GENERALIZED INVERSES OF BORDERED MATRICES*

R. B. BAPAT† AND BING ZHENG‡

Abstract. Several authors have considered nonsingular borderings $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ of B and investigated the properties of submatrices of A^{-1}. Under specific conditions on the bordering, one can recover any g-inverse of B as a submatrix of A^{-1}. Borderings A of B are considered, where A might be singular, or even rectangular. If A is $m \times n$ and if B is an $r \times s$ submatrix of A, the consequences of the equality $m + n - rank(A) = r + s - rank(B)$ with reference to the g-inverses of A are studied. It is shown that under this condition many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. We also consider g-inverses of the bordered matrix when certain rank additivity conditions are satisfied. It is shown that any g-inverse of B can be realized as a submatrix of a suitable g-inverse of A, under certain conditions.

Key words. Generalized inverse, Moore-Penrose inverse, Bordered matrix, Rank additivity.

AMS subject classifications. 15A09, 15A03.

1. Introduction. Let A be an $m \times n$ matrix over the complex field and let A^* denote the conjugate transpose of A. We recall that a generalized inverse G of A is an $n \times m$ matrix which satisfies the first of the four Penrose equations:

$$(1)\ AXA = A \qquad (2)\ XAX = X \qquad (3)\ (AX)^* = AX \qquad (4)\ (XA)^* = XA.$$

For a subset $\{i, j, \dots\}$ of the set $\{1, 2, 3, 4\}$, the set of $n \times m$ matrices satisfying the equations indexed by $\{i, j, \dots\}$ is denoted by $A\{i, j, \dots\}$. A matrix in $A\{i, j, \dots\}$ is called an $\{i, j, \dots\}$-inverse of A and is denoted by $A^{(i,j,\dots)}$. In particular, the matrix G is called a $\{1\}$-inverse or a g-inverse of A if it satisfies (1). As usual, a g-inverse of A is denoted by A^-. If G satisfies (1) and (2) then it is called a reflexive inverse or a $\{1, 2\}$-inverse of A. Similarly, G is called a $\{1, 2, 3\}$-inverse of A if it satisfies (1),(2) and (3). The Moore-Penrose inverse of A is the matrix G satisfying (1)-(4). Any matrix A admits a unique Moore-Penrose inverse, denoted $A^\dagger$. If A is $n \times n$ then G is called the group inverse of A if it satisfies (1), (2) and $AG = GA$. The matrix A has group inverse, which is unique and denoted by $A^\sharp$, if and only if $rank(A) = rank(A^2)$. We refer to [4], [6] for basic results on g-inverses.

Suppose

$$A = \begin{matrix} & q_1 & q_2 \\ p_1 & B & C \\ p_2 & D & X \end{matrix} \tag{1.1}$$

*Received by the editors on 15 July 2002. Accepted for publication on 14 January 2003. Handling Editor: Steve J. Kirkland.

†Indian Statistical Institute, New Delhi, 110016, India (rbb@isid.ac.in).

‡ Department of Mathematics, Shanghai University, Shanghai 200436, People's Republic of China (dzhengbing@eastday.com). The present work was done while this author was visiting the Indian Statistical Institute during October 2001-September 2002. He would like to express his thanks for the warm hospitality he enjoyed during his visit. Supported by the Chinese Scholarship Council.

is a partitioned matrix. We say that A is obtained by bordering B. We will generally partition a g-inverse A^- of A as

$$A^- = \begin{array}{c} \\ q_1 \\ q_2 \end{array} \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \end{array}, \tag{1.2}$$

which is in conformity with A^*.

We say that the g-inverses of A have the "block independence property" if for any g-inverses

$$A_i^- = \begin{pmatrix} E_i & F_i \\ G_i & Y_i \end{pmatrix},\ i = 1, 2$$

of A, $\begin{pmatrix} E_1 & F_1 \\ G_1 & Y_2 \end{pmatrix}$, $\begin{pmatrix} E_1 & F_1 \\ G_2 & Y_1 \end{pmatrix}$ etc. are also g-inverses of A.

If A is an $m \times n$ matrix, then the following function will play an important role in this paper:

$$\psi(A) = m + n - rank(A).$$

An elementary result is given next. For completeness, we include a proof.

LEMMA 1.1. *If B is a submatrix of A, then $\psi(B) \leq \psi(A)$.*

Proof. Let

$$A = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} B & C \\ D & X \end{pmatrix} \end{array}.$$

Then

$$\begin{aligned} rank(A) &\leq rank\,(\,B \quad C\,) + rank\,(\,D \quad X\,) \\ &\leq rank(B) + rank(C) + p_2 \\ &\leq rank(B) + q_2 + p_2. \end{aligned}$$

From this inequality, we get $\psi(B) \leq \psi(A)$. □

Note that a matrix B with $rank(B) = r$ can be completed to a nonsingular matrix A of order n if and only if $\psi(B) \leq n$ [10, Theorem 5]. As another example of a result concerning ψ, if

$$A = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} B & C \\ D & O \end{pmatrix} \end{array}$$

is a nonsingular matrix of order n, $n = p_1 + p_2 = q_1 + q_2$, then A^{-1} is of the form

$$A^{-1} = \begin{array}{c} \\ q_1 \\ q_2 \end{array} \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} E & F \\ G & O \end{pmatrix} \end{array}$$

if and only if $\psi(B) = \psi(A)$. This will follow from Theorem 3.1.

