A Remark on the structure of symmetric quantum dynamical semigroups on von Neumann algebras

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Abstract

We study the structure of the generator of a symmetric, conservative quantum dynamical semigroup with norm-bounded generator on a von Neumann algebra equipped with a faithful semifinite trace. For von Neumann algebras with abelian commutant (i.e. type I von Neumann algebras), we give a necessary and sufficient algebraic condition for the generator of such a semigroup to be written as a sum of square of self-adjoint derivations of the von Neumann algebra. This generalizes some of the results obtained by Albeverio, Høegh-Krohn and Olsen ([1]) for the special case of the finite dimensional matrix algebras. We also study similar questions for a class of quantum dynamical semigroups with unbounded generators.

1   Introduction

In [1], among other things trace-symmetric conservative quantum dynamical semigroups on the algebra M_n of $n \times n$ matrices are studied. Our purpose in the present note is to study similar questions for a more general (possibly infinite dimensional) class of von Neumann algebras, equipped with some faithful semifinite trace. We give a complete characterization of the structure of the generator of a symmetric (w.r.t. the above-mentioned trace) conservative quantum dynamical semigroup with bounded generator, and also obtain some results for certain semigroups with unbounded generators.

First of all, we shall recast some of the results obtained in [1] in a slightly new language. We recall that the basic fact used in section 2 of [1] is the

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following structure-theorem for the generator $\mathcal{L}$ of a trace-symmetric conservative quantum dynamical semigroup T_t on M_n :

$$\mathcal{L}(a) = \frac{1}{2} \sum_{j=1}^{k} [\beta_j, [\beta_j, a]]; a \in M_n;$$

where $k \geq 1$ is some integer and $\beta_j \in M_n$ with $\beta_j^* = -\beta_j; tr(\beta_j) = 0$. The above result is a simple corollary of the results obtained by Lindblad [15]. Let the self-adjoint derivation $[\beta_j, .] : M_n \to M_n$ be denoted by δ_j. Then we can write the above as $\mathcal{L} = \frac{1}{2} \sum_j \delta_j^2$. We shall study in the next section necessary and sufficient conditions under which the generator of a quantum dynamical semigroup (q.d.s. henceforth)on a von Neumann algebra can be written in the above form (with possibly infinitely many derivations, where the sum should converge in an appropriate topology). In particular, we shall show that for a general von Neumann algebra with a faithful semifinite trace, symmetry and conservativity alone do not suffice for such a structure, but one needs additional conditions. Then, finally, we shall extend our discussion to some special classes of semigroups with unbounded generators too.

2   Structure and dilation of symmetric conservative q.d.s. on more general von Neumann algebras : bounded generator case

Let $\mathcal{A} \subseteq \mathcal{B}(h)$ (where h is a separable Hilbert space) be a von Neumann algebra with a normal, faithful, semifinite trace τ on it. Let $L^p, 1 \leq p \leq \infty$ denote the well-known L^p-spaces with respect to the trace τ (cf [20], [16]). Let T_t be a conservative q.d.s. on $\mathcal{A}$, with norm-bounded generator $\mathcal{L}$. We also assume that T_t is symmetric w.r.t. τ, in the sense that $\tau(T_t(a)b) = \tau(aT_t(b))$ for all $a, b \in \mathcal{A}$ with $a, b \geq 0$. By the standard interpolation principle, T_t can be uniquely extended as a continuous map from L^p to L^p for each p. We shall not notationally distinguish among these maps, denoting all of them by the same symbol T_t (see, e.g. [4] and the references therein for more discussion on this).

It is clear that the generator $\mathcal{L}$ of the above q.d.s. will also satisfy the symmetry condition $\tau(\mathcal{L}(a)b) = \tau(a\mathcal{L}(b))$ for all $a, b \in L^1 \bigcap L^\infty$, and then by using the duality $(L^1)^* \cong L^\infty$, it is easy to see that $\mathcal{L}$ extends as a

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continuous linear map from L^1 to L^1, and $\tau(\mathcal{L}(a)b) = \tau(a\mathcal{L}(b))$ holds for all $a \in L^1, b \in L^\infty$.

