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A NOTE ON THE RESOLVABILITY OF BALANCED INCOMPLETE BLOCK DESIGNS

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INTRODUCTION

1. If v varieties or treatments are to be compared in randomized blocks of k experimental units ($k < v$), then the block differences can be simply eliminated if the arrangement is such that every variety occurs in r blocks, and every two varieties occur together in λ blocks. Such an arrangement is known as a *balanced incomplete block design* with parameters v, b, r, k, λ. These parameters obviously satisfy the equations

$$bk = vr \qquad .. \quad (0\cdot10)$$

$$\lambda(v-1) = r(k-1) \qquad .. \quad (0\cdot11)$$

Fisher[4] has also proved that the inequality

$$b \geqq v \qquad .. \quad (0\cdot12)$$

must hold. These designs were first introduced in experimental studies by F. Yates[6], and a number of papers have since then appeared dealing with the problem of the construction of these designs[1, 2, 3].

It is advantageous to separate if possible the b blocks of a balanced incomplete block design into r sets of n blocks each ($b = nr$), so that each variety occurs once among the blocks of a given set (*i.e.* each set contains a complete replication). When this separation is possible the design may be called a *resolvable balanced incomplete block design*. In this case it is possible to analyse the results of the experiment in two different ways, firstly as an ordinary randomised block design, and secondly as a balanced incomplete block design. It is therefore possible to obtain an unbiased estimate of the reduction in the error variance per plot, due to the smaller size of the incomplete blocks, and hence to obtain an estimate of the gain of information (if any) by using the incomplete block design instead of an ordinary randomised block design. It is therefore a problem of some interest to study the conditions under which a design with given parameters is resolvable.

2. It has been shown in §1, that *for a resolvable balanced incomplete block design, Fisher's inequality* (0·12) *can be replaced by the more stringent inequality*

$$b \geqq v+r-1 \qquad .. \quad (0{\cdot}21)$$

For a balanced incomplete block design it is known that when the equality in (0·12) holds, the design has the property that any two blocks have the same number of varieties in common; and conversely for a design with this property, the equality must hold in (0·12). It has been shown in §2, that *if for a resolvable balanced incomplete block design the equality in* (0·21) *holds, then two blocks belonging to different sets have the same number of varieties in common.* Balanced incomplete block designs with this property may be called *affine resolvable.* Conversely *for an affine resolvable balanced incomplete block design, the equality must hold in* (0·21).

For a resolvable balanced incomplete block design the number of varieties common to two blocks of different sets has been shown to be k^2/v. This number must therefore be integral. It follows that for designs with parameters

(i)	$v=6$,	$b=10$,	$r=5$,	$k=3$,	$\lambda=2$
(ii)	$v=10$,	$b=18$,	$r=9$,	$k=5$,	$\lambda=4$
(iii)	$v=28$,	$b=36$,	$r=9$,	$k=7$,	$\lambda=2$
(iv)	$v=14$,	$b=26$,	$r=13$,	$k=7$,	$\lambda=6$

though a combinatorial solution is known to exist, we can never separate the blocks into sets, such that each set contains a complete replication.

Finally the parameters of an affine resolvable balanced incomplete block design have been expressed in terms of two integral parameters n and t, and it has been shown that under certain conditions these designs can be constructed by the help of Finite Geometry, but that there exist also other (non-geometrical) affine resolvable designs.

§ 1

1. Consider a balanced incomplete block design with parameters v, b, r, k, λ. If it is resolvable, then

$$v = nk, \qquad b = nr \qquad .. \quad (1{\cdot}10)$$

and the b blocks are divisible into r sets of n blocks each, such that the blocks of a given set give a complete replication (*i.e.* all the varieties occur exactly once among the blocks of a given set). Let the blocks belonging to the i-th set (S_i) be

$$B_{i1}, B_{i2} \;\ldots\; B_{in} \qquad (i = 0, 1, \ldots r-1)$$

Let us take any particular block say B_{01} of the set (S_0), and let l_{ij} be the number of varieties common to the block B_{01} and the block B_{ij} of the set (S_i), for $i = 1, 2, \ldots r-1$, $j = 1, 2, \ldots n$. Let m denote the mean and s^2 the variance of the set of $n(r-1)$ quantities l_{ij}

Each of the k varieties occuring in B_{01} is replicated r times. If a variety occurs in B_{01} it cannot occur in the other blocks of the set (S_0). Hence it occurs just $r-1$ times among the blocks of the sets $(S_1), (S_2), \ldots (S_{r-1})$. Hence

$$\Sigma\, l_{ij} = k(r-1) \qquad .. \quad (1{\cdot}11)$$

$$\therefore\; m = \frac{k}{n} = \frac{k^2}{v} \qquad .. \quad (1{\cdot}12)$$

Again the $k(k-1)/2$ pairs involved in B_{01} each appear $\lambda - 1$ times in the sets (S_1), (S_2), ..., (S_{r-1}). Hence

$$\tfrac{1}{2}\sum l_{ij}(l_{ij}-1) = \tfrac{1}{2}(\lambda-1)k(k-1) \quad .. \quad (1\cdot 13)$$

$$\therefore \sum l^2_{ij} = k\{(r-1)+(\lambda-1)(k-1)\} \quad .. \quad (1\cdot 14)$$