Several authors ([4], [5], [8], [10], [11], [12]) have considered nonsingular borderings A of B and investigated the properties of submatrices of A^{-1}. Under specific conditions on the bordering, one can recover a special g-inverse of B as a submatrix of A^{-1}. It turns out that in all such cases the condition $\psi(B) = \psi(A)$ holds. The main theme of the present paper is to investigate borderings A of B, where A might be singular, or even rectangular. We show that if $\psi(A) = \psi(B)$ is satisfied then many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. For example, any g-inverse of B can be obtained as a submatrix of A^- where A is a bordering of B with $\psi(A) = \psi(B)$. This will be shown in Section 4. In Section 5 we show how to obtain the Moore-Penrose inverse and the group inverse by a general, not necessarily nonsingular, bordering. In the next two sections we consider general borderings A of B and obtain some results concerning A^-.

We say that rank additivity holds in the matrix equation $A = A_1 + \cdots + A_k$ if $rank(A) = rank(A_1) + \cdots + rank(A_k)$. Let $R(A)$ and $N(A)$ denote the range space of A and the null space of A respectively. We will need the following well-known result.

Lemma 1.2. [2] *Let A, B be $m \times n$ matrices. Then the following conditions are equivalent:*

(i) $rank(B) = rank(A) + rank(B - A)$.
(ii) *Every B^- is a g-inverse of A.*
(iii) $AB^-(B - A) = O,\ (B - A)B^-A = O$ *for any* B^-.
(iv) *There exists a B^- that is a g-inverse of both A and $B - A$.*

It follows from the proof of Lemma 1.1 that if $\psi(B) = \psi(A)$ then rank additivity holds in

$$\begin{pmatrix} B & C \\ D & X \end{pmatrix} = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & X \end{pmatrix} \tag{1.3}$$

and in

$$\begin{pmatrix} B & C \\ D & X \end{pmatrix} = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & X \end{pmatrix}. \tag{1.4}$$

In Section 2 we discuss necessary and sufficient conditions for the block matrix $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ to be a g-inverse of $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$ under the assumption of rank additivity in (1.3) and (1.4). In section 3, necessary and sufficient conditions for the block matrix $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ to be a g-inverse of $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$ are considered under the assumption $\psi(A) = \psi(B)$. Certain related results are also proved. Some additional references on g-inverses of bordered matrices as well as generalizations of Cramer's rule are [1], [14], [16], [17].

2. G-inverses of a bordered matrix . Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ be a block matrix which is a bordering of B. In this section we will study some necessary and sufficient conditions for a partitioned matrix $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$, conformal with A^*, to be a g-inverse of A.

THEOREM 2.1. *Let* $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. *Then rank additivity holds in* (1.3) *and* (1.4) *and* $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *is a g-inverse of A if and only if the following conditions hold.*

(i) $BEB = B, CGC = C, DFD = D, XGC = DFX = -DEC, X = XYX - DEC.$
(ii) $CYD, BFX, CYX, XGB, XYD, BEC, DEB, CGB, BFD$ *are null matrices.*
Furthermore, if $EBE = E$, *then* $X = XYX$.

Proof. "Only if" part: Assume rank additivity in (1.3) and (1.4) and that H is a g-inverse of A. Then by (ii) of Lemma 1.2, H is also a g-inverse of each summand matrix in (1.3) and (1.4). Using the definition of g-inverse, we easily get $BEB = B$, $CGC = C$, $DFD = D$, $XYD = O$, $CYX = O$, and

$$DFX + XYX = X, \ XGC + XYX = X. \tag{2.1}$$

On the other hand, by (iii) of Lemma 1.2, we have

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & C \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow BEC = O,$$

$$\begin{pmatrix} O & C \\ O & O \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CGB = O,$$

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & O \\ D & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow BFD = O, \ BFX = O,$$

$$\begin{pmatrix} O & C \\ O & O \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & O \\ D & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CYD = O, \ CYX = O,$$

$$\begin{pmatrix} O & C \\ O & X \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & O \\ D & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CYD = O, \ XYD = O.$$

$$\left.\begin{matrix} \begin{pmatrix} O & O \\ D & X \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & C \\ O & X \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \end{matrix}\right\} \Rightarrow XGB = O, \ DEB = O,$$

$$\left.\begin{matrix} \begin{pmatrix} O & O \\ D & X \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & C \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ D & O \end{pmatrix}\begin{pmatrix} E & F \\ G & Y \end{pmatrix}\begin{pmatrix} O & C \\ O & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \end{matrix}\right\} \Rightarrow XGC = DFX = -DEC. \tag{2.2}$$

Also, (2.1) and (2.2) imply $X = XYX - DEC$.

"If" part: If the conditions (i) and (ii) hold, then it is easy to verify that H is a g-inverse of each summand matrix in (1.3) and (1.4). By (iv) in Lemma 1.2, rank additivity holds in (1.3) and (1.4). It is also easily verified that H is a g-inverse of A.