Now, it is well-known ([3]) that $\mathcal{L}$ can be written as $\mathcal{L}(a) = \psi(a) + ka + ak + i[H, a]$, where $\psi : \mathcal{A} \to \mathcal{A}$ is a normal completely positive (CP) map, k, H are self-adjoint elements of $\mathcal{A}$. Since it is clear that $\tau((ak + ka + i[H, a])b) = \tau(a(kb + bk - i[H, b]))$ for all $a \in L^1$, $b \in L^\infty$, we conclude that ψ satisfies $\tau(\psi(a)b) = \tau(a\psi(b)) + 2i\tau(a[H, b])$. In particular, $\tau(\psi(a)) = \tau(a\psi(1))\forall a \in L^1$.

We shall now prove the main result of this section.

Theorem 2.1 *I. Let $H_j; j = 1, 2, ...$ be elements of $\mathcal{A}$ such that $H_j^* = -H_j$, and assume that $\sum_{j=1}^\infty H_j^2$ weakly converges. Then there is a normal norm-bounded map $\mathcal{L}$ from $\mathcal{A}$ to itself given by*

$$(1) \qquad \mathcal{L}(a) = \left(\frac{1}{2}\sum_j [H_j, [H_j, a]]\right);$$

(where the series in the right hand side above converges weakly) such that $T_t = e^{t\mathcal{L}}$ is a symmetric conservative q.d.s. on $\mathcal{A}$. Furthermore, the following algebraic relation is satisfied :

$$(2) \qquad \partial(az, az) = z^*\partial(a, a)z \forall a \in \mathcal{A}, z \in \mathcal{Z};$$

where $\partial(x, y) := \mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y)$ and $\mathcal{Z}$ denotes the centre of $\mathcal{A}$. (II) (partial converse)
Under the additional assumption that $\mathcal{A}$ is of type I, the converse of (I) holds, i.e. given any symmetric conservative q.d.s. T_t with bounded generator $\mathcal{L}$ on $\mathcal{A}$ satisfying the algebraic relation (2), there exist $H_j, j = 1, 2, ...$ satisfying the conditions of (I) and such that (1) is satisfied.

Proof :-
(I) The proof of this part is more or less standard and can be found in the literature, e.g., [5]. We, however, give a proof here for the sake of completeness. Let $H_0 = \sum_j H_j^2$, which is clearly a self-adjoint element of $\mathcal{A}$. To show that $\sum_j[H_j, [H_j, a]]$ weakly converges for all $a \in \mathcal{A}$, it suffices to prove it for all nonnegative elements a. Fix $a \in \mathcal{A}_+$, and define $b_n = -\sum_{j=1}^n H_j a H_j = \sum_{j=1}^n H_j^* a H_j$. Clearly, $0 \le b_n \uparrow$, and $b_n \le \|a\|(-\sum_{j=1}^\infty H_j^2)$, thus $\sup_n \|b_n\| < \infty$. So, b_n must be weakly convergent.

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Since $\sum_{j=1}^{n}[H_j,[H_j,a]] = -2\sum_{j=1}^{n}H_jaH_j + (\sum_{j=1}^{n}H_j^2)a + a(\sum_{j=1}^{n}H_j^2)$, it follows that $\sum_j[H_j,[H_j,a]]$ is weakly convergent. Furthermore, let $\mathcal{K}$ be any separable Hilbert space with an orthonormal basis $\{e_j, j = 1,2,...\}$, and $R : h \to h \otimes \mathcal{K}$ be a bounded linear map defined by $Ru := \sum_j H_j u \otimes e_j$ (that R is well-defined and bounded follows easily from the fact that $\sum_j H_j^2$ is weakly convergent, and it is easy to see that $\|R\|^2 = \|\sum_j H_j^2\|$). It is then easily verified that $\mathcal{L}(a) = R^*(a \otimes 1_{\mathcal{K}})R - \frac{1}{2}R^*Ra - a\frac{1}{2}R^*R$, from which we immediately conclude that $\mathcal{L}$ is a bounded normal map and $e^{t\mathcal{L}}$ is a conservative q.d.s. It remains to prove that T_t is symmetric. First we note that since τ is normal, and for $a, b \in L^1 \cap \mathcal{A}_+$, $0 \leq = -\sum_{j=1}^{n}\sqrt{b}H_jaH_j\sqrt{b} \uparrow -\sum_{j=1}^{\infty}\sqrt{b}H_jaH_j\sqrt{b}$, it follows that $\tau(\sum_{j=1}^{\infty}H_jaH_jb) = \tau(\sum_{j=1}^{\infty}\sqrt{b}H_jaH_j\sqrt{b}) = \sum_{j=1}^{\infty}\tau(\sqrt{b}H_jaH_j\sqrt{b}) = \sum_{j=1}^{\infty}\tau(aH_jbH_j) = \tau(a\sum_{j=1}^{\infty}H_jbH_j)$. The same equality will clearly hold for all $a, b \in L^1 \cap L^\infty$, and from this a straightforward calculation enables us to verify that $\mathcal{L}$ is symmetric. The symmetry of T_t then follows from the fact that $T_t(a)$ is given by the norm-convergent series $\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{L}^n(a)$. Finally, the relation (2) is verified by a direct calculation using the form of $\mathcal{L}$ given by (1). This completes the proof of (I).