Now from (0·11) and (1·10)

$$\lambda = \frac{r(k-1)}{nk-1} \quad .. \quad (1\cdot 15)$$

Substituting in (1·14) we have

$$\Sigma l^2_{ij} = \frac{k\{(nk-1)(r-k)+r(k-1)^2\}}{nk-1} \quad .. \quad (1\cdot 16)$$

$$\therefore s^2 = \frac{\Sigma(l_{ij}-m)^2}{n(r-1)}$$

$$= \frac{\Sigma l^2_{ij}}{n(r-1)} - m^2 \quad .. \quad (1\cdot 17)$$

$$= \frac{k\{(nk-1)(r-k)+r(k-1)^2\}}{n(r-1)(nk-1)} - \frac{k^2}{n^2}$$

$$= \frac{k^2(n-1)\{r(n-1)-(nk-1)\}}{n^2(r-1)(nk-1)}$$

Hence from (1·1)

$$s^2 = \frac{k(v-k)(b-v-r+1)}{n^2(r-1)(v-1)} \quad .. \quad (1\cdot 18)$$

2. Since s^2 is non-negative we have the following interesting result :—

(A) *If a balanced incomplete block design with parameters v, b, r, k, λ is resolvable, the following inequality must hold :*

$$b \geqq v + r - 1 \quad .. \quad (1\cdot 20)$$

Since $r > 1$, this is a more stringent inequality than Fisher's inequality (0·12), which holds irrespective of the assumption of resolvability.

§ 2.

1. If the r sets, (S_0), (S_1) ... (S_{r-1}), into which the b blocks of a resolvable balanced incomplete block design are divisible (each set containing a complete replication) are such that any two blocks belonging to different sets, have always the same number of varieties in common, then the design is said to be *affine resolvable*.

As an example of an affine resolvable balanced incomplete block design, we may take the orthogonal series design.

$$v = n^2, \ \ b = n(n+1), \ \ r = n+1, \ \ k = n, \ \ \lambda = 1 \quad .. \quad (2\cdot 10)$$

where n is a prime power p^e.

The design can either be written down by the help of a complete set of orthogonal Latin squares, or by taking the n^2 points of the Affine Geometry EG(2, p^n) as our varieties, and the n^2+n lines as our blocks. Then the blocks corresponding to the same pencil of parallel lines form a set. Clearly each variety occurs just once among the blocks of a set, and any two blocks belonging to different sets have just one variety in common.

2. Let us consider a resolvable balanced incomplete block design. Then adopting the notation of §1, we see that for affine resolvability

$$l_{ij} = m, \quad (i = 0, 1, \ldots r-1\,; \;\; j = 1, 2, \ldots n) \qquad .. \quad (2\cdot20)$$

Hence s^2 vanishes, and we have

$$b = v + r - 1 \qquad .. \quad (2\cdot21)$$

Conversely if for a resolvable balanced incomplete design, the condition (2·21) is satisfied, then s^2 must vanish, and consequently (2·20) must hold. Hence the design is affine resolvable. Since $m = k/n = k^2/v$ is the number of varieties common to any two blocks of different sets this number must be integral. These results may be stated as follows:

(B) *If a balanced incomplete block design with parameters v, b, r, k, λ is affine resolvable then*

$$b = v + r - 1 \qquad .. \quad (2.21)$$

Conversely if for a resolvable balanced incomplete block design the above condition holds, then the design is affine resolvable. The number of varieties common to any two blocks of different sets is k^2/v, so that k^2 must be divisible by v.

This result may be compared with the following which is known to hold for any balanced incomplete block design.

If a balanced incomplete block design with parameters v, b, r, k, λ has the property that any two blocks have the same number of varieties in common, then the design is symmetrical *i.e.*

$$b = v \qquad .. \quad (2\cdot22)$$

Conversely if the design is symmetrical, *i.e.* the condition (2·22) holds, then any two blocks have exactly the same number of varieties in common, *viz.* λ.

3. The result of the last paragraph enables us to prove the non-resolvability of a number of balanced incomplete block designs.