If $EBE = E$, then $DEC = O$ and so $X = XYX$. □

We note certain consequences of Theorem 2.1.

COROLLARY 2.2. *Let* $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. *Then rank additivity holds in* (1.3) *and* (1.4) *and the matrix* $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ *is a g-inverse of A if and only if the following conditions hold.*
(i) $BEB = B$, $CGC = C$, $DFD = D$, $DEC = -X$.
(ii) BEC, DEB, CGB, BFD *are null matrices.*
Furthermore if $EBE = E$, *then* $X = O$.

COROLLARY 2.3. *Let* $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$. *Then* $R(B) \cap R(C) = \{0\}$, $R(B^*) \cap R(D^*) = \{0\}$ *and* $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *is a g-inverse of A if and only if the following conditions hold.*
(i) $BEB = B$, $CGC = C$, $DFD = D$.
(ii) $CYD, DEC, BEC, DEB, CGB, BFD$ *are null matrices.*
In this case, the g-inverses of A have the block independence property.

REMARK 2.4. As the conditions $R(B) \cap R(C) = \{0\}$, $R(B^*) \cap R(D^*) = \{0\}$ together with $X = O$ imply rank additivity in (1.3) and (1.4), Corollary 2.3 is a direct consequence of Theorem 2.1. In particular, conditions (i) and (ii) indicate that the block matrices in $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ can be independently chosen if it is a g-inverse of A. In other words, the g-inverses of $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$ have the block independence property. Thus Corollary 2.3 complements the known result (see Theorem 3.1 in [15] and Lemma 5(1.2e) in [7]) that the g-inverses of A have the block independence property if and only if

$$\begin{aligned} rank(A) &= rank\begin{pmatrix} B \\ D \end{pmatrix} + rank(C) \\ &= rank\begin{pmatrix} B & C \end{pmatrix} + rank(D). \end{aligned}$$

The next result can also be viewed as a generalization of Corollary 2.3. This type of rank additivity has been considered, for example, in [13].

THEOREM 2.5. *Let* $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ *and suppose*

$$rank(A) = rank(B) + rank(C) + rank(D) + rank(X).$$

Then $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *is a g-inverse of* A *if and only if the following conditions hold.*
(i) $BEB = B,\ CGC = C,\ DFD = D,\ XYX = X$.
(ii) $BFX, CYD, CYX, DFX, XGB, XGC, XYD, BEC, BFD, CGB, DEB$, DEC *are null matrices.*

Proof. Note that the condition $rank(A) = rank(B) + rank(C) + rank(D) + rank(X)$ implies rank additivity in

$$A = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & X \end{pmatrix}.$$

Now the proof is similar to that of Theorem 2.1. □

A generalization of Theorem 2.5 is stated next; the proof is omitted.

Theorem 2.6. *Let* $A = (\,A_{i,j}\,),\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$ *be an* $m \times n$ *block matrix. If* $rank(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} rank(A_{i,j})$, *then* $G = (\,G_{l,s}\,),\ l = 1, 2, \cdots, n,\ s = 1, 2, \cdots, m$ *is a g-inverse of* A *if and only if the following equations hold.*

$$A_{i,j}G_{j,l}A_{l,s} = \begin{cases} A_{i,j} & (i,j) = (l,s) \\ O & (i,j) \neq (l,s) \end{cases}.$$

3. G-inverses of a block matrix A **with** $\psi(A) = \psi(B)$**.** Let A and H be matrices of order $m \times n$ and $n \times m$ respectively, partitioned as follows:

$$(3.1) \qquad A = \begin{matrix} & q_1 & q_2 \\ p_1 & B & C \\ p_2 & D & X \end{matrix} \quad \text{and} \quad H = \begin{matrix} & p_1 & p_2 \\ q_1 & E & F \\ q_2 & G & Y \end{matrix},$$

where $p_1 + p_2 = m$ and $q_1 + q_2 = n$. By $\eta(A)$ we denote the row nullity of A, which by definition is the number of rows minus the rank of A. If $m = n$, A is nonsingular, $H = A^{-1}$ and if A and H are partitioned as in (3.1) then it was proved by Fiedler and Markham [10], and independently by Gustafson [9], that

$$(3.2) \qquad \eta(B) = \eta(Y).$$

The following result, proved in [3], will be used in the sequel. We include an alternative simple proof for completeness.