(II.) To prove the converse, we shall proceed as follows. Since $\mathcal{A}$ is of type I, without loss of generality we can choose h such that the commutant of $\mathcal{A}$ in $\mathcal{B}(h)$ is abelian. Now, we shall very briefly recall the arguments given in [8], [7] to show that there exist a separable Hilbert space $\mathcal{K}_1$ and a bounded linear map R from h to $\mathcal{K}_1$ such that $\mathcal{L}(a) = R^*(a \otimes 1_{\mathcal{K}_1})R - \frac{1}{2}R^*Ra - a\frac{1}{2}R^*R + i[H, a]$, where $H \in \mathcal{A}$ is self-adjoint, and furthermore $R_i \in \mathcal{A} \forall i$, where R_i are defined by $\langle R_iu, v \rangle = \langle Ru, v \otimes e_i \rangle$ ($\{e_i\}$ is some orthonormal basis of $\mathcal{K}_1$). By the well-known result of Christensen and Evans ([3]), we can choose a separable Hilbert space $\mathcal{K}_0$ and $S \in \mathcal{B}(h, \mathcal{K}_0)$, a normal $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K}_0)$ and $H \in \mathcal{A}_{s.a.}$ such that $\mathcal{L}(a) = S^*\rho(a)S - \frac{1}{2}S^*Sa - a\frac{1}{2}S^*S + i[H, a]$, with the minimality condition that the linear span of $\{\alpha(x)u, x \in \mathcal{A}, u \in h\}$ (where $\alpha(x) = Sx - \rho(x)S$) is dense in $\mathcal{K}_0$. Then, we construct (c.f., e.g., [7], [9]) a normal $*$-homomorphism ρ' of the commutant $\mathcal{A}'$ of $\mathcal{A}$ in $\mathcal{K}_0$ which satisfies $\rho'(a)\alpha(x)u = \alpha(x)au\forall a \in \mathcal{A}', x \in \mathcal{A}, u \in h$. It is easy to verify by using the algebraic relation (2) that ρ' and ρ agree on the centre $\mathcal{Z}$. We note that as $\mathcal{A}'$ is abelian, $\mathcal{Z} = \mathcal{A}'$. We choose a separable Hilbert space $\mathcal{K}_1$ such that $\rho(x) = \Sigma_1^*(x \otimes 1_{\mathcal{K}_1})\Sigma_1$, where $\Sigma_1 \in \mathcal{B}(\mathcal{K}_0, h \otimes \mathcal{K}_1)$ is an isometry (such a choice is possible by the general result on the structure of a normal $*$-homomorphism of a von Neumann algebra), and let $R = \Sigma_1 S$. Then by some simple arguments as in [8], [7] we show that the required conditions

hold.

