

A GAUSSIAN CORRELATION INEQUALITY FOR SYMMETRIC CONVEX SETS¹

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If $n(x)$ is the standard normal density on R^n and if $A = -A$ and $B = -B$ are convex subsets of R^n then

$$\int_{A \cap B} n(x) d^2x \geq (\int_A n(x) d^2x)(\int_B n(x) d^2x).$$

1. Summary and introduction. A function $h(x)$ defined for $x \in R^n$ is called quasi-concave if for any $x_1, x_2 \in R^n$ and $0 < \lambda < 1$, $h(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{h(x_1), h(x_2)\}$. Our main result is

THEOREM 1. *Let $f(x)$ and $g(x)$ be even smooth quasi-concave functions of $x \in R^n$. Suppose also that the gradients $\nabla f(x)$ and $\nabla g(x)$ never vanish for $x \neq 0$. Then for any nonnegative $\phi(x) = \phi(|x|)$ that is a decreasing function of $|x|$,*

$$(1.1) \quad \int_{R^n} \nabla f(x) \cdot \nabla g(x) \phi(x) d^n x \geq 0,$$

provided only that the integral converges. Here $\nabla f(x) \cdot \nabla g(x)$ denotes the scalar product $\sum_{i=1}^n f_{x_i}(x) g_{x_i}(x)$.

From Theorem 1 we deduce

THEOREM 2. *Let $n(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$, $x \in R^n$, denote the standard normal probability density on R^n . If A and B are balanced (i.e., $A = -A$ and $B = -B$) convex subsets of R^n then*

$$(1.2) \quad \int_{A \cap B} n(x) d^n x \geq (\int_A n(x) d^n x)(\int_B n(x) d^n x).$$

Theorem 2 and the more detailed Theorem 3 represent improvements of the earlier results of Khatri [5], Šidák [10, 11] and others. See especially [4], Section 3, and the references given there.

Our proof of Theorem 1 does not seem to generalize to R^n with $n > 2$. However, if (1.1) is true in R^n then the deduction of Theorem 2 is also valid in R^n with $n > 2$.

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2. Proof of Theorem 1. Because $f(x)$ is even and quasi-concave, the set $F(\lambda) \equiv \{x: f(x) \geq \lambda\}$ is for each λ a closed convex balanced subset of R^n . Moreover from the assumption that $\nabla f(x) \neq 0$ for $x \neq 0$ it follows whenever $\{0\} \neq F(\lambda)$ that the boundary $F'(\lambda)$ of $F(\lambda)$ is either empty or $F'(\lambda) = \{x: f(x) = \lambda\}$ is a

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smooth balanced convex curve in R^2 . Similar comments hold for the sets $G(\mu) = \{x: g(x) \geq \mu\}$ and $G'(\mu) = \{x: g(x) = \mu\}$.

For each $x \neq 0$ we denote by $\theta(x)$, $0 \leq \theta(x) < \pi$, the angle between the vectors $\nabla f(x)$ and $\nabla g(x)$. Thus $\cos \theta(x) = \eta_1(x) \cdot \eta_2(x)$ where $\eta_1(x) = -\nabla f(x) / \|\nabla f(x)\|^{-1}$ is the outward pointing normal to $F'(f(x))$ at x and $\eta_2(x) = -\nabla g(x) / \|\nabla g(x)\|^{-1}$ is the normal to $G'(g(x))$ at x . Set $A = \{x \in R^2: \frac{1}{2}\pi < \theta(x) < \pi\}$ and for $x \in A$ define $y = \alpha(x)$ to be the first point of $G'(g(x))$ which can be reached from x by traversing the curve $F'(f(x))$ in the counterclockwise direction (see Figure 1).

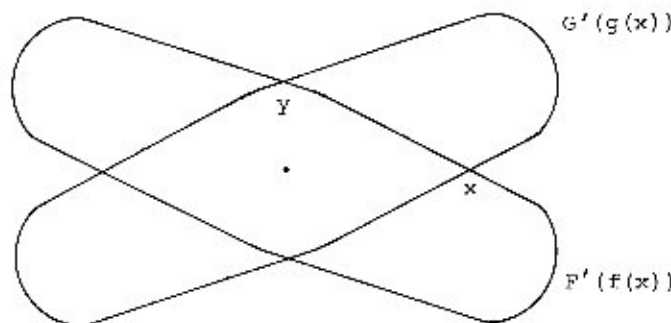


FIG. 1.

Elementary geometric considerations now show for each $x \in A$ that:

$$(2.1) \quad |y| \leq |x| \quad \text{and hence} \quad \phi(x) \leq \phi(|y|).$$

$$(2.2) \quad \text{The angle } \theta(y) \text{ is acute with } \cot \theta(y) \geq -\cot \theta(x) > 0.$$

Define $A_0 = \{x \in A: \theta(\alpha(x)) \neq 0\}$ and consider the map $\Phi: R^2 \rightarrow R^2$ with $\Phi(x) = (f(x), g(x))$. The Jacobi determinant of Φ is

$$(2.3) \quad \begin{aligned} |J_\Phi(x)| &= |f_{x_1}(x)g_{x_2}(x) - f_{x_2}(x)g_{x_1}(x)| \\ &= \|\nabla f(x)\| \|\nabla g(x)\| \sin(\theta(x)). \end{aligned}$$

For $x \in A$, $\Phi(\alpha(x)) = \Phi(x)$ and thus $\Phi(A - A_0) = \Phi(\alpha(A - A_0))$ contains only critical values of the function Φ . By the theorem of Sard and Brown (e.g., [6]), $\Phi(A - A_0)$ has zero Lebesgue measure. But (2.3) shows $|J_\Phi(x)| \neq 0$ for $x \in A$ and hence $A - A_0$ has zero Lebesgue measure.