Lemma 3.1. *Let* A *and* H *be matrices of order* $m \times n$ *and* $n \times m$ *respectively, partitioned as in* (3.1). *Assume* $rank(A) = r$ *and* $rank(H) = k$. *Then the following assertions are true.*
(i) *If* $AHA = A$, *then*

$$-(m - r) \le \eta(Y) - \eta(B) \le n - r.$$

(ii) *If* $HAH = H$, *then*

$$-(n - k) \le \eta(B) - \eta(Y) \le m - k.$$

Proof. (i) According to a result on bordered matrices and g-inverses [11, Theorem 1], there exist matrices P, Q and Z of order $m\times(m-r)$, $(n-r)\times n$ and $(n-r)\times(m-r)$ respectively, such that the matrix

$$S=\begin{pmatrix} A & P \\ Q & Z \end{pmatrix}$$

is nonsingular and the submatrix formed by the first n rows and the first m columns of $T=S^{-1}$ is W. Thus we may write

$$S=\begin{matrix} \\ p_1 \\ p_2 \\ n-r \end{matrix}\begin{matrix} q_1 \quad q_2 \quad m-r \\ \begin{pmatrix} B & C & P_1 \\ D & X & P_2 \\ Q_1 & Q_2 & Z \end{pmatrix}\end{matrix} \quad \text{and} \quad T=\begin{matrix} \\ q_1 \\ q_2 \\ m-r \end{matrix}\begin{matrix} p_1 \quad p_2 \quad n-r \\ \begin{pmatrix} E & F & U_1 \\ G & Y & U_2 \\ V_1 & V_2 & W \end{pmatrix}\end{matrix}.$$

Since S is nonsingular, we have, using (3.2),

$$\eta(B)=\eta(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix})=q_2+m-r-rank\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}.$$

Now by Lemma 1.1

$$rank(Y)\le rank\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\le rank(Y)+m+n-2r,$$

and hence

$$-(m-r)\le\eta(Y)-\eta(B)\le n-r.$$

The result (ii) follows from (i). □

The following result, proved using Lemma 3.1, will be used in the sequel.

THEOREM 3.2. *Let* $A=\begin{pmatrix} B & C \\ D & X \end{pmatrix}$ *with* $\psi(A)=\psi(B)$. *Then for any g-inverse* $A^-=\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *of* A, $Y=O$.

Proof. Assume the sizes of the block matrices in A to be as in (3.1). By Lemma 3.1 we have

$$-(m-r)\le\eta(Y)-\eta(B)\le n-r.$$

It follows that

$$-m+r\le q_2-rank(Y)-p_1+rank(B).$$

Using $\psi(A)=\psi(B)$ and the inequality above, $rank(Y)=0$ and hence $Y=O$. □

THEOREM 3.3. *Let* $A=\begin{pmatrix} B & C \\ D & X \end{pmatrix}$. *Then* $\psi(A)=\psi(B)$ *and* $H=\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *is a g-inverse of* A *if and only if the following equations hold.*

(i) $Y = O,\ BEB = B,\ GC = I,\ DF = I.$
(ii) $DEC = -X.$
(iii) $BEC = O,\ DEB = O,\ BF = O,\ GB = O.$
Furthermore, if $EBE = E$, *then* $X = O$.

Proof. If H is a g-inverse of A with $\psi(A) = \psi(B)$, then by Theorem 3.2, we know $Y = O$. From the proof of Lemma 1.1, the condition $\psi(A) = \psi(B)$ also indicates rank additivity in (1.3) and (1.4). Note that C and D are also of full column rank and of full row rank respectively under the condition $\psi(A) = \psi(B)$. Then the proof of the theorem is similar to that of Theorem 2.1. □

The proof of the following result is also similar and is omitted.

THEOREM 3.4. *Let* $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ *and consider the statements:*
(i) $Y = O,\ BEB = B,\ GC = I,\ DF = I,\ BF = O,\ GB = O.$
(ii) $EB + FD$ *is hermitian.*
(iii) $BE + CG$ *is hermitian.*
(iv) $EBE + FDE = E$ (v) $EBE + ECG = E.$
Then
(a) $\psi(A) = \psi(B)$ *and* $H \in A\{1,2,3\}$ *if and only if* (i), (ii), (iv) *hold,* $DEC = -X,\ EC = FDEC$ *and* $DEB = O$.
(b) $\psi(A) = \psi(B)$ *and* $H \in A\{1,2,4\}$ *if and only if* (i), (iii), (v) *hold,* $DEC = -X$, $DE = DECG$ *and* $BEC = O$.
(c) $\psi(A) = \psi(B)$ *and* $H = A^{\dagger}$ *if and only if* (i)-(v) *hold,* $DE + XG = O$ *and* $EC + FX = O$.

The two previous results will be used in the proof of the next result.

THEOREM 3.5. *let* $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. *Then the following conditions are equivalent:*

(1) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^{\dagger} = \begin{pmatrix} B^{\dagger} & D^{\dagger} \\ C^{\dagger} & X^{\dagger} \end{pmatrix}$.

(2) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} B^{\dagger} & D^{\dagger} \\ C^{\dagger} & X^{\dagger} \end{pmatrix}$ *is a g-inverse of A.*

(3) $X = O,\ C^{\dagger}C = I,\ DD^{\dagger} = I, BD^{\dagger} = O,\ C^{\dagger}B = O.$

(4) $X = O,\ C^{\dagger}C = I,\ DD^{\dagger} = I, BD^{*} = O,\ C^{*}B = O.$

(5) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} E & D^{\dagger} \\ C^{\dagger} & X^{\dagger} \end{pmatrix}$ *is a g-inverse of A for some* $E \in B^{\{1,2\}}$.