Now we shall use the assumption of symmetry. We have earlier noted that $\tau(\mathcal{L}(a)b) = \tau(a\mathcal{L}(b)) \forall a \in L^1, b \in L^\infty$. Let us denote by $\psi : \mathcal{A} \to \mathcal{A}$ the CP map given by $\psi(a) = R^*(a \otimes 1)R = \sum_{i=1}^\infty R_i^* a R_i$, which is a weakly convergent sum. We claim that $\sum_i R_i R_i^*$ also weakly converges and $\psi(a) = \sum_j R_j^* a R_j = \sum_j R_j a R_j^* - 2i[H, a]$. It is clear from the discussion preceding the present theorem that $\tau(\psi(a)b) = \tau(a\{\psi(b) + 2i[H, b]\}) \forall a \in L^1 \cap L^\infty, b \in L^\infty$. Let $c_n = \sum_{i=1}^n R_i R_i^*$. For any $a \in L^1 \cap L^\infty$ with $a \geq 0$, we have that $\tau(ac_n) = \sum_{i=1}^n \tau(aR_i R_i^*) = \sum_{i=1}^n \tau(R_i^* a R_i) \leq \tau(\psi(a)) = \tau(a\psi(1))$. Since any element in $L^1 \cap L^\infty$ can be canonically decomposed into a linear combination of nonnegative elements of $L^1 \cap L^\infty$, by a standard argument we show that $|\tau(ac_n)| \leq C\|a\|_1 \|\psi(1)\|_\infty$ for all $a \in L^1 \cap L^\infty$ (where C is some numerical constant independent of a, c_n). Since $L^1 \cap L^\infty$ is dense in L^1 in L^1-norm, and since $\|c_n\|_\infty = \sup_{a \in L^1, \|a\|_1 \leq 1} |\tau(ac_n)|$, we conclude that $\sup_n \|c_n\|_\infty < \infty$, and $0 \leq c_n \uparrow$, which proves that c_n is weakly convergent. Furthermore, by an exactly similar argument we can prove that $\sum_i R_i b R_i^*$ is weakly convergent for every $b \in L^\infty$. Now, by normality of τ, it is easy to see that $\tau(\sum_i a R_i b R_i^*) = \sum_i \tau(a R_i b R_i^*)$ for all $a, b \in L^1 \cap L^\infty, a, b \geq 0$, and hence the same equality will hold for all $a \in L^1, b \in \mathcal{A}$. Using this equality we verify that $\tau(a\psi(b)) = \tau(a\{\sum_j R_j b R_j^* - 2i[H, b]\})$ for all $a \in L^1, b \in \mathcal{A}$, and hence $\psi(b) = \sum_j R_j b R_j^* - 2i[H, b]$, which completes the proof of the claim. Finally, we choose $H_j, j = 1, 2, \ldots$ as follows :

$$H_{2j} = \frac{i}{2}(R_j + R_j^*), \quad H_{2j-1} = \frac{1}{2}(R_j - R_j^*).$$

Then a straightforward computation using the fact that $\sum_j R_j x R_j^* = \sum_j R_j^* x R_j + 2i[H, x]$ enables us to conclude that $\mathcal{L}(x) = \sum_{j=1}^\infty \frac{1}{2}[H_j, [H_j, x]] + i[H', x] \forall x \in \mathcal{A}$, for some s.a. H'. Using again the symmetry of $\mathcal{L}$, it follows that $\tau([H', x]y) = \tau(x[H', y]) = -\tau([H', x]y) \forall x, y \in L^1 \cap L^\infty$, so that $[H', x] = 0 \forall x \in \mathcal{A}$. This completes the proof.

Remark 2.2 *It is shown in [8] that the condition (2) is satisfied by any q.d.s. with norm continuous generator in case $\mathcal{A}$ is a type I factor. On the other hand, for a function algebra, i.e. $\mathcal{A} = L^\infty(\Omega, \mu)$ for some measure space (Ω, μ), the condition (2) can never be satisfied by any nontrivial q.d.s. with norm continuous generator. Thus, for abelian $\mathcal{A}$, we conclude that symmetry does not imply the structure given by (1), since there are plenty of examples of symmetric nontrivial q.d.s. with bounded generator on such an algebra.*

However, if we consider q.d.s. with unbounded generator on nice function algebras, the above condition is nothing but locality of the generator (cf [8]).