When restricted to A_0 the function $\alpha(x)$ is easily seen to be continuous. Applying the chain rule to the identity $\Phi(\alpha(x)) = \Phi(x)$ we find $\alpha(x)$ is differentiable on A_0 and its Jacobi determinate $|J_{\alpha}(x)|$ satisfies

$$(2.4) \quad |J_{\alpha}(x)| = |J_\Phi(x)| |J_\Phi(\alpha(x))|^{-1}.$$

We can now prove the inequality (1.1). The sets A_0 and $\alpha(A_0)$ are disjoint. Because $\nabla f(x) \cdot \nabla g(x) \geq 0$ unless $x \in A$, and because $A - A_0$ has measure zero, (1.1) will follow if we show

$$(2.5) \quad \int_{A_0} \nabla f(x) \cdot \nabla g(x) \phi(x) d^2x + \int_{\alpha(A_0)} \nabla f(x) \cdot \nabla g(x) \phi(x) d^2x \geq 0.$$

Using in order (2.4), (2.3), (2.2) and (2.1) we have

$$\begin{aligned} \int_{\alpha(A_0)} \nabla f(x) \cdot \nabla g(x) \phi(x) d^p x &= \int_{A_0} \nabla f(\alpha(x)) \cdot \nabla g(\alpha(x)) \phi(\alpha(x)) |J_\alpha(x)| d^p x \\ &= \int_{A_0} \nabla f(\alpha(x)) \cdot \nabla g(\alpha(x)) \phi(\alpha(x)) |J_\alpha(x)| |J_\alpha(\alpha(x))|^{-1} d^p x \\ &= \int_{A_0} \cot(\theta(\alpha(x)) \phi(\alpha(x)) |J_\alpha(x)| d^p x \\ &\geq - \int_{A_0} \cot(\theta(x)) \phi(x) |J_\alpha(x)| d^p x \\ &= - \int_{A_0} \nabla f(x) \cdot \nabla g(x) \phi(x) d^p x, \end{aligned}$$

thus proving Theorem 1.

COMMENT. The inequality (1.1) may be extended to a wider class of functions $f(x)$ and $g(x)$ than quasi-concave. Following Sherman [9], we bring in the norm $\|u\|_* = \max\{\|u\|_1, \|u\|_\infty\}$ on bounded integrable functions and the $\|\cdot\|_*$ closed convex cone \mathcal{C} generated by the indicator (= characteristic) functions $1_A(x)$ of balanced convex sets $A \subset R^n$. Sherman showed \mathcal{C} is closed under convolution. More recently Davidovic, Karenbljum and Hacet [3], Prékopa [7], [8] and others have shown that the convolution $u * v$ of two logarithmically concave functions is also log concave. This shows that indicator functions of balanced convex sets may be approximated by smooth log concave functions $f(x)$ with $\nabla f(x) \neq 0$ for $x \neq 0$. Since even log concave functions are also quasi-concave we may state

COROLLARY 1. *The inequality (1.1) holds for smooth Sherman functions f and g in \mathcal{C} .*

A more detailed exposition of the proof of Theorem 1 may be found in Antell [2].

3. Proof of Theorem 2. We begin by stating a more detailed and general theorem. Let the random vector $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ be normally distributed with mean zero and covariance matrix Σ where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^* & \Sigma_{22} \end{pmatrix}$$

and $\Sigma_{11} = (EX_i X_j)$, $\Sigma_{22} = (EY_i Y_j)$ and $\Sigma_{12} = (EX_i Y_j)$. Set $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ and for balanced convex subsets A and B of R^n set

$$p(\Sigma) = P\{X \in A; Y \in B\}.$$

For each λ , $0 \leq \lambda \leq 1$, the matrix

$$\Sigma_\lambda = \begin{pmatrix} \Sigma_{11} & \lambda \Sigma_{12} \\ \lambda \Sigma_{12}^* & \Sigma_{22} \end{pmatrix}$$

is also a normal covariance matrix and we may consider the probability $p(\Sigma_\lambda)$ as a function of λ , $0 \leq \lambda \leq 1$.