(6) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} E & D^{\dagger} \\ C^{\dagger} & X^{\dagger} \end{pmatrix}$ *is a g-inverse of A for some E.*

(7) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^{\dagger} = \begin{pmatrix} B^{\dagger} & F \\ G & Y \end{pmatrix}$ *for some matrices F, G, Y.*

(8) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} B^{\dagger} & F \\ C^{\dagger} & Y \end{pmatrix}$ *is a $\{1,2,3\}$-inverse of A for some F, Y.*

(9) $\psi(A) = \psi(B)$ *and* $\begin{pmatrix} B^{\dagger} & D^{\dagger} \\ G & Y \end{pmatrix}$ *is a $\{1,2,4\}$-inverse of A for some G, Y.*

Proof. Clearly, (1) $\Rightarrow$ (2).

(2 $\Rightarrow$ (3): This follows from Theorem 3.3.

(3) $\Leftrightarrow$ (4): Since $BD^\dagger = O$ and $C^\dagger B = O$ are equivalent to $BD^* = O$ and $C^*B = O$ respectively, we have this implication.

(3) $\Rightarrow$ (1): Note that $BD^\dagger = O$ and $C^\dagger B = O$ imply $DB^\dagger = O$ and $B^\dagger C = O$. Then it is easy to verify that $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is $A^\dagger$ thus (1) holds.

Clearly, (1) $\Rightarrow$ (5) $\Rightarrow$ (6).

(6) $\Rightarrow$ (3): By Theorem 3.3, if $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is a g-inverse of A for some matrix E, then we have $X^\dagger = O$, $C^\dagger C = I$, $DD^\dagger = I$, $BD^\dagger = O$ and $C^\dagger B = O$. Note that $X^\dagger = O \Leftrightarrow X = O$, thus (3) holds.

(6) $\Rightarrow$ (1): This follows from (6) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

Obviously, (1) $\Rightarrow$ (7), (1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (9).

(7) $\Rightarrow$ (1): By Theorem 3.3, we have $X = O$, $Y = O$, $GC = I$, $DF = I$, $BF = O$ and $GB = O$. Clearly, $G \in C\{1,2,4\}$ and $F \in D\{1,2,3\}$. Using the hermitian property of the matrices $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $\begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}$, $\begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}\begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $BB^\dagger$ and $B^\dagger B$, it is easy to conclude that CG and FD are also hermitian. Thus $F = D^\dagger$ and $G = C^\dagger$. Note that $Y = X^\dagger = O$ and (1) is proved.

Similarly, using Theorem 3.4 we can show (8) $\Rightarrow$ (1) and (9) $\Rightarrow$ (1) and the proof is complete. □

4. Obtaining any g-inverse by bordering. By Theorem 3.3 if $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ with $\psi(A) = \psi(B)$ and if $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a g-inverse of A, then E is a g-inverse of B which also satisfies $DEC = -X$, $BEC = O$ and $DEB = O$. Such an E, hereafter, will be denoted by $E_{(C,D,X)}$. Note that $E_{(C,D,X)}$ is not uniquely determined by C, D, X, since A^- is not unique. In this section we will investigate the converse problem, that is: for a given g-inverse E of B, how to construct C, D and X so that $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a g-inverse of $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ with $\psi(A) = \psi(B)$ for some matrices of proper sizes. We first state some well-known lemmas to be used later; see, for example, [4], [6].

Lemma 4.1. *The following three statements are equivalent:* (i) *E is a g-inverse of B,* (ii) *BE is an idempotent matrix and $rank(BE) = rank(B)$, and* (iii) *EB is an idempotent matrix and $rank(EB) = rank(B)$.*

Lemma 4.2. *E is a $\{1,2\}$-inverse of B if and only if E is a g-inverse of B and $rank(E) = rank(B)$.*

Lemma 4.3. *Let $H = UV$ be a rank factorization of a square matrix. Then the following three statements are equivalent:* (i) *H is an idempotent matrix,* (ii) *$I - H$ is an idempotent matrix, and* (iii) *$VU = I$.*

Theorem 4.4. (i) *Let E be a g-inverse of the $p_1 \times q_1$ matrix B with $rank(B) = r$. Then there exist C, D, and X such that $E = E_{(C,D,X)}$, where $rank(C) \leq p_1 - r$ and $rank(D) \leq q_1 - r$.*

(ii) *If* $E = E_{(C,D,X)}$, *then there exist matrices* U, V, $\bar{U}$ *and* $\bar{V}$ *such that*

$$I - BE = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} G \\ V \end{pmatrix} \text{ and } I - EB = \begin{pmatrix} F & \bar{U} \end{pmatrix} \begin{pmatrix} D \\ \bar{V} \end{pmatrix}$$

are the rank factorizations of $I - BE$ *and* $I - EB$ *respectively.*
(iii) $rank(E_{(C,D,X)}) = rank(B) + rank(R)$, *where*

$$R = \begin{pmatrix} -X & DEU \\ \bar{V}EC & \bar{V}EU \end{pmatrix} \tag{4.1}$$

for some matrices U *and* $\bar{V}$ *as in* (ii).

