcal{M}$) will solve the LCP related weighted problem $P(v, M, q)$?
As the quadratic objective function in $P(v, M, q)$ need not be convex in general, it is clear that any solution to LCP($\tilde{q}, \mathcal{M}$) need not be a solution to LCP related weighted problem.

(ii) Is any optimal solution to $P(v, M, q)$ constructed using any $0 \neq v \geq 0$, a solution to LCP(q, M)?
The answer is no and an example to demonstrate this can be easily constructed. Now we ask the following question.

(iii) Is it true that at least one optimal solution of $P(v, M, q)$ for any nonzero nonnegative v will solve LCP(q, M)?

The following example shows that this is not true.

Example 2.1. Consider the following LCP(q, M) where

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } q = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$\mathcal{M}$ and $\tilde{q}$ is given by

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix} \text{ and } \tilde{q} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We choose the following normalized nonnegative vector v where $v_1 = v_2 = 0$ and $v_3 = 1$.

Note that the objective function $P(v, M, q)$ with v as stated above is convex. Lemke's algorithm applied to LCP($\tilde{q}$, $\mathcal{M}$) with $d = e$ provides the following solution.

$$w_1 = 0,\ w_2 = 2,\ w_3 = 1,\ w_4 = 0,\ w_5 = 0,\ w_6 = 1,$$
$$z_1 = 0,\ z_2 = 0,\ z_3 = 0,\ z_4 = 0,\ z_5 = 1,\ z_6 = 0.$$

Therefore the solution of $P(v, M, q)$ is given by $x_1 = 0,\ x_2 = 1,\ x_3 = 0$ and the value of the objective function is equal to 0.

This is an optimal solution of $P(v, M, q)$ but the solution set of LCP(q, M) is empty.

(iv) Suppose we take $v > 0$ and solve LCP related weighted problem $P(v, M, q)$. Is it true that a solution to LCP related weighted problem is a solution to LCP(q, M)?

The following theorem answers the question.

Theorem 2.1. *Let* $v > 0$. *The LCP related weighted problem* $P(v, M, q)$ *has an optimal solution with value* 0, *iff the LCP*(q, M) *has a solution.*

Proof. Suppose $P(v, M, q)$ has an optimal solution with value 0. Then $\exists$ a $x \geq 0$ such that $Mx + q \geq 0$. Since $\sum_j v_j f_j(x) = 0$, it follows that $f_j(x) = x_j(Mx + q)_j = 0$ as $v_j > 0$, $\forall\ j$. Thus x solves LCP(q, M).

Conversely, if x solves LCP(q, M), then $x \in X = \{x \mid x \geq 0,\ Mx + q \geq 0\}$ and $f_j(x) = 0$, $\forall\ j$. Hence $\sum_j v_j f_j(x) = 0$. Thus, it follows that x solves $P(v, M, q)$ with optimal value 0. $\square$

In [3], Kostreva and Wiecek also suggests that by solving LCP($\tilde{q}$, $\mathcal{M}$) by Lemke's algorithm one can obtain a solution to LCP(q, M). This approach may not work for $v > 0 (\geq 0)$ with $\sum_{i=1}^{n} v_i = 1$. LCP related weighted problem $P(v, M, q)$ may have a nonconvex objective function and consequently a solution (which is accessible by Lemke's algorithm) may only be a point of local optimum. While solving LCP($\tilde{q}$, $\mathcal{M}$) by Lemke's method with appropriate d it is possible that whatever d is used, we only compute a point of local optimum and hence we do not obtain a solution to LCP(q, M). This is demonstrated in our counterexample.

Example 2.2. Consider the following LCP(q, M) where

$$M = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 5 & 2 \\ 2 & -5 & 0 \end{bmatrix} \text{ and } q = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$$

$\mathcal{M}$ and $\tilde{q}$ is given by

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -3 & 5 & 2 \\ 0 & 0 & 0 & 2 & -5 & 0 \\ -1 & 3 & -2 & \frac{2}{3} & -2 & \frac{2}{3} \\ 3 & -5 & 5 & -2 & \frac{10}{3} & -1 \\ 0 & -2 & 0 & \frac{2}{3} & -1 & 0 \end{bmatrix} \text{ and } \tilde{q} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ \frac{1}{3} \\ -1 \end{bmatrix}.$$

We choose the following normalized nonnegative vector $v = (v_1\ v_2\ v_3)^t$ where $v_1 = v_2 = v_3 = \frac{1}{3}$. Lemke's algorithm applied to LCP($\tilde{q}$, $\mathcal{M}$) with $d = (3\ 8\ 5\ 1\ 1\ 3)^t$ provides the following solution.

$$w_1 = 1.5,\ w_2 = 0,\ w_3 = 0,\ w_4 = 0,\ w_5 = 1,\ w_6 = 0,$$
$$z_1 = 0,\ z_2 = 0,\ z_3 = 1.1,\ z_4 = 1.5,\ z_5 = 0,\ z_6 = 1.8.$$

Therefore the solution of $P(v, M, q)$ is given by $x_1 = 1.5,\ x_2 = 0,\ x_3 = 1.8$ and the value of the objective function is equal to 0.75. This is not a solution of the corresponding LCP(q, M) as it is not complementary since $x_1 > 0$ and $y_1 > 0$. Note that if any other d is chosen, Lemke's algorithm either produces the same solution or it terminates in a ray. However LCP(q, M) has a unique solution

$$z_1 = 9,\ z_2 = 3,\ z_3 = 5.5,\ w_1 = 0,\ w_2 = 0,\ w_3 = 0.$$

Note that finding an appropriate d (if it exists) for which a solution is computed by Lemke's algorithm is an open problem. So solving LCP($\tilde{q}$, $\mathcal{M}$) with an appropriate d does not seem to be a promising approach to solve LCP(q, M) as suggested in [3]. Further the implicit claim that LCP($\tilde{q}$, $\mathcal{M}$) for any $v > 0$ with an arbitrary M, can be solved by Lemke's algorithm is also not correct. Now the option left is to solve LCP($\tilde{q}$, $\mathcal{M}$) by some enumerative method and definitely one of the solutions of LCP($\tilde{q}$, $\mathcal{M}$) will solve LCP(q, M). But in that case it is better to solve a smaller size LCP(q, M) rather than solving LCP($\tilde{q}$, $\mathcal{M}$).

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