Remark 2.3 *Under the assumption that $\mathcal{A}$ is type I, a q.d.s. T_t on $\mathcal{A}$ with bounded generator $\mathcal{L}$ admits an Evans-Hudson (EH) dilation with only classical Brownian motion as noise, in the sense that the EH dilation j_t satisfies a q.s.d.e. of the form $dj_t(x) = j_t(\mathcal{L}(x))dt + \sum_{i=1}^{\infty} j_t(\theta_i(x))(da_i(t) + da_i^\dagger(t)); j_0(x) = x$, where $\theta_i : \mathcal{A} \to \mathcal{A}$ are bounded linear maps, if and only if the algebraic relation (2) holds and T_t is symmetric. The proof of this fact is an immediate consequence of the preceding theorem and the results about existence, uniqueness and homomorphism property of EH flows, as derived, e.g., in [9], [17]. For more details regarding the dilation problem of symmetric q.d.s., the reader is referred to [2] and [14].*

3 A special class of q.d.s. with unbounded generator on $\mathcal{B}(h)$

It is not so easy to generalize the above results to the case of q.d.s. with unbounded generator due to a number of algebraic and analytic difficulties. The first source of problem is the absence of any general structure theorem like the theorem of Christensen-Evans which holds in the case of bounded generators. However, for symmetric q.d.s. the situation is much better, because in this case a Christensen-Evans type structure of the generator can be obtained with an unbounded closable operator R. But the algebraic difficulty will remain since we cannot ensure that $R^* \pi(x) R$ will be affiliated to the von Neumann algebra $\mathcal{A}$ (see [10] for some examples where it is affiliated, although in many other interesting examples it will indeed not be affiliated!). Furthermore, if we embed the range of R into a factorizable Hilbert space of the form $h \otimes k$ for some k, then we do not have much control on the domain of R_i^* (where we fix an orthonormal basis $\{e_i\}$ of k and write $R = \sum R_i \otimes e_i$, with domain of each R_i containing domain of R), so that $R_i + R_i^*$ may not be densely defined. Thus, it is far from straightforward how to look for a sufficient condition to obtain a structure as in the bounded generator case. However, we shall present in what follows a special case in which such a condition can be given.

Let h be a separable Hilbert space, $\mathcal{A} = \mathcal{B}(h)$, and τ be the usual trace of $\mathcal{B}(h)$. Let us choose and fix some countable total subset of vectors $\mathcal{E}$ of h and let $\mathcal{F}_0$ denote the norm-dense $*$-subalgebra of $\mathcal{B}_0(h)$(the set of compcat operators on h) generated algebraically by all finite-rank operators of the form $|u><v|$ for $u, v \in \mathcal{E}$. Let $\mathcal{F} = \mathcal{F}_0 \oplus 1$. we have the following :

Proposition 3.1 *Let $(T_t)_{t \geq 0}$ be a conservative q.d.s. on $\mathcal{B}(h)$ such that the generator $\mathcal{L}$ has $\mathcal{F}$ in its domain and $\mathcal{L}(1) = 0$, $\mathcal{L}(\mathcal{F}_0) \subseteq \mathcal{F}_0$. Assume furthermore that $\tau(\mathcal{L}(a)b) = \tau(a\mathcal{L}(b))$ for $a, b \in \mathcal{F}$. Then there exist $*$-derivations $\alpha_j, j = 1, 2, ...$ defined on $\mathcal{F}$ such that $\alpha_j = \alpha_j^\dagger, \forall j$ (where $\alpha_j^\dagger(x) = \alpha_j(x^*)^*$), and $< \xi, \mathcal{L}(y)\eta > = \frac{1}{2}\sum_j < \xi, \alpha_j^2(y)\eta >$ for any $y \in \mathcal{F}$ and ξ, η belonging to the domain consisting of finite linear combinations of vectors in $\mathcal{E}$.*