Proof. For a given g-inverse E of B, we use rank factorizations of $I - BE$ and $I - EB$, by which there exist C, D, X, F, G, U, $\bar{U}$, V, and $\bar{V}$ satisfying the following identities

$$I - BE = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} G \\ V \end{pmatrix}, \tag{4.2}$$

$$I - EB = \begin{pmatrix} F & \bar{U} \end{pmatrix} \begin{pmatrix} D \\ \bar{V} \end{pmatrix}, \tag{4.3}$$

$$X = -DEC.$$

To prove (i), we only need to show that these C, D, X, F and G along with $Y = O$ satisfy the conditions (i),(ii) and (iii) in Theorem 3.3. In fact, from (4.2) and (4.3), we have, in view of Lemma 4.3, that $\begin{pmatrix} G \\ V \end{pmatrix} \begin{pmatrix} C & U \end{pmatrix} = I$ and $\begin{pmatrix} D \\ \bar{V} \end{pmatrix} \begin{pmatrix} F & \bar{U} \end{pmatrix} = I$, implying

$$GC = I \text{ and } DF = I.$$

Again from (4.2) and (4.3), we have, by $(I - BE)B = O$ and $B(I - EB) = O$,

$$\begin{pmatrix} G \\ V \end{pmatrix} B = O \text{ and } B \begin{pmatrix} F & \bar{U} \end{pmatrix} = O, \tag{4.4}$$

and by $BE(I - BE) = O$ and $(I - EB)EB = O$,

$$BE \begin{pmatrix} C & U \end{pmatrix} = O \text{ and } \begin{pmatrix} D \\ \bar{V} \end{pmatrix} EB = O. \tag{4.5}$$

Now by (4.4), $GB = O$ and $BF = O$, and by (4.5), $BEC = O$ and $DEB = O$.

(ii) Let $E = E_{(C,D,X)}$. By Theorem 3.3, $BEC = O$, which means $R(C) \subseteq N(BE) = R(I - BE)$. Note that C is of full column rank under the condition $\psi(A) = \psi(B)$. Thus there exists a matrix U so that $R(\begin{pmatrix} C & U \end{pmatrix}) = R(I - BE)$

and the matrix $\begin{pmatrix} C & U \end{pmatrix}$ is of full column rank. Hence, there exists a matrix of full row rank which can be partitioned as $\begin{pmatrix} G \\ V \end{pmatrix}$ such that

$$I - BE = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} G \\ V \end{pmatrix}.$$

On the other hand, $DEB = O$ implies $N(I - EB) = R(EB) \subseteq N(D)$. So there exists a matrix $\bar{V}$ such that $\begin{pmatrix} D \\ \bar{V} \end{pmatrix}$ is of full row rank and

$$N(I - EB) = N(\begin{pmatrix} D \\ \bar{V} \end{pmatrix}).$$

From this we conclude that there exists a matrix of full column rank which can be partitioned as $\begin{pmatrix} F & \bar{U} \end{pmatrix}$ such that

$$I - EB = \begin{pmatrix} F & \bar{U} \end{pmatrix} \begin{pmatrix} D \\ \bar{V} \end{pmatrix}.$$

Now we prove (iii). If $E = E_{(C,D,X)}$, then from the proof of (ii) there exist matrices U, V, $\bar{U}$ and $\bar{V}$ such that (4.2) and (4.3) hold. Hence $BE \begin{pmatrix} C & U \end{pmatrix} = O$ and $\begin{pmatrix} D \\ \bar{V} \end{pmatrix} EB = O$. Therefore we have

$$\begin{aligned} \begin{pmatrix} B \\ D \\ \bar{V} \end{pmatrix} E \begin{pmatrix} B & C & U \end{pmatrix} &= \begin{pmatrix} B & O & O \\ O & DEC & DEU \\ O & \bar{V}EC & \bar{V}EU \end{pmatrix} \\ &= \begin{pmatrix} B & O \\ O & R \end{pmatrix}, \end{aligned}$$

where $R = \begin{pmatrix} D \\ \bar{V} \end{pmatrix} E \begin{pmatrix} C & U \end{pmatrix}$.

On the other hand,

$$\begin{aligned} \begin{pmatrix} E & F & \bar{U} \end{pmatrix} \begin{pmatrix} B & O \\ O & R \end{pmatrix} \begin{pmatrix} E \\ G \\ V \end{pmatrix} &= EBE + \begin{pmatrix} F & \bar{U} \end{pmatrix} R \begin{pmatrix} G \\ V \end{pmatrix} \\ &= EBE + (I - EB)E(I - BE) \\ &= E. \end{aligned}$$

Thus we have $rank(E_{(C,D,R)}) = rank(B) + rank(R)$. □

Theorem 4.4(i) and its proof show that for a given matrix B and its g-inverse E we can find matrices C of full column rank with $R(C) \subseteq N(BE)$ and D of full row rank with $R(EB) \subseteq N(D)$, as well as $X = -DEC$, F and G such that matrix

$\begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a g-inverse of $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$ with $\psi(A) = \psi(B)$. Furthermore, we have the following.