It is known [1, 3] that we can form balanced incomplete block designs with the parameters

$$v = 6, \quad b = 10, \quad r = 5, \quad k = 3, \quad \lambda = 2 \qquad .. \quad (2.30)$$

$$v = 10, \quad b = 18, \quad r = 9, \quad k = 5, \quad \lambda = 4 \qquad .. \quad (2\cdot31)$$

$$v = 28, \quad b = 36, \quad r = 9, \quad k = 7, \quad \lambda = 2 \qquad .. \quad (2\cdot32)$$

$$v = 14, \quad b = 26, \quad r = 13, \quad k = 7, \quad \lambda = 6 \qquad .. \quad (2\cdot33)$$

Since in each of the above cases the condition (2·21) holds, it follows that if there exists a resolvable design with parameters given by (2·30), (2·31), (2·32) or (2·33), it would be affine resolvable. Hence k^2/v would be integral. But k^2/v is non-integral in each case, being 3/2, 5/2, 7/4 and 7/2 respectively. Hence

(C) *No resolvable balanced incomplete block design with parameters given by* (2·30), (2·31), (2·32) or (2·33) *can exist.*

On the other hand if the parameters v, b, r, k, λ have any one of the following values

$$v = 8, \quad b = 14, \quad r = 7, \quad k = 4, \quad \lambda = 3 \qquad .. \quad (2·34)$$

$$v = 12, \quad b = 22, \quad r = 11, \quad k = 6, \quad \lambda = 5 \qquad .. \quad (2·35)$$

then not only is the condition (2·21) satisfied but k^2/v is integral, being 2 in the first case and 3 in the second case.

A resolvable (necessarily affine resolvable) balanced incomplete block design with parameters given by (2·34) is already known[3]. An affine resolvable design with parameters given by (2·35) is given below. The varieties are numbered 1, 2, ... 12 and blocks occupying the same row belong to the same set.

(1, 3, 4, 5, 9, 11),	(2, 6, 7, 8, 10, 12)
(2, 4, 5, 6, 10, 1)	(3, 7, 8, 9, 11, 12)
(3, 5, 6, 7, 11, 2)	(4, 8, 9, 10, 1, 12)
(4, 6, 7, 8, 1, 3),	(5, 9, 10, 11, 2, 12)
(5, 7, 8, 9, 2, 4),	(6, 10, 11, 1, 3, 12)
(6, 8, 9, 10, 3, 5),	(7, 11, 1, 2, 4, 12)
(7, 9, 10, 11, 4, 6),	(8, 1, 2, 3, 5, 12)
(8, 10, 11, 1, 5, 7),	(9, 2, 3, 4, 6, 12)
(9, 11, 1, 2, 6, 8),	(10, 3, 4, 5, 7, 12)
(10, 1, 2, 3, 7, 9),	(11, 4, 5, 6, 8, 12)
(11, 2, 3, 4, 8, 10),	(1, 5, 6, 7, 9, 12)

4. Let v, b, r, k, λ be the parameters of an affine resolvable balanced incomplete block design. Then from (1·10), (1·12) and (B).

$$v = mn^2, \; b = nr, \; k = mn \qquad .. \quad (2·40)$$

where m and n are integers. Now from (2·21)

$$nr = mn^2 + r - 1$$

where

$$r = \frac{mn^2 - 1}{n - 1} \qquad .. \quad (2·41)$$

Again from (0·11), (2·40) and (2·41)

$$\lambda = \frac{mn - 1}{n - 1} \qquad .. \quad (2·42)$$

Since λ is integral we must have

$$m = (n - 1)\, t + 1 \qquad .. \quad (2·43)$$

Thus finally using (2·40) — (2·43) we have the following result :—

(D) *The parameters* v, b, r, k, λ *of an affine resolvable balanced incomplete block design can be expressed in terms of only two integral parameters* n *and* t *in the following manner* :

$$v = n^2(n-1)t + n^2, \quad b = n(n^2 t + n + 1), \quad r = n^2 t + n + 1,$$
$$k = n(n-1)t + n, \quad \lambda = nt + 1 \qquad \ldots \quad (2{\cdot}44)$$

When $t = 0$, we get the orthogonal series design (2·10). As has been noted it can be written down by the help of the Affine Geometry EG(2, p^e) when $n = p^e$.

In general if t is connected with n by the relation

$$t = 1 + n + n^2 \ldots + n^{N-1} \qquad (2{\cdot}45)$$

then we get the design

$$v = n^{N+2},\; b = n + n^2 + n^3 + \ldots + n^{N+2},\; r = 1 + n + n^2 + \ldots n^{N+1},$$
$$k = n^{N+1},\; \lambda = 1 + n + n^2 + \ldots + n^N$$

To construct this design we need only take, the varieties as points of EG($N + 2$, p^e), and the blocks as the ($N + 1$)—flats of this geometry.

There exist however designs belonging to (2.44), but not included in the class of geometrical designs (2·45), for which the actual combinatorial solution can be written down. For example putting $n = 2, t = 2$, we get the design with parameters (2·35), the combinatorial solution for which has already been given.

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