Proof :
By standard arguments as in [19], [18], we can construct a separable Hilbert space $\mathcal{K}$, a $*$-homomorphism $\pi : \mathcal{B}_0(h) \to \mathcal{B}(k)$, and a π-derivation $\delta : \mathcal{F} \to \mathcal{B}(h, \mathcal{K})$ such that $\mathcal{L}(x^*y) - \mathcal{L}(x)^*y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y) \forall x, y \in \mathcal{F}$. Now, since every representation of $\mathcal{B}_0(h)$ is unitarily equivalent to a direct sum of the trivial representation, we can choose $\mathcal{K}$ to be of the form $h \otimes k$ for some separable Hilbert space k and $\pi(x) = \Sigma^*(x \otimes 1_k)\Sigma$ for some unitary Σ. But then by replacing δ by $\Sigma\delta$, we can assume without loss of generality that $\pi(x) = (x \otimes 1)$ and δ is an $(x \otimes 1)$-derivation, i.e. $\delta(x)u = \sum_i \delta_i(x)u \otimes e_i, x \in \mathcal{F}, u \in h$, where $\{e_i\}$ is an orthonormal basis of k and $\delta_i : \mathcal{F} \to \mathcal{B}(h)$ is a derivation for each i. In fact, $\delta_i(\mathcal{F}) \subseteq \mathcal{B}_0(h)$, because $\mathcal{L}$ maps $\mathcal{F}$ into $\mathcal{B}_0(h)$ (as $\mathcal{L}(1) = 0$ and $\mathcal{L}$ maps $\mathcal{F}_0$ into $\mathcal{F}_0$) and $0 \leq \delta_i(x)^*\delta_i(x) \leq \mathcal{L}(x^*x) - \mathcal{L}(x^*)x - x^*\mathcal{L}(x)$, which is a finite-rank operator. Hence $\delta_i(x)^*\delta_i(x)$ is compact, i.e. $\delta_i(x)$ is compact. Now, let us consider the following derivations defined on $\mathcal{F}$, given by, $\alpha_{2j} = \frac{1}{2}(\delta_j + \delta_j^\dagger); \alpha_{2j-1} = \frac{i}{2}(\delta_j^\dagger - \delta_j), j = 1, 2, ...$ Clearly $\alpha_j^\dagger = \alpha_j$. We claim that for $x, y \in \mathcal{F}$, $\sum_j \alpha_j(x)\alpha_j(y)$ weakly converges and equals $\mathcal{L}(xy) - \mathcal{L}(x)y - x\mathcal{L}(y)$. The proof is more or less the same as done earlier in context of q.d.s. with bounded generator. First we verify using symmetry of $\mathcal{L}$ that for nonnegative $y \in \mathcal{F}_0$, selfadjoint $x \in \mathcal{F}$, $0 \leq \tau(y \sum_{j=1}^n \delta_j(x)\delta_j^\dagger(x)) \leq \tau(\sum_{j=1}^\infty \delta_j^\dagger(x)y\delta_j(x)) = \tau(y(\mathcal{L}(x^2) - \mathcal{L}(x)x - x\mathcal{L}(x)))$, and from this (since $\mathcal{F}_0$ is clearly dense in the set of trace-class operators in trace-norm) conclude that the weak convergence of $\sum_j \delta_j(x)\delta_j^\dagger(x)$ holds for all self-adjoint $x \in \mathcal{F}$, and hence for all $x \in \mathcal{F}$. The claim is then proved by a usual polarization argument. Moreover, we also get that $\sum_j \delta_j(x)\delta_j^\dagger(y) = \sum_j \delta_j^\dagger(x)\delta_j(y) \forall x, y \in$

$\mathcal{F}$. This follows by showing that $\tau(\sum_j \delta_j(x)\delta_j^\dagger(y)z) = \tau(\sum_j \delta_j^\dagger(x)\delta_j(y)z)$ for all $z \in \mathcal{F}_0$. Then an easy computation as in the earlier section enables us to verify the claim.

We now choose an orthonormal basis $\xi_k, k = 1, 2, \ldots$ of h such that each of this basis vector belongs to the span of the total set $\mathcal{E}$ mentioned in the statement of the proposition, and we denote by $\mathcal{A}_n$ the algebra generated by $|u><v|, u, v \in span\{\xi_1, \ldots \xi_n\}$ (thus $\mathcal{A}_n$ is isomorphic with the algebra of $n \times n$ matrices). Since $\mathcal{L}(\mathcal{F}_0) \subseteq \mathcal{F}_0$, clearly for each m, there is some n with the property that $\mathcal{L}(\mathcal{A}_m) \subseteq \mathcal{A}_n$. It is easy to see that for any j, α_j also maps $\mathcal{A}_m$ into $\mathcal{A}_n$. Take any self-adjoint element $x \in \mathcal{A}_m$ and any vector ξ orthogonal to $\mathcal{V}_n := span\{\xi_1, \ldots \xi_n\}$. Since $\delta_j(x)^*\delta_j(x) \leq \mathcal{L}(x^2) - \mathcal{L}(x)x - x\mathcal{L}(x) \in \mathcal{A}_n$, we have $\delta_j(x)\xi = 0$, and similarly from the inequality $\delta_j(x)\delta_j(x)^* \leq \mathcal{L}(x^2) - \mathcal{L}(x)x - x\mathcal{L}(x)$ it follows that $\delta_j(x)^*\xi = 0$. Thus $\alpha_j(x)\xi = 0$ for any $\xi \in \mathcal{V}_n^\perp$, and as $\alpha_j(x)$ is self-adjoint, it follows that $\alpha_j(x)(\mathcal{V}_n) \subseteq \mathcal{V}_n$, i.e. $\alpha_j(x) \in \mathcal{A}_n$ for all selfadjoint $x \in \mathcal{A}_m$, and hence the same holds for any $x \in \mathcal{A}_m$.