COROLLARY 4.5. *Let B and its g-inverse E be given. Then the matrix $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ satisfies $\psi(A) = \psi(B)$ and has a g-inverse of the form $\begin{pmatrix} E & F \\ G & O \end{pmatrix}$ if and only if C is of full column rank with $R(C) \subseteq N(BE)$ and D of full row rank with $R(EB) \subseteq N(D)$. In this case, $X = -DEC$, $F \in D\{1,3\}$, $G \in C\{1,4\}$, $BF = O$ and $GB = O$.*

Proof. Necessity: This follows from Theorem 3.3.

Sufficiency: The proof of sufficiency is similar to that of Theorem 4.4(i), (ii). □

As a special case we recover the following known result.

COROLLARY 4.6. [11, Theorem 1] *Let E be a g-inverse of B. Then for any matrix C of full column rank with $R(C) = N(BE)$ and any matrix D of full row rank with $N(D) = R(EB)$, the matrix*

$$A = \begin{pmatrix} B & C \\ D & -DEC \end{pmatrix}$$

is nonsingular and

$$A^{-1} = \begin{pmatrix} E & F \\ G & O \end{pmatrix},$$

where $F \in D\{1,3\}$, $BF = O$, $G \in C\{1,4\}$ and $GB = O$.

5. Moore-Penrose inverse and group inverse by bordering. For a given g-inverse E of B, Corollary 4.5 shows that C and D can be chosen with the conditions $R(C) \subseteq N(BE)$ and $R(D^*) \subseteq N((EB)^*)$ so that $A = \begin{pmatrix} B & C \\ D & -DEC \end{pmatrix}$ satisfies $\psi(A) = \psi(B)$ and has a g-inverse of the form $\begin{pmatrix} E & F \\ G & O \end{pmatrix}$. Further, Corollary 4.6 provides an approach to border the matrix B into a nonsingular matrix such that in its inverse, the block matrix on the upper left corner is E. We now show how to border the matrix if E is the Moore-Penrose inverse or the group inverse of B.

THEOREM 5.1. *Let B be given. Then the matrix $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ satisfies $\psi(A) = \psi(B)$ and has a g-inverse of the form $\begin{pmatrix} B^\dagger & F \\ G & O \end{pmatrix}$ if and only if C has full column rank with $R(C) \subseteq N(B^*)$ and D has full row rank with $R(D^*) \subseteq N(B)$. In this case, $X = -DB^\dagger C = O$ and*

$$A^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}.$$

Proof. Note that $N(BB^\dagger) = N(B^*)$ and $N((EB)^*) = N(B^\dagger) = N(B)$, and the necessity and sufficiency follow from Corollary 4.5.

It is easy to verify that $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}$ is a g-inverse of A. Thus by Corollary 3.5(2), we have

$$A^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix},$$

where $X = -DB^\dagger C = O$. □

Combining Corollary 4.6 with Theorem 5.1, we have

COROLLARY 5.2. [5] *Let B be a $p_1 \times q_1$ matrix with $rank(B) = r$. Suppose the columns of $C \in C_{p_1-r}^{p_1\times(p_1-r)}$ are a basis of $N(B^*)$ and the columns of $D^* \in C_{q_1-r}^{q_1\times(q_1-r)}$ are a basis for $N(B)$. Then the matrix*

$$A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$$

is nonsingular and its inverse is

$$A^{-1} = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}.$$

If B is square and has group inverse, we can get a bordering $\begin{pmatrix} B & * \\ * & O \end{pmatrix}$ of B such that it has a g-inverse in the form $\begin{pmatrix} B^\sharp & * \\ * & O \end{pmatrix}$. Part (ii) of the following result is known. We generalize it to any bordering, not necessarily nonsingular, in part (i).

THEOREM 5.3. *Let B be $n \times n$ and with index 1. Then*
(i) *there exist matrices C of full column rank with $R(C) \subseteq N(B)$ and D of full row rank with $R(B) \subseteq N(D)$ which satisfy $DC = I$ such that $\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}$ is a g-inverse of $\begin{pmatrix} B & C \\ D & O \end{pmatrix}$ with $\psi(A) = \psi(B)$;*
(ii) ([8], [14], [17]) *for any matrix C of full column rank with $R(C) = N(B)$ and any matrix D of full row rank with $R(B) = N(D)$, the matrix*

$$A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$$

is nonsingular and

$$A^{-1} = \begin{pmatrix} B^\sharp & C(DC)^{-1} \\ (DC)^{-1}D & O \end{pmatrix}.$$

Proof. (i): Consider the rank factorization of $I - BB^\sharp$ given by

$$I - BB^\sharp = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} D \\ V \end{pmatrix}.$$

Note that $BB^\sharp = B^\sharp B$, and we have

$$I - B^\sharp B = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} D \\ V \end{pmatrix}.$$

Obviously $R(C) \subseteq N(A)$ and $R(B) \subseteq N(D)$. As in the proof of Theorem 4.4(i), we conclude that $\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}$ is a g-inverse of $\begin{pmatrix} B & C \\ D & O \end{pmatrix}$ with $\psi(A) = \psi(B)$, since $X = -DB^\sharp C = O$.