THEOREM 3. *Under the above conditions, if $\text{rank}(\Sigma_{12}) \leq 2$ then $p(\Sigma_\lambda)$ is an increasing function of λ , $0 \leq \lambda \leq 1$.*

REMARKS. The probabilities $P\{X \in A\}$ and $P\{Y \in B\}$ do not depend on λ . Moreover, $\lim_{\lambda \rightarrow 0} p(\Sigma_\lambda) = P\{X \in A\}P\{Y \in B\}$. Thus Theorem 3 implies

$$(3.1) \quad P\{X \in A; Y \in B\} \geq P\{X \in A\}P\{Y \in B\}.$$

If one takes $n = 2$ and $\Sigma_{11} = \Sigma_{22} = \Sigma_{33} = I$, then (3.1) becomes (1.2) and we see that Theorem 2 is a special case of Theorem 3.

The proof of Theorem 3 is conceptually simpler in the special case when $n = 2$ and $\Sigma_{11} = \Sigma_{22} = \Sigma_{33} = I$. We advise considering only this special case on the first reading.

PROOF. Without loss of generality we may assume $\text{rank}(\Sigma_{11}) = \text{rank}(\Sigma_{22}) = n$. Then by introducing canonical variates (see [1], Chapter 12), we may further assume that $\Sigma_{11} = \Sigma_{22} = I$ and that Σ_{12} is diagonal with nonnegative entries. Let $\Sigma_{12} = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $1 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq 0$. Since $\text{rank}(\Sigma_{12}) \leq 2$ we have $\lambda_3 = \lambda_4 = \dots = \lambda_n = 0$.

The functions $1_A(x)$ and $1_B(y)$ are both even and log concave. By the results of Davidovic et al., we may approximate 1_A and 1_B by smooth even log concave functions. Thus it suffices to show for arbitrary smooth even rapidly decreasing log concave functions $F(x)$ and $G(y)$ that the expectation

$$\mathcal{E}(\lambda) = EF(X)G(Y)$$

corresponding to the covariance matrix Σ_λ is an increasing function of λ , $0 \leq \lambda \leq 1$.

When conditioned on X_1, X_2, Y_1, Y_2 the variables $F(X)$ and $G(Y)$ are independent. Moreover,

$$\begin{aligned} E\{F(X) | X_1, X_2, Y_1, Y_2\} &= E\{F(X) | X_1, X_2\} \\ &= f(X_1, X_2), \end{aligned}$$

where

$$f(x_1, x_2) = (2\pi)^{-1(n-2)/2} \int F(x) \exp(-\frac{1}{2} \sum_{j=3}^n x_j^2) dx_3 \cdots dx_n,$$

and

$$\begin{aligned} E\{G(Y) | X_1, X_2, Y_1, Y_2\} &= E\{G(Y) | Y_1, Y_2\} \\ &= g(Y_1, Y_2), \end{aligned}$$

where

$$g(y_1, y_2) = (2\pi)^{-1(n-2)/2} \int G(y) \exp(-\frac{1}{2} \sum_{j=3}^n y_j^2) dy_3 \cdots dy_n.$$

Thus

$$\mathcal{E}(\lambda) = Ef(X_1, X_2)g(Y_1, Y_2).$$

Now Prékopa [8] has shown that if $H(x_1, \dots, x_k, y_1, \dots, y_l)$ is log concave then $h(x_1, \dots, x_k) = \int H(x_1, \dots, x_k, y_1, \dots, y_l) dy_1 \cdots dy_l$ is log concave. Thus, both of the above functions $f(x_1, x_2)$ and $g(y_1, y_2)$ are log concave.

The conditional density of (Y_1, Y_2) given $(X_1, X_2) = (x_1, x_2)$ is

$$p(\lambda, x_1, x_2; y_1, y_2) = \prod_{j=1}^2 [2\pi(1 - (\lambda\lambda_j)^2)]^{-1} \exp\left\{-\frac{1}{2} \frac{(\lambda\lambda_j x_j - y_j)^2}{1 - (\lambda\lambda_j)^2}\right\}.$$

Setting $n(x_1, x_2) = (2\pi)^{-1} \exp\{-(x_1^2 + x_2^2)/2\}$ and

$$g(\lambda, x_1, x_2) = \int p(\lambda, x_1, x_2; y_1, y_2) g(y_1, y_2) dy_1 dy_2$$

we find

$$\mathcal{E}(\lambda) = \int f(x_1, x_2) g(\lambda, x_1, x_2) n(x_1, x_2) dx_1 dx_2.$$

Let $\lambda = e^t$, $-\infty < t \leq 0$. Then direct computation gives

$$\frac{\partial}{\partial t} g(e^t, x_1, x_2) = \left(\sum_{j=1}^2 \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right) g(e^t, x_1, x_2)$$

and an easily justified integration by parts yields

$$(3.2) \quad \frac{d}{dt} \mathcal{E}(e^t) = \int \nabla f(x_1, x_2) \cdot \nabla g(e^t, x_1, x_2) n(x_1, x_2) dx_1 dx_2.$$

Since $p(\lambda; x_1, x_2; y_1, y_2)$ is log concave in (x_1, x_2, y_1, y_2) the results of Prékopa show that $g(e^t, x_1, x_2)$ is log concave in (x_1, x_2) . Since n is a radially decreasing function, Corollary 1 of Theorem 1 implies $(d/dt)\mathcal{E}(e^t) \geq 0$ and completes the proof of Theorem 3.

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