But being a derivation from a finite dimensional matrix-algebra into another, there must exist $H_{m,n} \in \mathcal{A}_n$ which implements $\alpha_j|_{\mathcal{A}_m}$, and thus we must have that $\tau(\alpha_j(x)) = 0 \forall x \in \mathcal{A}_m$. Now it is simple to verify that for $x, y \in \mathcal{A}_m$, $2\tau(x\mathcal{L}(y)) = -\sum_j \tau(\alpha_j(x)\alpha_j(y)) = \sum_j \tau(x\alpha_j^2(y))$, where in the last step of the above equality we have used the fact that $\tau(\alpha_j(x\alpha_j(y))) = 0$. This proves that on the algebraic direct sum of $\mathcal{A}_m, m = 1, 2, \ldots$ $\mathcal{L}$ is indeed of the form $\frac{1}{2}\sum_j \alpha_j^2$, in the sense that the sum $\sum_j \frac{1}{2} < \xi, \alpha_j^2(y)\eta >$ converges to $< \xi, \mathcal{L}(y)\eta >$ for every y in the algebraic direct sum of $\mathcal{A}_m, m = 1, 2, \ldots$, and for ξ, η belonging to the dense set consisting of finite linear combinations of $\xi_k, k = 1, 2, \ldots$.

Example :

Let us now give an example where the assumptions of the above Proposition are valid. Let $h = L^2(R^N)$, $\mathcal{D}$ be some countable dense subset of h consisting of elements from $C_c^\infty(R^N)$ (the set of smooth functions with compact supports), and let $\mathcal{E} = \bigcup_{n \geq 0} \bigcup_{\alpha:|\alpha|=n} \{f^{(\alpha)} : f \in \mathcal{D}\}$, where α denotes a multi-index, say $\alpha = (\alpha_1, \ldots \alpha_N), \alpha_j \geq 0, |\alpha| = \alpha_1 + \ldots \alpha_N$, and $f^{(\alpha)} := (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_N})^{\alpha_N} f$ and $f^{(0, \ldots 0)} = f$. Let $H_j, j = 1, \ldots, N$ denote the anti-selfadjoint operator $\frac{\partial}{\partial x_j}$ on h, and we consider the formal generator $\mathcal{L}$ given by $\mathcal{L} = \frac{1}{2}\sum_{j=1}^N [H_j, [H_j, .]]$ on the linear span of $\{|u><v|, u, v \in \mathcal{E}\}$. By the results of [13], it is indeed possible to obtain a conservative q.d.s. T_t

on $\mathcal{B}(h)$ whose generator extends $\mathcal{L}$, and it is easy to see that the hypotheses of the above Proposition are satisfied.