(ii): By Corollary 4.6, the nonsingularity of the matrix $\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}$ under the conditions $R(C) = N(A)$ and $R(B) = N(D)$ can be easily seen. We now prove that for any matrix C of full column rank with $R(C) = N(B)$ and any matrix D of full row rank with $R(B) = N(D)$, DC is nonsingular.

In fact, if $DCx = O$, then $Cx \in R(C)$ and $Cx \in N(D)$. Since $R(C) = N(B)$, $N(D) = R(B)$ and $R(B) \cap N(B) = \{0\}$, we have $Cx = O$ and therefore $x = 0$. Thus DC is nonsingular.

By Lemma 4.3, $C(DC)^{-1}D$ is an idempotent matrix and

$$I - BB^\sharp = I - B^\sharp B = C(DC)^{-1}D$$

is a rank factorization. From Corollary 4.6, we know that

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & C \\ (DC)^{-1}D & O \end{pmatrix}$$

are nonsingular and in fact

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix}^{-1} = \begin{pmatrix} B^\sharp & C(DC)^{-1} \\ D & O \end{pmatrix}.$$

Note that

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} = \begin{pmatrix} B & C \\ D & O \end{pmatrix} \begin{pmatrix} I & O \\ O & (DC)^{-1} \end{pmatrix}.$$

The result follows immediately from the two equations preceding the one above. □

Remark 5.4. Theorem 5.3(ii) can be used to compute the group inverse of the matrix $(I-T)^\sharp$ which plays an important role in the theory of Markov chains, where T is the transition matrix of a finite Markov chain. For an n-state ergodic chain, it is well-known that the transition matrix T is irreducible and that $rank(I-T) = n-1$ [6, Theorem 8.2.1]. Hence by Theorem 5.3(ii) we can compute the group inverse $(I-T)^\sharp$ of $I-T$ by a bordered matrix.

Let c be a right eigenvector of T and d a left eigenvector, that is c and d satisfy $Tc = c$ and $d^*T = d^*$, respectively. Then the bordered matrix $\begin{pmatrix} I-T & c \\ d^* & 0 \end{pmatrix}$ is nonsingular and

$$\begin{pmatrix} I-T & c \\ d^* & 0 \end{pmatrix}^{-1} = \begin{pmatrix} (I-T)^\sharp & \frac{c}{d^*c} \\ \frac{d}{d^*c} & o \end{pmatrix}.$$

Thus the group inverse $(I - T)^\sharp$ can be obtained by computing the inverse of a nonsingular matrix.

REFERENCES

[1] J.K. Baksalary and G.P.H. Styan. Generalized inverses of partitioned matrices in Banachiewicz-Schur form. *Linear Algebra Appl.*, 354:41–47, 2002.

[2] R.B. Bapat. *Linear Algebra and Linear Models.* Springer-Verlag, New York, 2000.

[3] R.B. Bapat. Outer inverses: Jacobi type identities and nullities of submatrices. *Linear Algebra Appl.*, 361:107–120, 2003.

[4] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications.* Wiley, New York, 1974.

[5] J.W. Blattner. Bordered matrices. *Jour. Soc. Indust. Appl. Math.*, 10:528–536, 1962.

[6] S.L. Campbell and C.D. Meyer Jr. *Generalized Inverses of Linear Transformations.* Pitman, London, 1979.

[7] Chen Yonglin and Zhou Bingjun. On g-inverses and nonsingularity of a bordered matrix $\begin{pmatrix} A & B \\ C & O \end{pmatrix}$. *Linear Algebra Appl.*, 133:133–151, 1990.

[8] N. Eagambaram. $(i, j, \cdot, k)$-inverse via bordered matrices. *Sankhy: The Indian Journal of Statistics, Ser. A*, 53:298–308, 1991.

[9] W.H. Gustafson. A note on matrix inversion. *Linear Algebra Appl.*, 57:71–73, 1984.

[10] M. Fiedler and T.L. Markham. Completing a matrix when certain entries of its inverse are specified. *Linear Algebra Appl.*, 74:225–237, 1986.

[11] K. Nomakuchi. On the characterization of generalized inverses by bordered matrices. *Linear Algebra Appl.*, 33:1–8, 1980.

[12] K. Manjunatha Prasad and K.P.S. Bhaskara Rao. On bordering of regular matrices. *Linear Algebra Appl.*, 234:245–259, 1996.

[13] Yongge Tian. The Moore-Penrose inverse of $m \times n$ block matrices and their applications. *Linear Algebra Appl.*, 283:35–60, 1998.

[14] G. Wang. A Cramer rule for finding the solution of a class of singular equations. *Linear Algebra Appl.*, 116:27–34, 1989.

[15] Musheng Wei and Wenbin Guo. On g-inverses of a bordered matrix: revisited. *Linear Algebra Appl.*, 347:189–204, 2002.

[16] Y. Wei. Expression for the Drazin inverse of a 2x2 block matrix. *Linear and Multilinear Algebra*, 45:131–146, 1998.

[17] Y. Wei. A characterization for the W-weighted Drazin inverse and a Cramer rule for the W-weighted Drazin inverse solution. *Appl Math. Comput.*, 125:303–310, 2002.