We conclude with a few remarks on the possibility of obtaining a similar result for q.d.s. with unbounded generators on von Neumann or C^* algebras other than the special cases considered here. There is a serious problem to even conceive of an appropriate generalization. In [11], an interesting class of symmetric q.d.s. on general C^* or von Neumann algebras with an additional assumption of covariance w.r.t. the action of a (possibly noncommutative and noncompact) Lie group has been studied in detail, and results regarding the structure of the generator as well as the existence and homomorphism property of EH dilation have been obtained. For technical reasons the authors in [11] had to work with the canonical embedding of the algebra in the L^2-space of the given trace, and the methods would not apply if any other embedding was considered. Thus, if we want to imitate the techniques used in this present article to obtain structure theorems for q.d.s. with unbounded generators, it is in principle possible to give some sufficient conditions on the generator in order to be able to write it as a series of squares of derivations only in the case where either the algebra or its commutant is abelian. But remembering that to apply the set-up of [11], we must take the commutant w.r.t. the embedding in the L^2-space of the trace, which is by the Tomita-Takesaki theory anti-isomorphic to the algebra itself, it is easily seen that we are confined to the case of abelian algebras only ! Thus, the question how to adapt the technique of section 2 to the case of unbounded generators for general algebras remains open; either one has to improve or modify the results of section 2 by weakening the assumption of commutativity of the algebra or its commutant, or to improve the techniques of [11] to accommodate more general situations.

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References

[1]   S. Albeverio, R. Høegh-Krohn and G. Olsen, *Dynamical semi-groups and Markov processes on C^*-algebras*, J. Reine Angew. Math. **319**(1980), 25-37.

[2] R. Alicki and M. Fannes, *Dilations of quantum dynamical semigroups with classical Brownian motion*, Comm. Math. Phys. **108**(1987), 353-361.

[3] E. Christensen and D.E. Evans, *Cohomology of operator algebras and quantum dynamical semigroups*, J. London Math. Soc. **20** (1979), 358–368.

[4] F. Cipriani and J-L. Sauvageot, *Derivations as Square Roots of Dirichlet Forms*, preprint, Dipartimento di Matematica, Politenico di Milano (2001).

[5] E. B. Davies and J. M. Lindsay, *Superderivations and symmetric Markov semigroups*, Comm. Math. Phys. **157**(1993), no.2, 359-370.

[6] M.P. Evans and R.L. Hudson, *Multidimensional quantum diffusions*, in 'Quantum Probability and Applications III', eds. L. Accardi and W. von Waldenfels, Lecture Notes in Mathematics **1303**, Springer-Verlag, Heidelberg (1988).

[7] D. Goswami, *Quantum Stochastic Dilation of Completely Positive Semigroups and Flows*, PhD thesis, Indian Statistical Institute, Delhi (1999).

[8] D. Goswami, J.M. Lindsay, K.B. Sinha and S.J. Wills, *Dilation of Markovian Cocycles on a von Neumann Algebra*, preprint, to appear in Pacific Jour. Math.

[9] D. Goswami and K.B. Sinha, *Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra*, Comm. Math. Phys. **205** (1999), 377–405.

[10] D. Goswami and K. B. Sinha, *Minimal quantum dynamical semigroup on a von Neumann algebra*, Inf. Dim. Anal. Quan. Prob. Related Topics **2**(1999), no. 2, 221-239.

[11] D. Goswami and K.B. Sinha, *Stochastic Dilation of Symmetric Completely Positive Semigroups* , submitted to Jour. Funct Anal.

[12] R.L. Hudson and K.R. Parthasarathy, *Quantum Itô's formula and stochastic evolutions*, Comm. Math. Phys. **93** (1984), 301–323.

[13] E. B. Davies, *Quantum dynamical semigroups and the neutron diffusion equation*, Rep. Math. Phys. **11**(1977), no. 2, 169-188.

[14] B. Kümmerer and H. Maassen, *The essentially commutative dilation of dynamical semigroups on M_n*, Comm. Math. Phys. **109**, no. 1 (1987), 1-22.

[15] G. Lindblad, *On the generators on quantum dynamical semigroups*, Comm. Math. Phys. **48**(1976), no. 2, 119-130.

[16] E. Nelson, *Notes on non-commutative integration*, Jour. Funct. Anal. **15**(1974), 103-116.

[17] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel (1992).

[18] J-L. Sauvageot, *Markov quantum semigroups admit covariant Markov C^*-dilations*, Comm. Math. Phys. **106** (1986), 91–103.

[19] J-L. Sauvageot, *Tangent bimodule and locality for dissipative operators on C^*-algebras*, in 'Quantum Probability and Applications IV,' edited by L. Accardi and W. von Waldenfels, Springer Lecture Notes in Mathematics **1396** Berlin, (1989), 322–338.

[20] I. Segal, *A non-commutative extension of abstract integration*, Ann. of Math.(2) **57** (1953), 401-457.

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